Thus we have an increasing Ω-sequence $A_1, A_2, \dots, A_\omega, \dots, A_\alpha, \dots$ and we have to show that $\lim\limits_{\alpha\to\Omega} A_\alpha = C\Lambda$.

Otherwise there would exist a set $A_\Omega \neq C\Lambda$ (and obviously $\neq \Lambda$) such that $A_\alpha < A_\Omega$ for all $\alpha < \Omega$. But it is easily seen that for every α there is a β such that

$$E_\alpha - A_\beta = \Lambda \quad \text{or} \quad E_\alpha + A_\beta = C\Lambda;$$

hence (for all α),

$$\text{either} \quad E_\alpha - A_\Omega = \Lambda \quad \text{or} \quad E_\alpha + A_\Omega = C\Lambda;$$

it would follow, by the lemma, that (5.1) is of measure 0, contrary to the hypothesis, q. e. d.

Remark. Theorem 7^c can be proved in exactly the same way; all that's necessary, is to replace „measure 0" by „first category", and „positive outer measure" by „second category" in the above proof.

A positive, lower semi-continuous, non-degenerate function on a metric space.

By

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§ 1. Introduction. Many terms in analysis such as the order of a zero or pole of an analytic function, the multiplicity of a branch point, a critical point, an extremal, the index of a critical point or extremal, a non-degenerate critical point or extremal, etc. admit useful topological definition. See [1]) Kuratowski (1), Seifert and Threlfall (1), Morse (3), (6). These definitions are not precisely equivalent to the definitions of analysis, but are equivalent in certain well determined consequences. The topological definitions give the principal relations in analysis in the large their proper setting, and lead to the simplest proofs. Artificial distinctions such as those between a critical point of a function of n variables and an extremal of an integral disappear.

It is, however, desirable that a topological theory which is to unify many forms of analysis of historic importance shall choose its definitions and axioms not so much in a subjective mood of abstract generality, as from the viewpoint of ready availability and interpretation in analysis. It is with these ideals in mind that the definitions and axioms necessary for the theory of a positive, lower semi-continuous, non-degenerate function F on a metric space S are presented.

The principal term in analysis to be topologically characterized, both in the small and in its global setting, is that of a non-degenerate critical point or extremal. Historically a critical point of a function f of class C'' of n variables $(x_1, \dots, x_n)$ is non-degenerate if the Hessian of the function fails to vanish at the critical point. An extremal g which satisfies the self-adjoint conditions of a boundary problem in the large is termed non-degenerate in the sense of analysis

[1]) References are listed at the end of the paper. References to Morse are indicated by the letter M.

if the accessory problem [M (2), p. 25] involving the Jacobi differential equations and accessory linear boundary conditions has no characteristic root which is null. A typical problem is that of finding geodesics which join two fixed points A and B on a given compact, connected, coordinate m-manifold M. See § 9. A geodesic g in such a problem is non-degenerate in the above sense if and only if B is not conjugate to A on g. See M (4).

A function all of whose critical points (including critical extremals in a boundary problem) are non-degenerate, is itself called non-degenerate. There is a sense in which degenerate functions have a measure zero among all admissible functions, and in this sense non-degenerate functions represent the general case. For an example see M (2), p. 233.

Three axioms on S and F are presented here for the first time. The topological definition of a non-degenerate critical point under the guise of Property C (§ 2) is new, and is shown in § 5 to be equivalent to an earlier definition of an homotopic critical point when the function F satisfies our three axioms. The complete set of relations between the numbers M_k of critical points of index k and the Betti numbers R_k of S is derived.

The singular homology theory of Eilenberg (1) is used. Its adequacy is a consequence of the topological non-degeneracy of the critical points. A Vietoris or related topology would be required to obtain similar results, were the function degenerate.

It is shown that the three axioms can be verified in a typical field of analysis. A purely topological application is made in determining the Betti numbers of the space of paths joining two fixed points on an m-sphere ($m>2$). When the function F is not assumed to be topologically non-degenerate there are two other levels upon which theories have been built. (See M (3) and (5)). These levels are briefly contrasted.

§ 2. Definitions. The space S shall be an abstract metric space with points $p, q, \ldots$ with distances $pq=qp$, $pq>0$ unless $p=q$, $pp=0$ and $pq \leqslant pr+rq$. The function F is defined over S and is to be numerically valued, positive and single-valued. We admit the value $+\infty$. The subsets of points p of S for which $F(p)\leqslant c$ or $F(p)<c$, c finite, will be denoted by S_c and S_{c-} respectively. The points of S_{c-} are said to be *below* c, and the points p on the locus $F(p)=c$ at the *F-level* c.

If $F(p)=c<\infty$, an *F-neighborhood* of p is a neighborhood of p relative to any set S_b for which $b>c$. If p is in a subset A of S, an *e-neighborhood* of p relative to A is the set of points q in A for which $pq<e$. The use of an F-neighborhood instead of an ordinary neighborhood will make no formal difference in the proofs. The reason for the introduction of F-neighborhoods is that the principal assumptions made concerning F and S have been verified in very difficult theories such as minimal surface theory, provided F-neighborhoods are used.

Let A be a subset of S. Let I represent an interval, $0\leqslant t\leqslant 1$, for a variable t termed the *time*. A deformation D of A on S is defined by a continuous mapping of $A\times I$ into S,

$$D: A\times I \to S,$$

in which the submapping of A into S defined by D when $t=0$, shall be the identity. The image of a pair (x,t), $x\in A$, $t\in I$, under D, is a point $D(x,t)$ in S. We say that $D(x,t)$ *replaces* x at the time t. For fixed x the path in S defined by $D(x,t)$, $0\leqslant t\leqslant 1$, is termed the *trajectory* of x, and $D(x,1)$ the *final image* x^1 of x. For fixed t, $D(x,t)$ defines a continuous mapping

$$D^t: A\to S.$$

The *initial* mapping D^0 is the identical mapping of A onto A; D^1 is the *terminal* mapping of A into S.

Let D_1 be a deformation of A in which the final image of A under D_1 is A^1. Let D_2 be a deformation of a subset of S which contains A^1. Then the *product deformation* $D_3=D_2D_1$ of A is defined by setting

$$D_3(x,t)=D_1(x,2t) \qquad (0\leqslant t\leqslant \tfrac{1}{2})$$
$$D_3(x,t)=D_2(x^1,2t-1) \qquad (\tfrac{1}{2}<t\leqslant 1),$$

where x is in A, and x^1 is the final image of x under D_1. A product deformation

$$d_n=D_nD_{n-1}\ldots D_1=D_n[D_{n-1}\ldots D_1] \qquad (d_1=D_1)$$

of A is inductively defined whenever the final image of A under d_m ($m=1,\ldots,n-1$) is in the subset of S over which D_{m+1} is defined.

We shall need a *composite deformation* $D_2 \cdot D_1$ defined as follows. Let D_1 be a deformation of A in which A^t is the image of A at the time t; $0 \leqslant t \leqslant 1$. Let D_2 be a deformation of the union of the sets A^t, $0 \leqslant t \leqslant 1$, with $D_2(x,1) \equiv x$ for $x \in A$. The composite deformation $D_3 = D_2 \cdot D_1$ of A is defined by setting

$$D_3(x,t) = D_2[D_1(x,t),1] \qquad (0 \leqslant t \leqslant 1) \qquad (x \in A).$$

This definition does not involve $D_2(p,t)$ when $t<1$. However in the application the assumption that D_2 is a proper F-deformation will involve $D_2(p,t)$ when $t<1$.

A continuous mapping T of A into S will be said to be *deformable* into the identity if T is the terminal mapping D^1 of some deformation D of A. If the union of the trajectories of points of A under D is a set B, T will be said to be *deformable into the identity in B*.

An F-deformation of A. A continuous deformation of A which replaces each point x by a point x^t at the time t is called an F-deformation of A if

$$F(x^t) \leqslant F(x) \qquad (0 \leqslant t \leqslant 1).$$

An F-deformation of A is termed *proper* if

$$F(x^t) < F(x)$$

whenever $x^t \neq x$.

The identity mapping of A onto A for each t defines a special F-deformation which is trivially proper. If F is defined in the (u,v) plane with $F(u,v) = u^2+v^2$, the replacing of each point (u,v) by the point $[u(1-t), v(1-t)]$ defines a proper F-deformation of the (u,v)-plane into the origin, holding the origin fast.

The essential topological properties of a non-degenerate critical point or extremal motivate the next definition.

Points σ with Property C. A point σ of S, $F(\sigma)$ finite, will be said to have Property C if there exists a proper F-deformation D_σ of some F-neighborhood $U(\sigma)$ of σ such that

(C'). *D_σ leaves σ invariant and deforms $U(\sigma)$ into a topological r-disc $K(\sigma)$ on S $(r \geqslant 0)$ which contains σ as an interior*[2]) *point and is below $F(\sigma)$ except for σ when $r>0$, and which reduces to σ when $r=0$.*

[2]) The point σ is to be an interior point of $K(\sigma)$ in the sense that the antecedent of σ on the defining euclidean r-disc k shall be an interior point of k.

(C''). *The terminal mapping D_σ^1 of $K(\sigma) \cap U(\sigma)$ defined by D_σ is F-deformable in $K(\sigma)$ into the identity, holding σ fast.*

When σ has the Property C, the integer r is termed its index.

When the index $r=0$, σ affords a proper, relative minimum to F. For the deformation of $U(\sigma)$ into σ displaces every point p of $U(\sigma)$ not σ, so that $F(\sigma) < F(p)$, since D_σ is proper.

