The mapping $V \to V'$ is 1—1 since V' cannot be a subclass of different homology classes V. Moreover each homology class V' is in some class V, so that the mapping is „onto" the appropriate group of Y.

Finally the mapping is a homomorphism; that is, if U and V are rel. k-homology classes in X then

$$U' + V' = (U+V)'. \tag{11.4}$$

To establish (11.4) it is sufficient to show that there is a chain common to both members of (11.4). To that end let x and y be chains in U' and V' respectively. Then $x+y$ is in $U'+V'$. But $x+y$ is also in $U+V$ and in Y, and so in $(U+V)'$. Thus (11.4) holds and the lemma follows.

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Axiomatic and algebraic aspects of two theorems on sums of cardinals.

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Introduction.

We shall concern ourselves in this paper with two theorems of the arithmetic of cardinal numbers which express, in two different ways, the fact that the sum of an infinite series of cardinals is the least upper bound of the sequence of partial sums. The theorems in question can be formulated as follows:

$\mathfrak{T}_1$. *Given any infinite sequence of cardinals* $\mathfrak{a}_0, \mathfrak{a}_1, \dots, \mathfrak{a}_n, \dots$ *and a cardinal* $\mathfrak{b}$, *if*

$$\mathfrak{b} < \sum_{n<\infty} \mathfrak{a}_n,$$

then there is a natural number p *such that*

$$\mathfrak{b} \leqslant \sum_{n<p} \mathfrak{a}_n.$$

For the purpose of this discussion, the symbol $\leqslant$ in the conclusion of $\mathfrak{T}_1$ could be changed to $<$.

$\mathfrak{T}_2$. *Given any infinite sequence of cardinals* $\mathfrak{a}_0, \mathfrak{a}_1, \dots, \mathfrak{a}_n, \dots$ *and a cardinal* $\mathfrak{b}$, *if*

$$\sum_{n<p} \mathfrak{a}_n \leqslant \mathfrak{b}$$

for every natural number p, *then*

$$\sum_{n<\infty} \mathfrak{a}_n \leqslant \mathfrak{b}.$$

Our discussion will aim to exhibit some essential differences between $\mathfrak{T}_1$ and $\mathfrak{T}_2$, both from an axiomatic and an algebraic point of view. While $\mathfrak{T}_1$ proves to be equivalent to the axiom of choice in its most general form, $\mathfrak{T}_2$ turns out to be derivable from a rather special case of this axiom. Moreover, $\mathcal{T}_2$ can be considerably generalized and given a form in which it applies to a comprehensive class of abstract algebraic systems, while in the case of $\mathcal{T}_1$ no possibility of such a generalization is seen[1]).

§ 1. Axiomatic aspect of the problem.

For the basis of the discussion we may take the axiom system of Zermelo from which the axiom of choice is supposed to be removed. The part of the discussion referring to $\mathcal{T}_1$ requires, however, the assumption that this axiom system has been supplemented by the replacement axiom of Fraenkel — or, at least, by the following weaker axiom:

For every set A there is a family of sets F which contains (as an element) A and which, whenever it contains a set X, also contains the family of all subsets of X.

Since, on our axiomatic basis, use of the notion of a cardinal may give rise to some doubts, we are going to eliminate this notion from $\mathcal{T}_1$ and $\mathcal{T}_2$, and in fact to replace it by the relation $\sim$ of set-theoretical equivalence (equality of powers). Nevertheless, we shall find it convenient to apply the notion of a cardinal and that of an ordinal in proofs; this will be done, however, with sufficient care so as to remove any possible doubts of an axiomatic nature.

The reformulated propositions $\mathcal{T}_1$ and $\mathcal{T}_2$ will be referred to as $\mathcal{P}_1$ and $\mathcal{P}_2$; they can be given the following form:

$\mathcal{P}_1$. *Given any infinite sequence of sets $A_0, A_1, \ldots, A_n, \ldots$ and a set B, if there is a set D such that*

$$B \sim D \subset \sum_{n<\infty} A_n,$$

then either

$$B \sim \sum_{n<\infty} A_n,$$

or else there are a natural number p and a set C for which

$$B \sim C \subset \sum_{n<p} A_n.$$

[1]) For notions and results from the domain of set theory involved in the present discussion consult Fraenkel [1] and Sierpiński [1]; for those applying to Boolean algebras see Birkhoff [1] (the figures in square brackets refer to the bibliography at the end of the paper). In these works, as well as in Schönflies [1], historical references regarding the origin of notions and results involved may also be found.