Example 2.1. Suppose that S is the euclidean 2-plane with rectangular coordinates (x,y). Set $F(x,y) = x^2 - y^2$. Then the origin σ has Property C. We take D_σ as the deformation which replaces each point (x,y) by the point $[x(1-t),y]$ at the time t, $0 \leqslant t \leqslant 1$. The terminal image of an ordinary neighborhood

$$x^2 + y^2 \leqslant e^2 \qquad (e>0)$$

of σ is the 1-disc $-e \leqslant y \leqslant e$, $x=0$. On this 1-disc $F(x,y) < F(0,0)$ except at σ. Condition (C'') is satisfied in this case, since the terminal mapping D_σ^1 of the 1-disc is the identity.

The fact that D_σ^1 is the identical mapping of $K(\sigma)$ onto itself in the preceding example, extends to any non-degenerate critical point of a function $f(x_1,\dots,x_n)$ of class C''', provided D_σ is properly defined. See M (3), p. 46. In more general cases it seems that one must be content with a deformation of D_σ^1 into the identity. Topologically, however, a deformation into the identity has for our purposes the essential properties of the identity.

The F-reducibility of S at infinity. We shall say that S is F-reducible at infinity, if corresponding to any compact subset A of S there exists an F-deformation D^A of A into some subset S_c of S. The value of c will depend upon A in general, and in general there will exist no one F-deformation of S into a set S_c. An assumption that S is reducible at infinity conditions all subsets of S on which F is unbounded, even if F is never infinite on S.

Example 2.2. Let H be a closed set in the (x,y) plane which in terms of y and the polar angle θ has the form

$$\frac{\pi}{4} \leqslant \theta \leqslant \frac{3\pi}{4}, \qquad y \leqslant 1.$$

Let open circular discs with centers at the points $1-\frac{1}{n}$, $n=2,3,\ldots$, and with radii so small that the closed discs fail to intersect, be removed from H to form S. Let S have the metric of the (x,y) plane. Let

$$F=\frac{1}{1-y} \qquad (y\neq 1),$$

with $F=\infty$ when $y=1$. The space S is compact. Any continuous deformation of S in S must leave the point $(0,1)$ fixed. There is accordingly no F-deformation of S into a set S_c, so that S is not F-reducible at infinity.

Example 2.3. Let S' be the space S of Example 2.2 with the point $(0,1)$ removed. Let F be defined as in Example 2.2. Any compact subset A of S' is bounded from $(0,1)$, and admits an F-deformation into a subset S'_c of S', so that S' is F-reducible at infinity. There is no one F-deformation of S' into a subset S'_c.

§ 3. The three axioms on S and F. In the terms defined in the preceding section the three axioms on S and F upon which the theory will be built are as follows.

Axiom I. *The metric space S shall be arc-wise connected, the function F positive, and the sets S_c compact for each finite c.*

Axiom II. *S shall be F-reducible at infinity.*

Axiom III. *There shall exist a set (σ) of points σ in S with the following properties:*

(1) *Each point σ shall have the Property C.*

(2) *The number of points σ below any finite F-level c shall be finite.*

(3) *Corresponding to any finite constant c there shall exist a proper F-deformation Δ_c of S_c in which the points of (σ) in S_c are invariant and all other points of S_c are displaced, while the value of $F[E(x)]$ at the final image $E(x)$ of x in S_c, shall vary continuously with x in S_c.*

We shall now obtain the first consequences of these axioms. F could equally well be replaced by a function which is bounded below instead of being positive. The addition of a constant to F would change none of the properties to be deduced.

Axiom I implies that F is lower semi-continuous. In particular let $[p_n]$ be a sequence of points p_n in S which converge to a point p. In accordance with the definition of lower semi-continuity we must show that

$$\liminf F(p_n)\geqslant F(p). \tag{3.1}$$

Let $c=\lim.\inf. F(p_n)$. If $c=\infty$ (3.1) holds. If c is finite let e be any positive constant. There will exist a subsequence $[q_r]$ of $[p_n]$ such that $[q_r]$ is in S_{c+e}. The set S_{c+e} is closed in S, so that p is in S_{c+e}. That is $F(p)\leqslant c+e$. Since e is arbitrarily small $F(p)\leqslant c$, and the proof of the theorem is complete.

The deformation Δ_c^n. We shall make use of the n-fold product of Δ_c by itself, regarded as a deformation of S_c. Let $E^n(x)$ be the final image of x in S_c under Δ_c^n. A point y which is given as an image $E^n(x)$ of a point x of S_c will be said to have a *deformation index* n relative to Δ_c.

A value of F at any one of the points σ of the set (σ) of Axiom III will be termed *critical* and any other value *ordinary*. We shall prove the fundamental theorem.

Theorem 3.1. *Let a be a critical value of F and $(w_1,\ldots,w_k)=(w)$ the set of points of (σ) at the F-level a. If $c>a$ is any constant such that there are no critical values of F on the interval $a<F\leqslant c$, there exists an F-deformation of S_c into a subset of $(w)\cup S_{a-}$, holding each point of (σ) on S_c fast.*

To prove this theorem three lemmas are required.

Lemma 3.1. *If $[b,c]$ is a closed interval containing no critical values of F, then the difference*

$$\theta(x)=F(x)-F[E(x)] \qquad (E(x)=E^1(x)) \tag{3.2}$$

is bounded from zero for x on S_c-S_{b-}.

If the lemma were false there would exist a sequence of points $[p_r]$ on S_c-S_{b-} such that $\theta(p_r)$ converges to 0 with $1/r$. Without loss of generality we can suppose that p_r converges to a point p in S_c. There are two cases to be considered.

Case I. $F(p)<b$. Since Δ_c is an F-deformation $F[E(p)]\leqslant F(p)<b$. We have

$$\lim\theta(p_r)\geqslant b-\lim F[E(p_r)]=b-F(E(p))>0 \tag{3.3}$$

contrary to the choice of $[p_r]$; in (3.3) we have made use of the continuity of $F(E(x))$ in accordance with Axiom III.

Case II. $F(p)\geqslant b$. With $\theta(x)$ lower semi-continuous for x in S_c

$$\liminf \theta(p_r)\geqslant F(p)-F(E(p)). \tag{3.4}$$

Since $F(p)$ is on the interval $[b,c]$, p is not a point of σ, and in accordance with Axiom III, p is displaced into a point $E(p)\neq p$. Since Δ_c is a proper F-deformation the right member of (3.4) is then positive, contrary to the choice of $[p_r]$. We infer the truth of the lemma.

The preceding lemma implies the existence of an integer n so large that the final image of S_c under Δ_c^n is below the F-level b of the lemma. If as in the theorem $[a,c]$ is a closed interval in which a is the only critical value, b can be taken arbitrarily on the interval $a<b<c$, and we can infer that

$$\lim_{n=\infty} [\max_x F[E^n(x)]]=a \qquad (x \text{ in } S_c]. \tag{3.5}$$

The limit in (3.5) cannot exceed a by virtue of the preceding lemma, and it cannot be less than a because $F[E^n(w_i)]=a$ for every n.

An *F-neighborhood* of the set (w) of the theorem shall be any neighborhood of (w) relative to S_b where $b>a$. Recall that $F(w_i)=a$.

Lemma 3.2. *Let $[a,c]$ be a closed interval in which a is the only critical value. Let (w) be the subset of points of (σ) at the F-level a, and let $U(w)$ be an F-neighborhood of (w). If n is sufficiently large Δ_c^n deforms S_c into a subset of $U(w)\cup S_{a-}$.*

If the lemma were false there would exist a sequence $[p_r]$ of points p_r, on S_c but not below a, with deformation indices $n(r)>1$ relative to Δ_c, which become infinite with r, while p_r is never on $U(w)$. It follows from this choice of $[p_r]$ and from (3.5) that

$$\lim F(p_r)=a. \tag{3.6}$$

Let q_r be a point such that $E(q_r)=p_r$. Without loss of generality we can suppose that p_r and q_r converge respectively to points p and q in S_c. From (3.6) and the continuity of $F(E(x))$ for x in S_c, we infer that

$$a=\lim F(p_r)=\lim F[E(q_r)]=F[E(q)].$$

From the lower semi-continuity of F and from (3.5)

$$F(q)\leqslant \liminf F(q_r)\leqslant \limsup F(q_r)\leqslant a. \tag{3.7}$$

Since $F(q)\geqslant F[E(q)]=a$, (3.7) reduces to

$$F(q)=\lim F(q_r)=a. \tag{3.8}$$

From (3.8), (3.6) and the relation $E(q_r)=p_r$,

$$\lim \theta(q_r)=\lim F(q_r)-\lim F(p_r)=a-a=0. \tag{3.9}$$

We shall find a contradiction to (3.9). For that purpose we shall show that q is not a point w_i. Otherwise q_r would converge to w_i as $1/r$ tends to 0; then $E(q_r)=p_r$ would likewise converge to w_i, since $E(w_i)=w_i$, and $E(x)$ is continuous. [Axiom III]. Hence $F(p_r)=F[E(q_r)]$ would converge to $F[E(q)]=F(w_i)=a$. For r sufficiently large the points p_r would be in $U(w)$ contrary to the choice of $[p_r]$. Hence q is not a point w_i.