$\mathcal{P}_2$. *Given any infinite sequence of sets $A_0, A_1, \ldots, A_n, \ldots$ and a set B, if, for every natural number p, there is a set C such that*

$$\sum_{n<p} A_n \sim C \subset B,$$

then there is a set D for which

$$\sum_{n<\infty} A_n \sim D \subset B.$$

Since in our further discussion we shall always refer to $\mathcal{P}_1$ and $\mathcal{P}_2$, and not to $\mathcal{T}_1$ and $\mathcal{T}_2$, we can pass over the question of precise logical relations between $\mathcal{P}_1$ and $\mathcal{T}_1$, or $\mathcal{P}_2$ and $\mathcal{T}_2$. We can modify Propositions $\mathcal{P}_1$ and $\mathcal{P}_2$ by including in their hypotheses the condition that the sets $A_0, A_1, \ldots, A_n, \ldots$ are pairwise disjoint; it is easily seen, however, that the propositions thus obtained are equivalent to the original ones.

We are going to use the familiar set-theoretical symbolism to denote the membership relation, the set of all elements satisfying a given condition, the empty set, the set consisting of just one element, or just two elements, the inclusion relation between sets, the set-theoretical sum and product of two or more sets, the difference of two sets, the inverse of a biunique function (i. e., of a function f for which $f(x) = f(y)$ always implies $x = y$), the composition of two arbitrary functions, and the n^{th} iteration of a function (the 0^{th} iteration being the identity function). Given a set A and a function f, we denote by $\bar{f}(A)$ the set of all function values $f(x)$ correlated with those elements x of A for which f is defined, i. e., which belong to the domain of f; in symbols,

$$\bar{f}(A) = \mathop{E}_{f(x)} [x \in A].$$

Thus, $\bar{f}$ is a new function, which is defined over arbitrary sets. If

$$\bar{f}(A) = B$$

and the set A is included in the domain of f, we say that f maps A onto B. Thus, the formula

$$A \sim B$$

expresses the fact that there is a biunique function f which maps A onto B; this is clearly equivalent to saying that there is a biunique function g whose domain is A and whose counter-domain (range) is B.

By the *axiom of choice* we, of course, understand the statement that, for every family of non-empty pairwise disjoint sets, there is a set A which has with each set X of this family just one element in common. If the family is assumed to be denumerable, the corresponding statement will be referred to as the *restricted axiom of choice*. As is well known, the axiom of choice is equivalent to the so-called principle of choice by which, for every family of non-empty sets, there is a function f which correlates with every set X of this family an element $f(X)$ in X. Similarly, the restricted axiom of choice is equivalent to the restricted principle of choice. Two other statements, which are also known to be equivalent to the (general) axiom of choice, will be involved in our discussion: the well-ordering principle by which every set can be well ordered, and the trichotomy law by which, for any given sets A and B, either there is a set C such that

$$A \sim C \subset B,$$

or else there is a set D such that

$$B \sim D \subset A.$$

Theorem 1 [2]). *Proposition $\mathfrak{P}_1$ is equivalent to the axiom of choice.*

Proof: It is well known that $\mathfrak{P}_1$ follows from the axiom of choice. In fact, $\mathfrak{P}_1$ can easily be derived from $\mathfrak{P}_2$ by means of the trichotomy law and the Cantor-Bernstein equivalence theorem; and it will be seen from Theorem 2 below that $\mathfrak{P}_2$ can be established even without using the axiom of choice in its general form.

It remains to be shown that $\mathfrak{P}_1$ implies the axiom of choice. For this purpose we consider an arbitrary set A and we put

(1) $$A_0 = A \quad \text{and} \quad A_p = \sum_{n<p} A_n + \underset{X}{E}\,[X \subset \sum_{n<p} A_n] \quad \text{for } p = 1, 2, \ldots$$

We proceed to construct a set B which, together with $A_0, A_1, \ldots, A_n, \ldots$, will satisfy the hypothesis of $\mathfrak{P}_1$.

By a (binary) relation R we understand an arbitrary set of ordered couples; we assume that the notion of an ordered couple (x,y) has been defined in a familiar way by means of the formula

(2) $$(x,y) = \{\{x\}, \{x,y\}\}.$$

[2]) Theorem 1 was stated without proof (and in a slightly different form) in Lindenbaum-Tarski [1], p. 312.

Given a relation R, the set of all elements x such that, for some y, (x,y) or (y,x) belongs to R is called the field of R, in symbols, $F(R)$. R is said to be a well-ordering relation if the formulas $(x,y) \in R$ and $(y,x) \in R$ always imply $x = y$, and if every non-empty subset X of $F(R)$ contains an element x such that $(x,y) \in R$ for every $y \in X$. Thus, an example of a well-ordering relation is provided by the relation $\leqslant$ (and not $<$) between natural numbers.