Since q is not a point w_i and $F(q)=a$, q is not a point of (σ). By virtue of the lower semi-continuity of $\theta(x)$ for x in S_c, and the fact that Δ_c displaces every point of S_c not in (σ), (such as q), we infer that

$$\liminf \theta(q_r)\geqslant \theta(q)>0$$

contrary to (3.9). This establishes the lemma.

Lemma 3.3. *Let A be a subset of S and p a point of A. If D is a* (proper) *F-deformation on S of the closure of an e-neigborhood N_e^A of p relative to A, there exist a* (proper) *F-deformation D^* of all of A, which deforms points initially in A and sufficiently near p, as does D, and which reduces to the identity for points initially in $A-N_e^A$.*

Let e_1 be a positive constant $<e$. Under D^* points initially on the e_1-neigborhood $N_{e_1}^A$ of p relative to A, shall be deformed as under D, while points initially on $A-N_e^A$ shall be held fast. Of the remaining points of A let q be a point such that $e_1\leqslant qp<e$. The point q shall have the same image q^t under D^* as under D until the time t reaches $t(q)$, where $1-t(q)$ divides the interval $[0,1]$ in the ratio in which qp divides $[e_1,e]$. For $t\geqslant t(q)$, q^t shall remain fixed. In particular when $qp=e_1$ $t(q)=1$, and q is deformed as under D. When $qp=e$, $t(q)=0$, and q is held fast for all time. The resulting deformation D^* is continuous by virtue of the continuity of the mapping D of $\overline{N}_e^A\times I$ into S. The deformation D^* is clearly a (proper) F-deformation since D is, and the proof of the lemma is complete.

Proof of Theorem 3.1. Since each point w_i of the theorem is a point σ, with $F(\sigma)=a$, and σ has Property C, we can state the following. If b is any constant for which $b-a$ is positive and sufficiently small, and e is any sufficiently small positive constant, there exists a proper F-deformation D_{w_i} of the closure of an e-neighborhood $N(e,w_i,b)$ of w_i relative to S_b, which holds w_i fast, and caries $N(e,w_i,b)$ into a subset of w_i+S_{a-}. We suppose e so small that the neighborhoods $N(e,w_i,b)$, $[i=1,\ldots,k]$ are disjoint and contain no point of (σ) other than the respective points w_i. In accordance with the preceding lemma there exists a proper F-deformation $D^*_{w_i}$ of S_b which deforms points initially in $N(e_1,w_1,b)$, for some positive e_1, as does D_{w_i}, while points initially in $S_b-N(e,w_i,b)$ are held fast. The set

$$U(w)=\text{Union } N(e_1,w_1,b) \qquad (i=1,\ldots,k)$$

is an F-neighborhood of (w). By virtue of Lemma 3.2 the deformation Δ_c^n, for n sufficiently large, will carry S_c into a subset of $U(w)+S_{a-}$. For such an n the product deformation

$$D^*_{w_1}D^*_{w_2}\ldots D^*_{w_k}\Delta_c^n$$

will carry S_c into a subset of $(w)\cup S_{a-}$, holding the points of (σ) in S_a fast.

This completes the proof of Theorem 3.1.

Theorem 3.2. *Let $a<F<b$ be an interval on which there are no critical values $F(\sigma)$. Let $[\sigma]_a$ be the set of points in (σ) at the F-level a. The respective homology groups of S_{b-} and of $Y=S_{a-}\cup[\sigma]_a$ are isomorphic, with each homology class V in S_{b-} corresponding to the sub-class of V in Y.*

This theorem follows from Lemma 11.2 once statements (a) and (b) have been proved.

(a) *Each k-cycle x in S_{b-} is homologous in S_{b-} to a k-cycle in Y.*

(b) *A k-cycle z in Y which is bounding in S_{b-} is bounding in Y.*

Proof of (a). Let Δ_b be the deformation associated with S_b in Axiom III. Then for x in S_{b-}

$$\partial\hat{\Delta}_b x=\bar{\Delta}_b x-x \qquad (\text{in } S_{b-})$$

in accordance with (11.1). The value of F at the terminal image $E(p)$ under Δ_b, of p in S_b, varies continuously with p, and hence on the compact carier of x is at most a constant $c<b$. We may suppose that $a<c<b$. Set $x'=\bar{\Delta}_b x$. Since x' is in S_c, we may apply the deformation D of S_c affirmed to exist in Theorem 3.1; then D deforms S_c into Y. Using (11.1)

$$\partial\hat{D}x'=\bar{D}x'-x' \qquad (\text{in } S_c).$$

Thus x is homologous in S_{b-} to $\bar{D}x'$ in Y. This establishes (a).

Proof of (b). By the hypothesis of (b), $z=\partial w$, where z is a k-cycle in Y, and w a $(k+1)$-chain in S_{b-}. As in the proof of (a), the chains $w'=\bar{\Delta}_b w$ and $z'=\bar{\Delta}_b z$ are in some set S_c, with $a<c<b$. We have $z'=\partial w'$. On applying the deformation D of Theorem 3.1 which deforms S_c into Y, we have terminally

$$\bar{D}z'=\partial\bar{D}w' \quad \text{or} \quad \bar{D}z'\sim 0 \text{ in } Y.$$

But z in Y is deformed in Y under Δ_b into z', and z' is deformed in Y under D into $\bar{D}z'$, so that z bounds in Y with $\bar{D}z'$. Thus (b) holds and the theorem follows.

§ 4. Property C.

It will be convenient to order the points (σ) of Axiom III in agreement with their F-values, ordering points of σ with the same F-value arbitrarily. Let w be any point of the set (σ). Let $(\sigma)_w$ denote the finite subset of points of (σ) whose order is at most that of w. Let $F(w)=a$. Set

$$X(w)=S\cup[\sigma]_w. \tag{4.1}$$

Indicating order among the points (σ) by the signs $>$, $<$ in the usual sense, we have a sequence (possibly finite) of points,

$$\sigma_1<\sigma_2<\ldots, \tag{4.2}$$

and a sequence of sets

$$X(\sigma)\subset X(\sigma)\subset\ldots$$

We shall be concerned with rel. cycles in the respective sets $X(\sigma)$, taken mod. $X(\sigma)-\sigma$.

Lemma 4.1. *Given e, any rel. k-cycle z in $X(\sigma)$ is rel. homologous in $X(\sigma)$ to a rel. cycle in an e-neighborhood of σ.*

Let $B^n z$ be the n-th barycentric subdivision of z. By virtue of (11.2)

$$\partial\varrho z=B^1z-z-\varrho\partial z \qquad [\text{in } X(\sigma)].$$

Since ∂z is in $X(\sigma)-\sigma$, $\varrho\partial z$ is likewise, so that $z\sim B^1 z$ rel. in $X(\sigma)$. Hence $B^n z\sim z$ rel. in $X(\sigma)$. Let $B^n z$ be written in the form,

$$B^n z = g_i\tau_i = u+v \qquad (i=1,\ldots,n) \tag{4.3}$$

where g_i is in G, τ_i is a k-cell, and u is the sum of the terms $g_i\tau_i$ for which τ_i is in $X(\sigma)-\sigma$. Then it is trivial that $B^n z\sim v$ rel. in $X(\sigma)$. If, however, n is sufficiently large, the norm of each cell τ_i in (4.3) will be less than e, and v (possibly null) will be in the e-neighborhood of σ. Since $z\sim B^n z\sim v$ rel. in $X(\sigma)$, the lemma follows.

We shall refer to the topological r-disc $K(\sigma)$ associated with the point σ. Cf. Property C. For chains in $K(\sigma)$ the modulus will invariably be $K(\sigma)-\sigma$. The following lemma is proved exactly as was Lemma 4.1.

Lemma 4.2. *Any rel. cycle in $K(\sigma)$ is rel. homologous in $K(\sigma)$ to a cycle in an e-neighborhood of σ.*

The case where σ has the index $r=0$. Both Lemma 4.1 and 4.2 are trivial when $r=0$; in this case $K(\sigma)=\sigma$, and the modulus is empty. A rel. cycle in $K(\sigma)$ is then a cycle in $K(\sigma)$. The k-dimensional rel. homology group of $K(\sigma)$ consists of a null element except when $k=0$. When $k=0$ there is one non-trivial homology class.

The following theorem is central.

Theorem 4.1. *The n-th homology group of $X(\sigma)$ mod. $X(\sigma)-\sigma$ is isomorphic with the n-th homology group of $K(\sigma)$ mod. $K(\sigma)-\sigma$, with each relative homology class V in $X(\sigma)$ corresponding to the subclass of V in $K(\sigma)$.*

The proof of this theorem depends upon the validity of statements (a) and (b).

(a) *Each rel. homology class V of $X(\sigma)$ contains at least one rel. cycle in $K(\sigma)$.*

(b) *Each rel. k-cycle x in $K(\sigma)$ which is rel. bounding in $X(\sigma)$ is rel. bounding in $K(\sigma)$.*

Proof of (a). We refer to the F-neighborhood $U(\sigma)$ used in describing Property C. § 2. It follows from Lemma 4.1 that there exists a chain z of the homology class V in $U(\sigma)$. On making use of the deformation D_σ affirmed to exists when σ has Property C, (11.1) yields the relation

$$\partial\hat{D}_\sigma z = \bar{D}_\sigma z - z - \hat{D}_\sigma\partial z.$$

Here $\hat{D}_\sigma\partial z$ is in $X(\sigma)-\sigma$, since ∂z is in this modulus; $\bar{D}_\sigma z$ is in $K(\sigma)$, and $\hat{D}_\sigma z$ is in $X(\sigma)$. Thus z is rel. homologous in $X(\sigma)$ to $\bar{D}_\sigma z$ in $K(\sigma)$.