Two relations R and S are called similar if there is a biunique function f which maps $F(R)$ onto $F(S)$ in such a way that the formulas

$$(x,y) \in R \quad \text{and} \quad (f(x), f(y)) \in S$$

are equivalent for all elements x and y in $F(R)$. By an ordinal (number) relative to a given set X we understand a non-empty set consisting of all those relations whose fields are included in X and which are similar to a certain well-ordering relation R.

Let B_0 be the set of all ordinals relative to A_0; and, for $p = 1, 2, \ldots$, let B_p be the set of all ordinals relative to A_p which do not contain as an element any relation R with $F(R) \subset A_{p-1}$. Finally, let

$$B = \sum_{n<\infty} B_n.$$

We first show that

(3) $$B \subset \sum_{n<\infty} A_n.$$

In fact, x and y being elements of a set A_n, $n = 0, 1, 2, \ldots$, we see from (1) and (2) that $(x,y) \in A_{n+2}$; therefore every relation R with $F(R) \subset A_n$ belongs to A_{n+3}, and every ordinal relative to A_n belongs to A_{n+4}. Hence and from the definition of B formula (3) follows at once. This formula clearly implies that the sets $A_0, A_1, \ldots, A_n, \ldots$ and B indeed satisfy the hypothesis of $\mathfrak{P}_1$.

Suppose now that there are a set C and a natural number p such that

(4) $$B \sim C \subset \sum_{n<p} A_n.$$

By means of an argument analogous to that applied in the Burali-Forti antinomy we can show that this supposition leads to a contradiction. In fact, X and Y being any two ordinals in B, we agree to say that $X \leqslant Y$ if there are relations $R \in X$ and $S \in Y$ such that $R \subset S$. Let T be the relation consisting of all couples (X, Y)

with $X \in B$, $Y \in B$, and $X \leqslant Y$; if Z is an arbitrary ordinal in B, the set $T(Z)$ of all couples (X, Y) with $X \in B$, $Y \in B$, and $X \leqslant Y \leqslant Z$ is called a segment of T. By applying familiar arguments from the theory of well-ordered sets we can show that

(5) T is a well-ordering relation, and $F(T) = B$.

From (1), (4), and (5) we infer that there is a well-ordering relation whose field is included in A_p and which is similar to T. Consequently, there is an ordinal Z in B consisting of relations similar to T; and hence, as is well known, the segment $T(Z)$ is itself a relation similar to T. Thus, the well-ordering relation T proves to be similar to one of its segments; and this contradicts one of the fundamental results of the theory of well-ordering.

Since the sets $A_0, A_1, \ldots, A_n, \ldots$ and B satisfy the hypothesis of $\mathscr{P}_1$ and since, as we have just shown, there are no set C and natural number p for which (4) holds, we obtain by applying $\mathscr{P}_1$

$$B \sim \sum_{n<\infty} A_n.$$

Therefore, by (5), there exists a well-ordering relation R with

$$F(R) = \sum_{n<\infty} A_n;$$

and consequently, by (1), there is also a well-ordering relation S with

$$F(S) = A.$$

Thus, the set A being quite arbitrary, Proposition $\mathscr{P}_1$ turns out to imply the well-ordering principle. Hence, $\mathscr{P}_1$ also implies the axiom of choice; and the proof of Theorem 1 is complete.

Theorem 2. *Proposition $\mathscr{P}_2$ is derivable from the restricted axiom of choice.*

Proof: To simplify the argument it proves convenient to apply certain notions of the arithmetic of cardinal numbers. The notions involved will be understood in a relative sense, depending on an arbitrary set U; but this relativization will not reflect itself in the terminology and symbolism.