Proof of (b). Let V' be the rel. homology class in $K(\sigma)$ which contains x. Let e be positive and so small that points of $K(\sigma)$ within a distance e of σ are in the F-neighborhood $U(\sigma)$. By hypothesis in (b), $\partial w\equiv x$ for some $(k+1)$-chain w in $X(\sigma)$, where $\equiv$ indicates equality up to some chain in the modulus $X(\sigma)-\sigma$. Let w and z be subdivided barycentrically so many times (say r times) that the norms of the resulting chains are less than e. Since $\partial w\equiv x$ we have

$$\partial B^r w = B^r\partial w \equiv B^r x \qquad [\text{mod. } (X(\sigma)-\sigma)]. \tag{4.4}$$

From the chain $B^r w = g_i\tau_i$ let all $(k+1)$-cells τ_i in $X(\sigma)-\sigma$ be dropped, giving a $(k+1)$-chain w'. Let x' be similarly obtained from $B^r x$. Then

$$\partial w' \equiv x' \qquad [\text{mod. } (X(\sigma)-\sigma)]. \tag{4.5}$$

It is clear (cf. proof of Lemma 4.1), that $B^r x$ and hence x' is in V'. Moreover w' and x' are in $U(\sigma)$ by virtue of the choice of e, so that in accordance with Property C of σ one can apply $\bar{D}_\sigma$ to both members of (4.5) with the result

$$\partial\bar{D}_\sigma w' \equiv \bar{D}_\sigma x' \qquad [\text{mod. } (X(\sigma)\dot{-}\sigma)]. \tag{4.6}$$

The terminal mapping D_σ^1 of D_σ restricted to $K(\sigma)\cap U(\sigma)$, is F-deformable into the identity in $K(\sigma)$, holding σ fast. Cf. Property C. Hence

$$\bar{D}_\sigma x' \sim x' \qquad [\text{rel. in } K(\sigma)]. \tag{4.7}$$

From (4.6) and (4.7) we infer that $x'\sim 0$ rel. in $K(\sigma)$. With x' each other chain in V' is rel. bounding in $K(\sigma)$.

Statement (b) is accordingly true, and the theorem follows from Lemma 11.2.

Corollary 4.1. *If σ is a point (σ) of index r, the Betti numbers P_k of $X(\sigma)$ mod. $X(\sigma)-\sigma$ are δ_r^k.*

By virtue of the preceding theorem the desired Betti numbers are those of the r-disc $K(\sigma)$ mod. $(K(\sigma)-\sigma)$, and as is well-known must then equal δ_r^k.

§ 5. Homotopic critical points. The question has been left open as to whether the points of (σ) are the only points of S which satisfy Property C. This question can be answered with the aid of an earlier topological definition [M (3)] of a critical point as follows. A point p of S at which F is finite will be called homotopically ordinary if some F-neighborhood of p admits a proper F-deformation $(0 \leqslant t \leqslant 1)$ which ultimately displaces p. A point p at which F is finite, and which is not homotopically ordinary, will be termed a *homotopic critical point*. We begin with the following lemma.

Lemma 5.1. *A point p of S which has Porperty C is a homotopic critical point.*

Suppose that $F(p)=a$ and set $X=S_{a-}+p$. The proof of Theorem 4.1 and its Corollary shows that the Betti number P_k of X, mod. $X-p$, is δ_r^k, where r is the index of p. There is accordingly an r-cycle z in X, mod. $X-p$, which is rel. non-bounding. If the lemma were false there would exist a proper F-deformation D of some F-neighborhood of p which displaces p. By modifying D, as in the proof of Lemma 3.3, one can obtain a proper F-deformation D_0 defined over all of X, and identical with D in its deformation of points of X sufficiently near p. We have

$$\partial \hat{D}_0 z = \overline{D}_0 z - z - \hat{D}_0 \partial z \qquad \text{(in } X\text{)}.$$

Since the carrier of z is below a, except at most at p, and D_0 displaces p, $\overline{D}_0 z$ is below a. The chain $\hat{D}_0 \partial z$ is below a with ∂z. Thus z is rel. bounding in X. From this contradiction we infer the truth of the lemma.

The following theorem characterizes the set (σ) in Axiom III.

Theorem 5.1. *The set of homotopic critical points of F, the set of points of S with Property C, and the set (σ) are identical.*

It follows from the preceding lemma that each point of (σ) is a homotopic critical point. Conversely every point p not in σ, with $F(p)$ finite, is homotopically ordinary; for if $F(p)<c$, Axiom III provides a proper F-deformation Δ_c which displaces p. On the other hand a point p with Property C is a homotopic critical point (Lemma 5.1); while a point p without Property C is not a point of (σ) and, as has just been seen, is homotopically ordinary. This completes the proof of the theorem.

We shall refer to the points of (σ) as *critical* points, dropping the adjective homotopic. The reader will, of course, be aware that a point (x^0) may be a differential critical point of a function $f(x_1, \dots, x_n)$ without being a homotopic critical point. For example, the point $x=0$ is a differential critical point of x^3, but not a homotopic critical point. However, it is easy to show that a critical point of a function $f(x_1, \dots, x_n)$ of class C''' at which the Hessian is not zero, has Property C, and so is a homotopic critical point. See M (3), p. 46. The definition of a homotopic critical point is one that is applicable in any space of any dimension, finite or infinite.

§ 6. Critical points of linking or non-linking types. We shall make a classification of the critical points of (σ) which depends in part on their ordering at any given critical level. Suppose that σ is a critical point with index r. According to Corollary 4.1 there is exactly one non-trivial r-dimensional homology class V in $X(\sigma)$ mod. $X(\sigma)-\sigma$.

If V contains an „absolute" r-cycle λ_r, σ will be said to be of linking type, otherwise of non-linking type. In the latter case μ_r shall denote any rel. cycle in V.

According as σ is of linking or non-linking type, the cycle λ_r or the rel. cycle μ_r forms a rel. homology base for rel. r-cycles in $X(\sigma)$. The following lemma makes the origin of the term linking clear.

Lemma 6.1. *A necessary and sufficient condition that a critical point σ be of linking type is that the boundary of every rel. k-cycle z in $X(\sigma)$ bound in $X(\sigma)-\sigma$.*

The condition is proved necessary as follows. Let r be the index of σ. Suppose first that $k=r$. When σ is of linking type, λ_r is a rel. homology base for r-cycles in $X(\sigma)$ so that a relation

$$\text{(6.1)} \qquad \partial w = z - g\lambda_r - u \qquad [u \text{ in } X(\sigma)-\sigma]$$

holds with g in G, and w an $(r+1)$-chain in $X(\sigma)$. Relation (6.1) implies that $\partial z = \partial u$ so that ∂z bounds u in $X(\sigma)-\sigma$. If $k \neq r$, z is rel. bounding in $X(\sigma)$. [Cf. Corollary 4.1], and ∂z accordingly bounds in $X(\sigma)-\sigma$. [Cf. Lemma 11.1].

The condition is proved sufficient as follows. According to Corollary 4.1 there exists a rel. non-bounding r-cycle z in $X(\sigma)$. By hypothesis ∂z bounds a chain u in $X(\sigma)-\sigma$. Hence $z-u$ is an absolute cycle λ_r in the rel. homology class of z. Hence z is of linking type.

Lemma 6.2. *If the Betti numbers of $X(\sigma)-\sigma$ are finite*[3]) *and if σ is a critical point of index r, then in Case I, (σ of linking type) a minimal r-homology base for $X(\sigma)$ can be obtained from one for $X(\sigma)-\sigma$ by the addition of the linking r-cycle λ_r associated with σ, while in Case II, (σ of non-linking type) a minimal $(r-1)$-homology base for $X(\sigma)$ can be obtained from a suitably chosen minimal $(r-1)$-homology base for $X(\sigma)-\sigma$ by removing a suitable $(r-1)$-cycle. When $k \neq r$ in Case I, or $k \neq r-1$ in Case II, any minimal k-homology base for $X(\sigma)-\sigma$ is a minimal k-homology base for $X(\sigma)$.*

Case I. The proof in this case consists in showing that a minimal k-homology base for $X(\sigma)$ is obtained by adding $\delta_r^k\lambda_r$ to a minimal k-homology base B_k for $X(\sigma)-\sigma$. To that end let

w = a $(k+1)$-chain in $X(\sigma)$
z = a k-cycle in $X(\sigma)$
u = a k-cycle in $X(\sigma)-\sigma$
g = an element in G.

Recall that $\delta_r^k\lambda_r$ is a minimal k-homology base in $X(\sigma)$ mod. $X(\sigma)-\sigma$. Hence, given z,

$$\partial w = z - g\delta_r^k\lambda_r + u \qquad (r \text{ not summed}) \tag{6.1}$$

for a suitable choice of w, g, and u. Relation (6.1) shows that $\delta_r^k\lambda_r$ with B_k forms a k-homology base for $X(\sigma)$. It remains to show that this base is minimal.