Thus, by the power $\overline{\overline{A}}$ of a subset A of U we understand the family of all sets X such that $A \sim X \subset U$; the power of the empty set 0 is denoted by the same symbol 0. The family $\mathfrak{a}$ is called a cardinal number if there is a set $A \subset U$ for which $\mathfrak{a} = \overline{\overline{A}}$. The sum $\mathfrak{a}+\mathfrak{b}$

of two cardinals $\mathfrak{a}$ and $\mathfrak{b}$ is the cardinal containing all the sets $A+B$ with $A \in \mathfrak{a}$, $B \in \mathfrak{b}$, and $A \cdot B = 0$ (provided such sets exist). The sum $\sum_{n<p} \mathfrak{a}_n$, or $\sum_{n<\infty} \mathfrak{a}_n$, of a finite, or infinite, sequence of cardinals $\mathfrak{a}_0, \mathfrak{a}_1, \ldots$ is defined in an analogous way. Given two cardinals $\mathfrak{a}$ and $\mathfrak{b}$ we say that $\mathfrak{a} \leqslant \mathfrak{b}$ if there is a cardinal $\mathfrak{c}$ such that $\mathfrak{a}+\mathfrak{c} = \mathfrak{b}$; or, in other words, if there are sets A and B with $A \in \mathfrak{a}$, $B \in \mathfrak{b}$, and $A \subset B$. Elementary laws of addition — e. g., the commutative and associative laws, and the laws expressing fundamental properties of the cardinal 0 — are assumed to be known. It should be noticed that, in establishing these laws for sums of infinite sequences, we have to apply the restricted axiom of choice.

We are now going to derive a few lemmas, $\mathscr{A}$—$\mathscr{D}$, which are of a less obvious nature. In the proofs of these lemmas we shall again make use of the restricted axiom of choice; Lemma $\mathscr{A}$, however, will be obtained without the help of this axiom.

$\mathscr{A}$[3]. *If $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ are cardinals such that the sum $\mathfrak{a}+\mathfrak{c}$ exists and $\mathfrak{a}+\mathfrak{c} = \mathfrak{b}+\mathfrak{c}$, then there are cardinals $\mathfrak{a}'$, $\mathfrak{b}'$, and $\mathfrak{d}$ for which*

$$\mathfrak{a} = \mathfrak{a}'+\mathfrak{d}, \quad \mathfrak{b} = \mathfrak{b}'+\mathfrak{d}, \quad \textit{and} \quad \mathfrak{c} = \mathfrak{a}'+\mathfrak{c} = \mathfrak{b}'+\mathfrak{c}.$$

In fact, the hypothesis of this lemma implies the existence of four sets A, B, C, and C' (included in a given set U) with the following properties:

(1) $$\overline{\overline{A}} = \mathfrak{a}, \quad \overline{\overline{B}} = \mathfrak{b}, \quad \overline{\overline{C}} = \overline{\overline{C'}} = \mathfrak{c},$$

(2) $$A + C = B + C' \quad \text{and} \quad A \cdot C = B \cdot C' = 0.$$

By (1), there is a biunique function f whose domain is C and whose counter-domain is C'. We put:

(3) $$D_0 = D_0' = A \cdot B,$$

(4) $$D_n = (C - C') \cdot \overline{f^{-n}}(C' - C) \text{ and } D_n' = (C' - C) \cdot \overline{f^{n}}(C - C') \text{ for } n = 1, 2, 3, \ldots,$$

(5) $$A' = A - \sum_{n<\infty} D_n' \quad \text{and} \quad B' = B - \sum_{n<\infty} D_n,$$

(6) $$\mathfrak{a}' = \overline{\overline{A'}}, \quad \mathfrak{b}' = \overline{\overline{B'}}, \quad \text{and} \quad \mathfrak{d} = \overline{\overline{\sum_{n<\infty} D_n}}.$$

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[3]) Lemma $\mathscr{A}$ was stated without proof in Lindenbaum-Tarski [1], p. 301, Theorem 6. Several other results stated in the same work can easily be derived from this lemma; e. g., Theorems 7-9, 12-20, and 38, ibid., pp. 302 ff. See also W. Sierpiński, Fund. Math. **34** (1947), p. 116.

By (3) and (4), each of the sets D_n, $n=0,1,2,\ldots$, is included in the domain of the biunique function f^n, and we have

$$\overline{f^n}(D_n)=D'_n;$$

hence

(7) f^n maps D_n onto D'_n for $n=0,1,2,\ldots$

Moreover, the sets $D_0,D_1,\ldots,D_n,\ldots$ are pairwise disjoint. In fact, from (2)-(4) it follows directly that

$$D_0\cdot D_n=0 \quad \text{for } n>0.$$

If, on the other hand, $0<m<n$, then by (4) the set $\overline{f^m}(D_n)$ is included in the domain of f, i. e., in C, while $\overline{f^m}(D_m)$ is disjoint with C; therefore,

$$\overline{f^m}(D_m)\cdot\overline{f^m}(D_n)=0$$

and hence

$$D_m\cdot D_n=0.$$

For similar reasons the sets $D'_0,D'_1,\ldots,D'_n,\ldots$ are pairwise disjoint. Consequently, and in view of (7), we can construct a biunique function g which maps $\sum_{n<\infty}D_n$ onto $\sum_{n<\infty}D'_n$; in fact, it suffices to put

$$g(x)=f^n(x) \quad \text{for} \quad x\in D_n, \quad n=0,1,2,\ldots$$

Thus,

$$\sum_{n<\infty}D_n\sim\sum_{n<\infty}D'_n,$$

and hence, by (6),

(8) $$\mathfrak{d}=\overline{\overline{\sum_{n<\infty}D'_n}}.$$

By (2) and (4) we have

$$\sum_{n<\infty}D_n\subset B \quad \text{and} \quad \sum_{n<\infty}D'_n\subset A;$$

consequently, by (1), (5), (6), and (8),

(9) $$\mathfrak{a}=\mathfrak{a}'+\mathfrak{d} \quad \text{and} \quad \mathfrak{b}=\mathfrak{b}'+\mathfrak{d}.$$

Furthermore, we show by an easy induction that

(10) $$A'\subset\overline{f^n}(C')$$

holds for every natural number n. In fact, formulas (2), (3), and (5) imply

(11) $$A'\subset A-D_0=C'-C,$$

and hence (10) holds for $n=0$. Assume now that (10) holds for any given n. Since

$$C'=\overline{f}(C)\subset\overline{f}(C')+\overline{f}(C-C'),$$

our inductive premise gives

$$A'\subset\overline{f^{n+1}}(C')+\overline{f^{n+1}}(C-C').$$

Therefore, by (4) and (11),

$$A'\subset\overline{f^{n+1}}(C')+D'_{n+1},$$

and hence, by (5),

$$A'\subset\overline{f^{n+1}}(C');$$

i. e., (10) holds also for $n+1$. From (10) we see that A' is included in the domain of f^{-n} for $n=0,1,2,\ldots$; consequently, the set

(12) $$A'+\sum_{n<\infty}\overline{f^{-n-1}}(A')=\sum_{n<\infty}\overline{f^{-n}}(A')$$

is included in the domain of f^{-1}. Since we obviously have

$$\overline{f^{-1}}\Big[\sum_{n<\infty}\overline{f^{-n}}(A')\Big]=\sum_{n<\infty}\overline{f^{-n-1}}(A'),$$

we conclude that

$$\sum_{n<\infty}\overline{f^{-n}}(A')\sim\sum_{n<\infty}\overline{f^{-n-1}}(A'),$$

and therefore, in view of (11),

(13) $$\sum_{n<\infty}\overline{f^{-n}}(A')+\Big[C-\sum_{n<\infty}\overline{f^{-n-1}}(A')\Big]\sim\sum_{n<\infty}\overline{f^{-n-1}}(A')+\Big[C-\sum_{n<\infty}\overline{f^{-n-1}}(A')\Big].$$

Since the domain of f is C, we have

$$\sum_{n<\infty}\overline{f^{-n-1}}(A')\subset C.$$

Hence, with the help of (12), formula (13) can be simplified:

$$A'+C\sim C;$$

and, by (1), (6), and (11), we obtain

(14) $$\mathfrak{a}'+\mathfrak{c}=\mathfrak{c}.$$

In an entirely analogous way we derive

(15) $$\mathfrak{b}'+\mathfrak{c}=\mathfrak{c}.$$

Thus, the existence of cardinals $\mathfrak{a}'$, $\mathfrak{b}'$, and $\mathfrak{d}$ which satisfy (9), (14), and (15) has been established; and the proof of Lemma $\mathcal{A}$ is complete.

$\mathcal{B}$. *If* $\mathfrak{a}_0,\mathfrak{a}_1,\dots,\mathfrak{a}_n,\dots$ *and* $\mathfrak{b}_0,\mathfrak{b}_1,\dots,\mathfrak{b}_n,\dots$ *are two infinite sequences of cardinals such that*

$$\mathfrak{b}_n=\mathfrak{a}_n+\mathfrak{b}_{n+1} \quad \textit{for} \quad n=0,1,2,\dots,$$

then the sum $\sum_{n<\infty}\mathfrak{a}_n$ *exists and*

$$\sum_{n<\infty}\mathfrak{a}_n\leqslant\mathfrak{b}_0.$$

In fact, the restricted axiom of choice implies the existence of a sequence of sets $B_0,B_1,\dots,B_n,\dots$ such that

(16) $$\overline{\overline{B}}_n=\mathfrak{b}_n \quad \text{for} \quad n=0,1,2,\dots$$