There is no relation

$$\partial w = \lambda_r + u \qquad (k=r) \tag{6.2}$$

since λ_r would then be rel. bounding in $X(\sigma)$. Nor is there a relation

$$\partial w = u \qquad [k \neq r;\ u \not\sim 0 \text{ in } X(\sigma)-\sigma]. \tag{6.3}$$

For (6.3) would imply that w is a rel. cycle in $X(\sigma)$ with ∂w non-bounding in $X(\sigma)-\sigma$, so that by Lemma 6.1, σ could not be of linking type, contrary to fact. Thus neither (6.2) nor (6,3) can hold so that $\delta_r^k\lambda_r$, with B_k, forms a minimal k-homology base for $X(\sigma)$.

[3]) This hypothesis is of an inductive character and will presently be established.

Case II. For $k \neq r-1$ let B_k be a minimal k-homology base for $X(\sigma)-\sigma$. When $k=r-1$ in Case II, $\partial\mu_r$ is non-bounding in $X(\sigma)-\sigma$ by Lemma 6.1, so that there exist a minimal $(r-1)$-homology base for $X(\sigma)-\sigma$, composed of $\partial\mu_r$ and a set B_{r-1} of $(r-1)$-cycles. We shall show that B_k is a k-homology base for $X(\sigma)$ for all k.

Now $\delta_r^k\mu_r$ is a minimal k-homology base in $X(\sigma)$ mod. $X(\sigma)-\sigma$. Hence given z,

$$\partial w = z - g\delta_r^k\mu_r + u \qquad (r \text{ not summed}) \tag{6.4}$$

for a suitable choice of w, g, and u. In (6.4) every chain except $g\delta_r^k\mu_r$ is a cycle. Since μ_r is not a cycle, $g\delta_r^k=0$. This shows that a k-homology base for $X(\sigma)-\sigma$ is a base for $X(\sigma)$. Since $\partial\mu_r$ bounds in $X(\sigma)$, B_k is a k-homology base for $X(\sigma)$, even when $k=r-1$. It remains to prove that B_k is minimal in $X(\sigma)$.

Suppose that a relation

$$\partial w = g_i u_i \neq 0 \qquad (u_i \text{ in } B_k) \tag{6.5}$$

held. Then w would be a rel. non-bounding $(k+1)$-cycle in $X(\sigma)$, in accordance with Lemma 11.1. This is possible at most if $k+1=r$. But if $k+1=r$, $w-g\mu_r$ would be rel. bounding in $X(\sigma)$ for some g in G, by virtue of Corollary 4.1. Hence the absolute cycle

$$\partial w - g\partial\mu_r = g_i u_i - g\partial\mu_r$$

would be bounding in $X(\sigma)-\sigma$ by Lemma 11.1, contrary to the nature of B_{r-1}. Thus no relation (6.5) holds and B_k is minimal in $X(\sigma)$.

The theorem follows.

The Betti number $R_k(X,G)$ of the set X over G is the rank of the homology group of k-cycles in X over G. The numbers R_k alone, shall refer to $R_k(S,G)$.

Theorem 6.1. *The Betti numbers of the sets $X(\sigma)$ and S_{b-} are finite.*

The sets $X(\sigma)$. Let the critical points in (σ) be written in order $\sigma_1 < \sigma_2 < \ldots$ admitting the possibility that the number of such critical points is finite. Set

$$R_k[X(\sigma_m), G] = R_k^m.$$

Observe that $F(\sigma_1)$ is an absolute minimum of F and that $X(\sigma_1) = \sigma_1$.

Hence $R_k^1 = \delta_0^k$. Proceeding inductively we assume that R_k^{m-1} is finite, and seek to prove that R_k^m is finite. If $F(\sigma_{m-1}) = F(\sigma_m)$

$$X(\sigma_m) - \sigma_m = X(\sigma_{m-1}),$$

and the Betti numbers of $X(\sigma_m) - \sigma_m$ are R_k^{m-1}, thus finite by inductive hypothesis. In this case it follows from Lemma 6.2 that the numbers R_k^m are finite. In case $F(\sigma_{m-1}) < F(\sigma_m)$, if one sets $F(\sigma_m) = b$, then

$$X(\sigma_m) - \sigma_m = S_{b-}.$$

In accordance with Theorem 3.2, with $X(\sigma_{m-1}) = Y$ therein,

$$R_k(S_{b-}, G) = R_k^{m-1},$$

and the Betti numbers R_k^{m-1} of $X(\sigma_m) - \sigma_m$ are again finite by inductive hypothesis. As before Lemma 6.2 implies that R_k^m is finite.

The sets S_{b-}. We discard the trivial case in which $b \leqslant F(\sigma_1)$, and let σ_m be the last point in (σ) for which $F(\sigma_m) < b$. It follows from Theorem 3.2, with $X(\sigma_m) = Y$ therein, that $R_k(S_{b-}, G) = R_k^m$. But R_k^m has been proved finite, and the proof of the theorem is complete.

Now that it is known that the numbers R_k^m are finite, Lemma 6.2 with the aid of Theorem 3.2, gives the following theorem.

Theorem 6.2. *Let σ° and σ be successive critical points in (σ), with σ of index r. Then in Case I (σ of linking type) a minimal r-homology base for $X(\sigma)$ can be obtained from one for $X(\sigma^\circ)$ by the addition of a suitable r-cycle in $X(\sigma)$, while in Case II, (σ of non-linking type) a minimal $(r-1)$-homology base for $X(\sigma)$ can be obtained by removing a suitable $(r-1)$-cycle from a suitably chosen minimal $(r-1)$-homology base for $X(\sigma^\circ)$. When $k \neq r$ in Case I, or $k \neq r-1$ in Case II, any minimal k-homology base for $X(\sigma^\circ)$ is a minimal k-homology base for $X(\sigma)$.*

Corollary 6.1. *If σ_m is of index r, $(m > 1)$ then $R_k^m = R_k^{m-1}$ except that*

$$R_r^m - R_r^{m-1} = 1$$

in case σ_m is of linking type, while

$$R_{r-1}^m - R_{r-1}^{m-1} = -1,$$

in case σ_m is of non-linking type.

The following lemma will be used in § 7.

Lemma 6.3. *If for a given k there is a last critical point σ_m in (σ), among points of index k and $k+1$, then the Betti number R_k of S is R_k^m.*

This lemma will follow from Lemma 11.2 of the Appendix, once statements (a) and (b) are proved.

(a) *Any k-cycle x in S is homologous in S to a k-cycle in $X(\sigma_m)$.*

(b) *Any k-cycle z in $X(\sigma)_m$ which is bounding in S is bounding in $X(\sigma_m)$.*

Proof of (a). It follows from Axiom II that x is homologous on S to a k-cycle x' in some set S_{b-}. We can suppose that $b > F(\sigma_m)$. From the hypothesis of the lemma, from Theorem 3.2 and Theorem 6.2, we can make the inference (c). *A minimal k-homology base B_k for $X(\sigma_m)$ is a minimal k-homology base for S_{b-}*. It follows from (c) that x' is homologous in S_{b-} to a k-cycle in $X(\sigma_m)$, so that (a) is true.

Proof of (b). It follows from Axiom II that any k-cycle z in $X(\sigma_m)$ which is bounding in S is bounding in some set S_{b-} with $b > F(\sigma_m)$. If $z_1, \ldots, z_n$ is a minimal k-homology base for $X(\sigma_m)$ we have $z \sim g_i z_i$ in $X(\sigma_m)$. As stated in (c), $z_1, \ldots, z_n$ is a minimal k-homology base for S_{b-} as well as for $X(\sigma_m)$; hence each $g_i = 0$, since z is bounding in S_{b-}. Thus $z \sim 0$ in $X(\sigma_m)$ and (b) is true.

The lemma follows as stated.

§ 7. Relations between the type numbers $M_0, M_1, \ldots$ and the numbers $R_0, R_1, \ldots$ when each M_k and R_k is finite.

We set

$R_k =$ the k-dimensional Betti number of S over G.

$M_k =$ the number of critical points of index k.

$a_k =$ the number of critical points of index k of linking type.

$b_k =$ the number of critical points of index k of non-linking type. When each M_k and R_k is finite

$$M_k = a_k + b_k \tag{7.1$'$}$$
$$R_k = a_k - b_{k+1} \tag{7.1$''$}$$
$$\left\{\begin{array}{l} k = 0, 1, \ldots \\ b_0 = 0, \end{array}\right.$$

where (7.1)″ follows with the aid of Lemma 6.3. Points of index $k = 0$ are isolated relative minima of linking type, so that $b_0 = 0$. From (7.1) we obtain the basic relations

$$M_k - R_k = b_k + b_{k+1} \geqslant 0 \qquad (k = 0, 1, \ldots). \tag{7.2}$$

Given the numbers R_k with $R_0=1$, it is possible to construct an S, and define an F on S, such that $M_k=R_k$ for each k, so that no more than R_k critical points of index k are necessary in *all* cases.

Let S be a compact, connected coordinate m-manifold M of class C''' (Cf. § 9) with a Riemanian metric. On M let f be a function of class C''' of non-degenerate type (i. e. with a non-vanishing Hessian at each critical point). Then it can be shown [M (3) § 10] that each critical point has Property C, and so is a homotopic critical point. Deformation along the trajectories orthogonal to the level manifolds $f=$constant suffice to establish the existence of the deformations Δ_c of Axiom III, so that all three Axioms are satisfied. It is of interest to note that f has no homotopic critical points of index $r>m$, since there exists no topological r-disc on M for which $r>m$.

There exist m-manifolds M such that for no non-degenerate function f on M is $M_k=R_k$ for all k. In fact this is easily seen to be the case for a 3-dimensional manifold which has the Betti numbers of the 3-sphere but is not homeomorphic to the 3-sphere.

Theorem 7.1. *When the numbers R_k and M_k are finite and each critical point is of linking type, then $R_k=M_k$, $k=0,1,\ldots$*

This theorem is available to determine the numbers R_k in cases where other methods fail. It has been used on the space of paths joining two points on an m-sphere (Cf. § 9), on the space of closed curves, and on various other spaces such as the symmetric square of an m-sphere. (Cf. M (2), p. 191).

The excess numbers $E_k=M_k-R_k$. From (7.2) we find that

$$E_0-E_1+\ldots(-1)^m E_m=(-1)^m b_{m+1} \qquad (m=0,1,\ldots) \tag{7.3}$$

and so obtain the relations

$$\begin{cases} E_0 \geqslant 0 \\ E_1 \geqslant E_0 \\ E_2 \geqslant E_1-E_0 \\ E_3 \geqslant E_2-E_1+E_0 \\ \ldots \qquad \ldots \end{cases} \tag{7.4}$$

Given integers $R_k\geqslant 0$ and $M_k\geqslant 0$ satisfying (7.4) with $R_0=1$, it is possible to construct a space S and function F with the given R_k and M_k, so that *no relations beyond* (7.4) *exist between these numbers alone.*

The conditions (7.4) are derived from (7.1). Conversely conditions (7.4) imply that M_k and R_k $(k=0,1,\ldots)$ satisfy relations of the form (7.1). In fact if one defines b_{m+1} by (7.3) then (7.2) is satisfied; if (7.1)'' is then used to define a_k, (7.1)' is satisfied.

A particular consequence of (7.2) is that

$$E_{k-1}+E_{k+1}\geqslant E_k \qquad (k=1,2,\ldots)$$

§ 8. The case of infinite M_k. In simple variational problems in which S is the space of paths joining two fixed points A and B on a manifold M, and F is the integral of length on M, some M_k can be infinite, while every R_k is finite (see M (1), § 17).

When some of the M_k are finite, the following remarks can be made. If for a given k, M_k and M_{k+1} are finite, then for this k, (7.1) holds, using Lemma 6.3 as previously, so that $M_k\geqslant R_k$. If M_k is finite for $k=0,1,\ldots,m+1$, the first $m+1$ inequalities in (7.4) still hold. If M_{r-1}, M_r, M_{r+1}, M_{r+2} are finite, then as before

$$E_{r-1}+E_{r+1}\geqslant E_r, \tag{8.1}$$

even if all other M_k are infinite.

To prepare for the general theorem let

$M_k(\sigma)=$ the number of critical points of index k in $X(\sigma)$.

$a_k(\sigma)=$ the number of critical points of index k in $X(\sigma)$ of linking type.

$b_k(\sigma)=$ the number of critical points of index k in $X(\sigma)$ of non-linking type.

$R_k(\sigma)=$ the k-th Betti number of $X(\sigma)$.

The following relations are immediate:

$$M_k(\sigma)=a_k(\sigma)+b_k(\sigma) \tag{8.2$'$}$$

$$R_k(\sigma)=a_k(\sigma)-b_{k+1}(\sigma) \tag{8.2$''$}$$

The algebraic consequences of (8.2) are formally the same as those of (7.1). We shall need two lemmas.

Lemma 8.1. *Given k, with R_k finite, then for any critical point σ of sufficiently high order in (σ) there exists a minimal k-homology base B_k for S in $X(\sigma)$. For any such σ, and for the given k,*

$$R_k=R_k(\sigma)-q_k, \tag{8.3}$$

where $q_k\geqslant 0$, and there exists a minimal k-homology base for $X(\sigma)$ composed of B_k and of q_k cycles, z_i, $(i=1,\ldots,q_k)$ such that each z_i is bounding in $X(\sigma')$, provided $\sigma'>\sigma$ is a critical point of sufficiently high order in (σ).

Since R_k is assumed finite the existence of a base B_k for S in some $X(\sigma)$ follows from Axiom II and Theorem 3.2. Relation (8.3) then holds. As a consequence there will exist a minimal k-homology base for $X(\sigma)$, composed of B_k and k-cycles u_i $(i=1,\dots,q_k)$. Since B_k is also a base for S, there will exist a k-cycle v_i, linearly dependent (over G) on the cycles of B_k, and such that the k-cycle $z_i=u_i-v_i$ is bounding in S. It follows from Axiom II that for some critical point $\sigma'>\sigma$ of sufficiently high order in (σ), each z_i will bound in $X(\sigma')$. It is clear that B_k and the q_k cycles z_i again form a minimal k-homology base for $X(\sigma)$.

Note: The critical points σ and σ' in the Lemma depend upon k and the ordering of (σ), and their order may become infinite as k does.

Lemma 8.2. *Under the conditions of Lemma 8.1 there exist q_k critical points σ^i of index $k+1$ such that*

$$\sigma<\sigma^1<\sigma^2<\dots<\sigma^{q_k}\leqslant\sigma'.$$

The k-cycles z_i of Lemma 8.1 suffer no proper homology in $X(\sigma)$, but are bounding in $X(\sigma')$. If $q_k>0$ there is accordingly a critical point σ^1 of least order, with $\sigma<\sigma^1\leqslant\sigma'$, such that some proper sum $g^iz^i=y_1$ is bounding in $X(\sigma^1)$. With change of notation if necessary, we can suppose that $g^1\neq 0$. If σ^0 is the immediate predecessor of σ^1 in (σ) then the cycles $y_1,z_2,\dots,z_{q_k}$ (z_i as in Lemma 8.1) are part of a minimal k-homology base in $X(\sigma^0)$, while y_1 is bounding in $X(\sigma^1)$. It follows from Theorem 6.2 that σ^1 is of index $k+1$, and that the cycles $z_2,\dots,z_{q_k}$ are part of a minimal k-homology base in $X(\sigma^1)$.

If $\sigma^1=\sigma'$ then $q_k=1$ and the lemma is proved. If $q_k>1$ then $\sigma^1<\sigma'$, and one can similarly infer the existence of a critical point σ^2 of least order in (σ) with $\sigma^1<\sigma^2\leqslant\sigma'$, such that some proper sum $y_2=g^iz^i$, $i=2,\dots,q_k$ is bounding in $X(\sigma^2)$. One sees that σ^2 is of index $k+1$. Continuing one arrives at the existence of the q_k critical points σ^i of the lemma.

The k-th type number of a set of critical points is defined as the number of critical points of index k in the set.

Theorem 8.1. *Suppose that the first $r+1$ Betti numbers of S,*

$$R_0, R_1, \dots, R_r,$$

are finite. Let H be an arbitrary finite subset of the critical points of F with indices at most r. There exists a finite subset $K\supset H$ of the critical points of F with indices at most r, and with type numbers

$$m_0, m_1, \dots, m_r$$

such that if $e_k=m_k-R_k$, $k=0,\dots,r$, then

$$(8.4)\qquad \begin{cases} e_0\geqslant 0\\ e_1\geqslant e_0\\ e_2\geqslant e_1-e_0\\ \dots\ \dots\\ e_r\geqslant e_{r-1}-e_r+\dots(-1)^{r-1}e_0.\end{cases}$$

It is clear that Lemmas 8.1 and 8.2 can be satisfied for $k=0,1,\dots,r$ by a common critical point σ of arbitrarily high order in (σ), together with a critical point $\sigma'>\sigma$. We take σ so advanced in (σ) that $X(\sigma)$ contains each point in H. In accordance with Lemmas 8.1 and 8.2

$$(8.5)\qquad R_k(\sigma)=R_k+q_k \qquad (k=0,\dots,r)$$

with $q_k\geqslant 0$, and there exists a set H_{k+1} of q_k critical points σ^* of index $k+1$ with $\sigma<\sigma^*\leqslant\sigma'$. Let H' be the subset of those critical points of F in $X(\sigma)$ whose indices do not exceed r. We shall satisfy the theorem by setting

$$K=H'\cup H_1\cup\dots\cup H_r.$$

If m_k is the k-th type number of K

$$(8.6)\qquad m_k=M_k(\sigma)+q_{k-1} \qquad (k=0,\dots,r)$$

where $q_{-1}=0$. On setting $e_k=m_k-R_k$ it follows from (8.5) and (8.6) that

$$e_k=M_k(\sigma)-R_k(\sigma)+q_{k-1}+q_k \qquad (k=0,1,\dots,r)$$

and then from (8.2) that

$$(8.7)\qquad e_k=b_k(\sigma)+b_{k+1}(\sigma)+q_{k-1}+q_k.$$

The equalities (8.7) imply that for $k=0,\dots,r$

$$(8.8)\qquad e_0-e_1+\dots(-1)^ke_k=(-1)^k[b_{k+1}(\sigma)+q_k],$$

and (8.4) follows immediately.

Corollary 8.1. *If the Betti numbers R_k are finite $M_k \geqslant R_k$.*

From (8.7) we have $e_{k-1}+e_{k+1} \geqslant e_k$ for $k=1,2,\dots,r-1$. If all numbers R_k are finite the integer r in the theorem can be taken arbitrarily large, and the subset of critical points can be taken to include arbitrarily many of the M_k points of index k. Hence the corollary.

Corollary 8.2. *If the numbers R_k are finite then regardless of the finiteness of the numbers M_{k-1}, M_k, and M_{k+1}*

$$E_{k-1}+E_{k+1} \geqslant E_k \qquad (k=1,2,\dots).$$

That is, if M_k is infinite at least one of the numbers M_{k-1} or M_{k+1} must be infinite. We depart now from the assumption that R_k is finite, and prove the theorem.