Furthermore, by the hypothesis of Lemma $\mathcal{B}$ and with the help of the same axiom, we can correlate, with every natural number n, two sets A_n and B'_{n+1} as well as a biunique function f_{n+1} such that

(17) $$B_n=A_n+B'_{n+1},\quad A_n\cdot B'_{n+1}=0,$$

(18) $$\overline{\overline{A}}_n=\mathfrak{a}_n,\quad \overline{\overline{B'_{n+1}}}=\mathfrak{b}_{n+1},$$

and

(19) $$f_{n+1} \text{ maps } B_{n+1} \text{ onto } B'_{n+1}.$$

Now let g_0 be the identity function, and let g_{n+1} for $n=0,1,2,\dots$ be the composition of functions $f_1,f_2,\dots,f_{n+1}$ (in this order), so that

$$g_1=f_1,\quad \text{and} \quad g_{n+1}=g_nf_{n+1} \quad \text{for} \quad n=1,2,3,\dots$$

The functions $g_0,g_1,\dots,g_n,\dots$ are obviously biunique. From (17) and (19) we infer by an easy induction that each of the sets A_n is included in the domain of g_n, and that the sets

$$\overline{g_0}(A_0),\quad \overline{g_1}(A_1),\quad \dots,\quad \overline{g_n}(A_n),\dots$$

are mutually exclusive and are subsets of B_0. Consequently, by (18),

$$\overline{\overline{\sum_{n<\infty}\overline{g_n}(A_n)}}=\sum_{n<\infty}\mathfrak{a}_n \quad \text{and} \quad \sum_{n<\infty}\overline{g_n}(A_n)\subset B_0.$$

Hence and from (16) the conclusion of Lemma $\mathcal{B}$ follows at once.

$\mathcal{C}$[4]. *If* $\mathfrak{a}_0,\mathfrak{a}_1,\dots,\mathfrak{a}_n,\dots$ *and* $\mathfrak{b}$ *are cardinals such that*

$$\mathfrak{b}=\mathfrak{a}_n+\mathfrak{b} \quad \textit{for} \quad n=0,1,2,\dots,$$

then the sum $\sum_{n<\infty}\mathfrak{a}_n$ *exists and*

$$\mathfrak{b}=\sum_{n<\infty}\mathfrak{a}_n+\mathfrak{b}.$$

To prove this, we define a double sequence of cardinals $\mathfrak{a}_{n,p}$ and a sequence of cardinals $\mathfrak{b}_n$ by putting

(20) $$\mathfrak{a}_{n,p}=\mathfrak{a}_p \quad \text{and} \quad \mathfrak{b}_n=\mathfrak{b} \quad \text{for } n,p=0,1,2,\dots$$

By hypothesis we have

(21) $$\mathfrak{b}_n=\mathfrak{a}_n+\mathfrak{b}_{n+1} \quad \text{for } n=0,1,2,\dots$$

and

(22) $$\mathfrak{b}_n=\mathfrak{a}_{n,p}+\mathfrak{b}_{n+1} \quad \text{for } n,p=0,1,2,\dots$$

By applying Lemma $\mathcal{B}$, we conclude from (21) that the sum $\sum_{n<\infty}\mathfrak{a}_n$ exists. Similarly, for any given natural number p, (22) implies that the sum $\sum_{n<\infty}\mathfrak{a}_{n,p}$ exists and that

(23) $$\mathfrak{b}_0=\sum_{n<\infty}\mathfrak{a}_{n,p}+\mathfrak{c}$$

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[4]) Lemma $\mathcal{C}$ was proved in a different way by E. Zermelo; see Schoenflies [1], p. 40.

for some cardinal $\mathfrak{c}$. From (20) and (23) we obtain (with the help of the commutative and associative laws):

$$\mathfrak{b}=\mathfrak{b}_0=\sum_{n<\infty}\mathfrak{a}_{2n,p}+\Big(\sum_{n<\infty}\mathfrak{a}_{2n+1,p}+\mathfrak{c}\Big)=\sum_{n<\infty}\mathfrak{a}_{n,p}+\Big(\sum_{n<\infty}\mathfrak{a}_{n,p}+\mathfrak{c}\Big),$$

and consequently

$$\mathfrak{b}_p=\sum_{n<\infty}\mathfrak{a}_{n,p}+\mathfrak{b}_{p+1}\quad\text{for } p=0,1,2,\dots$$