Theorem 8.2. *If for a given k, R_k is infinite, M_k is infinite.*

If R_k is infinite it follows from Axiom II that $R_k(\sigma_m)$ must become infinite with m; for each non-bounding k-cycle in S has a homologous k-cycle on some $X(\sigma_m)$. But we have

$$M_k \geqslant M_k(\sigma_m) \geqslant R_k(\sigma_m)$$

on using (8.2), so that M_k must be infinite.

§ 9. Spaces S of paths. The application of the results of the preceding sections to spaces of paths is a by-product of variational theory in the large. To make this application it is merely necessary to verify Axioms I, II, III for the space of paths S and the length function J which replaces F. Complete details of this verification will be given at length in another place. In this paper we shall be content with defining the required deformations and giving references where the necessary preliminary analysis is found.

Let M be a compact connected Hausdorff space which is a coordinate manifold in the following sense. There shall exist a subset of the open sets of M termed „coordinate regions" which cover M and are topological images of regions in a Euclidean m-space of coordinates (x), (y), etc.; if coordinate regions N_x and N_y with coordinates (x) and (y) respectively intersect in N, then on N the transformations from coordinates (x) to (y) shall be non-singular and of class $C^{(4)}$. It is no additional restriction to suppose M metricized consistent with its topology.

The space S. Let A and B be two fixed points on M and let S be a component of the space of paths (in general singular) joining A to B in M, with the distance pq between two paths p and q defined as by Fréchet. A basic problem is to determine the topological characteristics of the space S, in particular its Betti numbers. These characteristics are independent of the choice of A and B on M; for there is a homeomorphic mapping of M onto M in which (A,B) goes into any prescribed pair (A',B') of distinct points of M.

On M let J be a curve integral

$$J=\int f(x,\dot{x})\,dt$$

be defined in terms of the respective local coordinates (x) of M. We suppose that $f(x,z)$ satisfies the usual conditions of being positive, positively homogeneous in the contravariant vectors $(z)\neq(0)$, and positively regular, and of class C''' in (x,z) in each local coordinate system. The value of J can be defined for each path p of S, whether p is locally rectifiable or not. Thus $J(p)$ may equal $+\infty$. An extremal joining A to B is called non-degenerate if B is not conjugate to A on g. It can be shown (M (2), p. 233) that for „almost all" pairs A, B on M, no degenerate extremal joins A to B, although there will in general be infinitely many extremals joining A to B; such pairs A, B are called *non-degenerate* and the function J termed non-degenerate. It can be shown that there exists a positive constant ϱ with the following properties. Any two points P and Q on M whose distance on M is less than ϱ can be joined by a unique extremal arc $E(P,Q)$ of absolute minumum type, which varies continuously in the sense of Fréchet with (P,Q). We term $E(P,Q)$ an *elementary extremal*.

The deformation $\theta(r)$. In M (3), p. 59, a proper J-deformation of any compact subset H of paths of S is defined. The parameters (r) are a set of positive numbers $(r_1,\dots,r_n)$ whose sum is 1. Paths p of S are referred to s, a parameter proportional to „reduced μ-length", with $0 \leqslant s \leqslant 1$, and the interval for s divided into n successive intervals of euclidean length $r_1,\dots,r_n$. If n is sufficiently large and each r_i sufficiently small, the n subarcs of p, represented by the respective subintervals of $0 \leqslant s \leqslant 1$ of lengths r_i, will have diameters on M less than ϱ, and will admit proper J-deformations into the elementary arcs joining their end points. In this way a proper J-deformation $\theta(r)$ of H will be defined. There may be paths in H

which are not displaced under $\theta(r)$; but (given H) if the set (r) be replaced by a new set $(r'_1,\dots,r'_n)$ in which $r_i - r'_i$ $(i=1,\dots,n-1)$ is positive and sufficiently small, the product deformation $\theta(r')\,\theta(r)$ will displace each path of H which is not an extremal. The sets S_c on which $J \leqslant c$ are easily proved compact, and if $H = S_c$ the product deformation

$$\Delta_c = \theta(r')\,\theta(r) \tag{9.1}$$

will satisfy Axiom III, provided the points (σ) of Axiom III are taken as the extremals joining A to B. The hypothesis of non-degeneracy insures that there are at most a finite number of these points (σ) on S_c.

Axiom II, that S is J-reducible at infinity, is readily verified. Given a compact subset H of S, a deformation $\theta(r)$ can be defined over H for a proper choice of (r), and will deform H into some set S_c.

Verification of Property C. Let σ be an extremal joining A to B on M, and suppose that $J(\sigma) < c$. Let (r) be chosen so that the deformation $\theta(r)$ is well defined over S_c. The final image σ^1 of σ under $\theta(r)$ is identical with σ as a path, but it may also be regarded as a broken extremal with vertices at the points

$$A, P_1, \dots, P_{n-1}, B$$

of σ determined by the set (r). We cut „transversally" across σ at the point P_i $(i=1,\dots,n-1)$ with a $(m-1)$-manifold M_i, regular and of class C'''. A broken extremal joining A to B composed of r elementary extremal arcs with vertices on the successive manifolds M_i will be called *canonical*. The existing variational theory [4]) in the large permits the definition of a proper J-deformation W_σ of any set of broken extremals of S whose vertices ($n-1$ in number) are sufficiently near the respective vertices of σ^1, into a topological k-disc $K(\sigma)$ in S of canonical extremals, holding $K(\sigma)$ fast, with σ an „inner" point of $K(\sigma)$, and with $K(\sigma)-\sigma$ below $J(\sigma)$. The deformation

$$D_\sigma = W_\sigma \theta(r) \tag{9.2}$$

will meet the requirements of Property C.

[4]) The deformation Z of M, (3), p. 70, deforms broken extremals with vertices sufficiently near those of σ^1 into canonical broken extremals. To obtain W_σ it remains to deform these canonical extremals into a topological k-disc of canonical extremals, and this is readily established on using the theorems of § 10 and § 16 of M (3).

It remains to show that the terminal mapping D^1_σ defined by D_σ when $t=1$, restricted to a sufficiently small neighborhood $N(\sigma)$ of σ, relative to $K(\sigma)$, admits a J-deformation into the identity in $K(\sigma)$, holding σ fast. The required deformation is obtained by restricting the composite deformation $W_\sigma \cdot D_\sigma$, defined in § 2, to a sufficiently small neighborhood $N(\sigma)$ of σ relative to $K(\sigma)$.

It follows from M (4) *and* M (3) § 10 *that the index k of the extremal σ equals the number of conjugate points of A on σ prior to B.*

We can now prove the following theorem.

Theorem 9.1. *The Betti numbers of the space S of paths joining two points A and B on an m-sphere M with $m > 2$ are*

$$R_k = 1 \qquad (k \equiv 0 \text{ mod. } m-1) \tag{9.3'}$$

$$R_k = 0 \qquad (k \not\equiv 0 \text{ mod. } m-1). \tag{9.3''}$$

Let the points A and B on M be taken as any two distincts points not the extremities of a diameter of M. Given any integer $i \geqslant 0$ there is a geodesic g_i leading from A to B which passes through the point A' opposite to A exactly i times. The point A' appears as a conjugate point of A of multiplicity $m-1$, so that there are in effect $i(m-1)$ conjugate points of A on g prior to B. Thus the index of g is $i(m-1)$. Moreover these geodesics g_i are the only geodesics which join A to B on M. With the function F on S defined by the length integral, we have

$$M_k = 1 \qquad (k \equiv 0 \text{ mod. } m-1) \tag{9.4'}$$

$$M_k = 0 \qquad (k \not\equiv 0 \text{ mod. } m-1). \tag{9.4''}$$

The Betti numbers of S are finite by virtue of Lemma 6.3, and the relations

$$M_k = a_k + b_k \qquad (k=0,1,\dots) \tag{9.5'}$$

$$R_k = a_k - b_{k+1} \tag{9.5''}$$

hold. If one substitutes the values of M_k given by (9.4) in (9.5), and makes use of the fact that $R_0 = 1$, $b_0 = 0$, and that all symbols in (9.5) represent non-negative integers it is found, for $m > 2$, that (9.3) gives the only solution for the numbers R_k.

We have established the theorem for a special choice of A and B. But it is easy to show [Cf. M (3), p. 51] that the k-homology group of a space of paths $S(A,B)$ joining A to B on M is isomorphic with the corresponding group of $S(A',B')$, where (A',B') is any pair of points on M, distinct or coincident. Thus the theorem holds regardless of the choice of A and B.

The case $m=2$. The theorem remains true even when $m=2$. In this case $R_k=1$ for every k. The above method of proof fails because the substitution in (9.5) of the values M_k given by (9.4) when $m=2$ does not determine the numbers R_k uniquely. A general proof of the theorem, including the case $m=2$, is given in M (2), p. 247.

The Betti numbers of the spaces of paths have been determined in many other cases. The determination of the Betti numbers for the space of closed curves associated with the restricted problem of three bodies would greatly advance celestial mechanics.

§ 10. The three levels of the theory. The theory should be developed at three levels.

Level 1. The theory of non-degenerate functions. This theory is illustrated by the present lectures. There is a sense in which such a theory is general. It is that the non-degenerate functions constitute „almost all" admissible functions, provided an appropriate and natural definition of measure is introduced.