Hence, by applying Lemma $\mathcal{B}$ again, we infer that

$$\sum_{p<\infty}\sum_{n<\infty}\mathfrak{a}_{n,p}$$

exists and that, for some cardinal $\mathfrak{d}$,

(24) $$\mathfrak{b}_0=\sum_{p<\infty}\sum_{n<\infty}\mathfrak{a}_{n,p}+\mathfrak{d}.$$

From (19) and (24) we obtain (again with the help of the commutative and the associative laws)

$$\mathfrak{b}=\sum_{n<\infty}\sum_{p<\infty}\mathfrak{a}_{n,p}+\mathfrak{d}=\sum_{p<\infty}\mathfrak{a}_{0,p}+\Big(\sum_{n<\infty}\sum_{p<\infty}\mathfrak{a}_{n+1,p}+\mathfrak{d}\Big)=\sum_{p<\infty}\mathfrak{a}_p+\Big(\sum_{n<\infty}\sum_{p<\infty}\mathfrak{a}_{n,p}+\mathfrak{d}\Big),$$

and finally

$$\mathfrak{b}=\sum_{n<\infty}\mathfrak{a}_n+\mathfrak{d}.$$

The proof of Lemma $\mathcal{C}$ has thus been completed.

$\mathfrak{D}$. *If* $\mathfrak{a}_0,\mathfrak{a}_1,\dots,\mathfrak{a}_n,\dots$, *and* $\mathfrak{b}$ *are cardinals such that, for every natural number* p, *the sum* $\sum_{n<p}\mathfrak{a}_n$ *exists and*

$$\sum_{n<p}\mathfrak{a}_n\leqslant\mathfrak{b},$$

then $\sum_{n<\infty}\mathfrak{a}_n$ *also exists and*

$$\sum_{n<\infty}\mathfrak{a}_n\leqslant\mathfrak{b}.$$

In fact, by hypothesis and with the help of the restricted axiom of choice, we obtain a sequence of cardinals $\mathfrak{c}_0,\mathfrak{c}_1,\dots,\mathfrak{c}_n,\dots$ such that

(25) $$\sum_{n<p}\mathfrak{a}_n+\mathfrak{c}_p=\mathfrak{b}\quad\text{for } p=0,1,2,\dots$$

Hence

$$\mathfrak{c}_p+\sum_{n<p}\mathfrak{a}_n=(\mathfrak{c}_{p+1}+\mathfrak{a}_p)+\sum_{n<p}\mathfrak{a}_n\quad\text{for } p=0,1,2,\dots$$

We apply to the latter formula Lemma $\mathcal{A}$ with

$$\mathfrak{a}=\mathfrak{c}_p,\quad\mathfrak{b}=\mathfrak{c}_{p+1}+\mathfrak{a}_p,\quad\text{and}\quad\mathfrak{c}=\sum_{n<p}\mathfrak{a}_n,$$

and, by again using the restricted axiom of choice, we obtain three sequences of cardinals $\mathfrak{a}'_p$, $\mathfrak{b}'_p$, and $\mathfrak{d}_p$ such that, for every natural number p,

(26) $$\mathfrak{c}_p=\mathfrak{a}'_p+\mathfrak{d}_p,\quad\mathfrak{c}_{p+1}+\mathfrak{a}_p=\mathfrak{b}'_p+\mathfrak{d}_p,$$

and

(27) $$\sum_{n<p}\mathfrak{a}_n=\mathfrak{a}'_p+\sum_{n<p}\mathfrak{a}_n=\mathfrak{b}'_p+\sum_{n<p}\mathfrak{a}_n.$$

By (25) and (27) we have

$$\mathfrak{b}=\mathfrak{a}'_p+\Big(\sum_{n<p}\mathfrak{a}_n+\mathfrak{c}_p\Big)=\mathfrak{b}'_p+\Big(\sum_{n<p}\mathfrak{a}_n+\mathfrak{c}_p\Big);$$

therefore

$$\mathfrak{b}=\mathfrak{a}'_p+\mathfrak{b}=\mathfrak{b}'_p+\mathfrak{b}\quad\text{for } p=0,1,2,\dots;$$

and hence, by Lemma $\mathcal{C}$, the sums $\sum_{n<\infty}\mathfrak{a}'_n$ and $\sum_{n<\infty}\mathfrak{b}'_n$ exist and

(28) $$\mathfrak{b}=\sum_{n<\infty}\mathfrak{a}'_n+\mathfrak{b}=\sum_{n<\infty}\mathfrak{b}'_n+\mathfrak{b}.$$