Level 2. The second type of theory is illustrated by M (3). The space S is again a connected metric space with $F>0$ thereon. The theory is based on two hypotheses, the *F-accessibility* of S and the *upper-reducibility* of F. Under the hypothesis of F-accessibility any non-bounding Vietoris k-cycle which is homologous to zero mod. S_{c+e}, for each positive e, is homologous to a k-cycle in S_c. The upper-reducibility of F is probably the best substitute for the upper semi-continuity of F which can be established for the classical integrals of variational theory. The fundamental theorem is as follows.

Theorem 10.1. *Let V be a homology class of Vietoris k-cycles in S. If there is a k-cycle of V in some set S_a, there is a least value of b such that there is a k-cycle of V in S_b. If c is this minimum value of b there is a homotopic critical point of F at the level c.*

Level 3. In the most general theory critical points are replaced by critical sets, which are of relatively unlimited complexity, and these critical sets are classified in terms of associated groups G_k, $(k=0,1,\ldots)$ of relative k-cycles („caps"). The preceding relations

between the numbers M_k, R_k, a_k, b_k, b_{k+1}, etc., are replaced by isomorphisms between special groups of $(k+1)$-caps and their boundaries. The theory can be brought to a finitary basis by the introduction of the notion of rank and span of a relative k-cycle. See M (5). Here every cycle is ignored whose essential characteristics (as defined) can be specified using a range (span) of values of F less than a fixed constant e in magnitude.

§ 11. Appendix on singular homology theory. If X is a topological space, singular chains in X over a discrete group G will be taken in the sense of Eilenberg (1). We shall impose the condition that G be a field, and this requires an additional assumption. Let a_i and b_i be elements in G, and z_i a singular k-chain, $(i=1,\ldots n,)$. Understanding that repeated indices of any sort are to be summed we require that

$$a(b_i z_i)=(ab_i)z_i \qquad (i=1,\ldots,n).$$

Singular cycles, both relative and absolute, are defined as by Eilenberg. The group of singular k-chains in X over G is denoted by

$$C^k(X,G) \qquad (k=0,1,\ldots).$$

Let X and Y be two topological spaces. Let a collection of homomorphisms, one for each dimension k

$$\tau: C^k(X,G)\to C^k(Y,G)$$

be given, where a k-chain z in X has the k-chain τz as its image in Y. If ∂ denotes the boundary operator and $\partial\tau=\tau\partial$, then τ is called a *chain transformation*. (See Eilenberg *op. cit.*).

Departing momentarily from Eilenberg, we define the *carrier* of a chain and the *norm* of a singular cell. With Eilenberg a singular cell τ in X is a continuous mapping

$$\tau: s\to X$$

of a non-degenerate euclidean simplex s with ordered vertices, into X. An appropriate definition of „equivalent" mappings is given. The geometric simplex upon which s is based, without any ordering of vertices is denoted by $|s|$. The *carrier* of τ is the image of $|s|$ in X. If $z=g_i\tau_i$ $(i=1,\ldots,n)$ is a k-chain in which the τ_i's are distinct cells and no $g_i=0$, the carrier of z shall be the union of the carriers of the τ_i. The *norm* of τ_i, in case X is a metric space, shall be the diameter of its carrier. The norm of the above chain $z=g_i\tau_i$ shall be the maximum of the norms of the τ_i.

Corresponding to any continuous mapping

$$\varphi: X \to Y$$

there is induced a chain transformation

$$\bar{\varphi}: C^k(X,G) \to C^k(Y,G) \qquad (k=0,1,\ldots).$$

Besides the property $\partial\bar{\varphi}=\bar{\varphi}\partial$ the essential characteristic of $\bar{\varphi}$ for us is that the carrier of $\bar{\varphi}z$ is in the image under φ of the carrier of z. Eilenberg uses the symbol φ both for the mapping and for the chain transformation.

Let D, $0\leqslant t\leqslant 1$, be a continuous deformation on S of a subset X of S. The chain transformation induced by the terminal mapping

$$D^1: X \to S$$

will be denoted by $\bar{D}$, dropping the superscript 1. Recall that the initial mapping D^0 is the identity. Corresponding to D there can be defined, in a variety of ways, a homomorphism

$$\hat{D}: C^k(X,G) \to C^{k+1}(X,G) \qquad (k=0,1,\ldots)$$

such that for any k-chain z in X

$$\partial\hat{D}z = \bar{D}z - z - \hat{D}\partial z, \tag{11.1}$$

with the very essential property that the carrier of $\hat{D}z$ is in the trajectory under D of the carrier of z. We term $\bar{D}z$ the *final image* of z under D.

The barycentric subdivision Bz of a chain z. Here one has a special chain-transformation

$$B: C^k(X,G) \to C^k(X,G) \qquad (k=0,1,\ldots)$$

with three essential properties: (1) the carrier of Bz is in the carrier of z; (2) the n-th iterate $B^n z = B^{n-1}(Bz)$ has a norm which for a given chain z tends to zero with $1/n$; (3) there is an associated homomorphism

$$\varrho: C^k(X,G) \to C^{k+1}(X,G) \qquad (k=0,1,\ldots)$$

such that

$$\partial\varrho z = Bz - z - \varrho\partial z, \tag{11.2}$$

where the carrier of ϱz is in the carrier of z. The notation Bz is ours.

Let A be a subset of X. A k-chain w in X whose boundary ∂w is in A is called a *k-cycle mod. A*. If u is a $(k-1)$-chain in X one writes $u\sim 0$ mod. A in X if there exists a k-chain w in X such that

$$\partial w = u - v \qquad (\text{with } v \text{ in } A). \tag{11.3}$$

Since $0=\partial\partial w=\partial u-\partial v$, u is a $(k-1)$-cycle mod. A. We state a lemma of frequent use.

Lemma 11.1. *If u is a $(k-1)$-chain ~ 0 mod. A, then ∂u bounds in A.*

This follows at once from (11.3), since v is a k-chain whose boundary equals ∂u, and v is in A.

If A is a subset of X invariably associated with X as a modulus, an n-cycle in X mod. A will be termed a rel. (relative) n-cycle in X, omitting reference to the constant modulus A. Similarly „bounding in X mod. A" is termed „rel. bounding in X". A cycle „homologous to zero in X mod. A" is termed „rel. homologous to zero". If Y is a subset of X it will be convenient to associate $A\cap Y$ with Y as a modulus, so that a rel. cycle in Y is a rel. cycle in X, but in general not conversely.

The following lemma is of a character common in topology. The conditions (a) and (b) are those most easily verified. The formulation is ours.

Lemma 11.2. *Let X be a topological space with an invariable modulus $A\subset X$, and Y a subspace of X with an invariable modulus $A\cap Y$. Let V be an arbitrary rel. homology class of k-cycles in X, and V' the subclass of chains of V in Y. If*

(a) *each rel. cycle in X is rel. homologous in X to a rel. cycle in Y and*

(b) *each rel. cycle in Y which is rel. bounding in X is rel. bounding in Y*

then V' is a rel. homology class of Y, and the mapping $V\to V'$ defines an isomorphism between the respective rel. homology groups of X and of Y.

Given V, V' is a rel. homology class in Y. For chains x and y in V' are in V, hence rel. homologous in X, and so by (b) rel. homologous in Y. If x is in V' and z is in the same rel. homology class in Y, then z is in V with x, and so in V'.

The mapping $V \to V'$ is 1—1 since V' cannot be a subclass of different homology classes V. Moreover each homology class V' is in some class V, so that the mapping is „onto" the appropriate group of Y.

Finally the mapping is a homomorphism; that is, if U and V are rel. k-homology classes in X then

$$U' + V' = (U + V)'. \tag{11.4}$$

To establish (11.4) it is sufficient to show that there is a chain common to both members of (11.4). To that end let x and y be chains in U' and V' respectively. Then $x+y$ is in $U'+V'$. But $x+y$ is also in $U+V$ and in Y, and so in $(U+V)'$. Thus (11.4) holds and the lemma follows.

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Axiomatic and algebraic aspects of two theorems on sums of cardinals.

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Introduction.

We shall concern ourselves in this paper with two theorems of the arithmetic of cardinal numbers which express, in two different ways, the fact that the sum of an infinite series of cardinals is the least upper bound of the sequence of partial sums. The theorems in question can be formulated as follows:

$\mathfrak{T}_1$. *Given any infinite sequence of cardinals* $\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_n, \ldots$ *and a cardinal* $\mathfrak{b}$, *if*

$$\mathfrak{b} < \sum_{n<\infty} \mathfrak{a}_n,$$

then there is a natural number p *such that*

$$\mathfrak{b} \leqslant \sum_{n<p} \mathfrak{a}_n.$$

For the purpose of this discussion, the symbol $\leqslant$ in the conclusion of $\mathfrak{T}_1$ could be changed to $<$.

$\mathfrak{T}_2$. *Given any infinite sequence of cardinals* $\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_n, \ldots$ *and a cardinal* $\mathfrak{b}$, *if*

$$\sum_{n<p} \mathfrak{a}_n \leqslant \mathfrak{b}$$

for every natural number p, *then*

$$\sum_{n<\infty} \mathfrak{a}_n \leqslant \mathfrak{b}.$$

Our discussion will aim to exhibit some essential differences between $\mathfrak{T}_1$ and $\mathfrak{T}_2$, both from an axiomatic and an algebraic point of view. While $\mathfrak{T}_1$ proves to be equivalent to the axiom of choice in its most general form, $\mathfrak{T}_2$ turns out to be derivable from