(25) and (28) imply (by the commutative and associative laws)

$$\mathfrak{b}=\Big(\sum_{n<p}\mathfrak{a}'_n+\sum_{n<\infty}\mathfrak{b}'_{n+p}+\mathfrak{c}_p\Big)+\Big(\sum_{n<\infty}\mathfrak{a}'_{n+p}+\sum_{n<p}\mathfrak{b}'_n+\sum_{n<p}\mathfrak{a}_n\Big).$$

Thus, by putting

(29) $$\mathfrak{b}_p=\sum_{n<p}\mathfrak{a}'_n+\sum_{n<\infty}\mathfrak{b}'_{n+p}+\mathfrak{c}_p\quad\text{for } p=0,1,2,\dots,$$

we obtain an infinite sequence of cardinals $\mathfrak{b}_0,\mathfrak{b}_1,\dots,\mathfrak{b}_p,\dots$ which are all $\leqslant\mathfrak{b}$. By (25), (28), and (29) we have

(30) $$\mathfrak{b}_0=\mathfrak{b}.$$

From (26) and (29) we conclude:

$$\mathfrak{b}_p=\sum_{n<p}\mathfrak{a}'_n+\sum_{n<\infty}\mathfrak{b}'_{n+p}+\mathfrak{a}_p+\mathfrak{d}_p=\sum_{n<p+1}\mathfrak{a}'_n+\sum_{n<\infty}\mathfrak{b}'_{n+p+1}+\mathfrak{b}'_p+\mathfrak{d}_p,$$

hence

$$\mathfrak{b}_p = \sum_{n<p+1} \mathfrak{a}'_n + \sum_{n<\infty} \mathfrak{b}'_{n+p+1} + \mathfrak{c}_{p+1} + \mathfrak{a}_p,$$

and therefore

(31) $$\mathfrak{b}_p = \mathfrak{a}_p + \mathfrak{b}_{p+1} \quad \text{for} \quad p = 0, 1, 2, \ldots$$

By Lemma $\mathcal{B}$, the conclusion of Lemma $\mathcal{D}$ follows immediately from (30) and (31).

Lemma $\mathcal{D}$ just established coincides essentially with Proposition $\mathcal{C}_2$ formulated in the introduction; and from this lemma we can now easily derive Proposition $\mathcal{P}_2$. In fact, consider any sets $A_0, A_1, \ldots, A_n, \ldots$, and B which satisfy the hypothesis of $\mathcal{P}_2$; as the universal set U (to which the notion of the cardinal is relativized) we take, e. g.,

$$U = \sum_{n<\infty} A_n + B.$$

We put

(32) $$A'_p = A_p - \sum_{n<p} A_n \quad \text{and} \quad \mathfrak{a}_p = \overline{\overline{A'_p}} \quad \text{for} \quad p = 0, 1, 2, \ldots$$

as well as

(33) $$\mathfrak{b} = \overline{\overline{B}}.$$

Obviously, by (32),

(34) $$\sum_{n<p} A_n = \sum_{n<p} A'_n \quad \text{for} \quad p = 0, 1, 2, \ldots, \infty.$$

We also easily see that the sets $A'_0, A'_1, \ldots, A'_n, \ldots$ are pairwise disjoint. Hence, by (32) and (34) (and with the help of the restricted axiom of choice),

(35) $$\overline{\overline{\sum_{n<p} A_n}} = \sum_{n<p} \mathfrak{a}_n \quad \text{for} \quad p = 0, 1, 2, \ldots, \infty.$$

By the hypothesis of $\mathcal{P}_2$, formulas (33) and (35) imply

$$\sum_{n<p} \mathfrak{a}_n \leqslant \mathfrak{b} \quad \text{for} \quad p = 0, 1, 2, \ldots,$$

and consequently, by Lemma $\mathcal{D}$,

(36) $$\sum_{n<\infty} \mathfrak{a}_n \leqslant \mathfrak{b}.$$

The conclusion of $\mathcal{P}_2$ is a direct consequence of (33), (35), and (36); and the proof of Theorem 2 is complete.

By comparing the results obtained in Theorems 1 and 2, we see that the implication $\mathcal{P}_1 \to \mathcal{P}_2$ can be established without the help of the axiom of choice. If the same were true for the implication in the opposite direction, $\mathcal{P}_2 \to \mathcal{P}_1$, then the axiom of choice in its general form would prove derivable from the restricted axiom of choice; but this has recently been shown not to be the case [5]).

The derivation of $\mathcal{P}_1$ is essentially non-effective in its character. The proof of $\mathcal{P}_2$ is not effective either; it can easily, however, be made effective by introducing in the hypothesis of $\mathcal{P}_2$ an infinite sequence of biunique functions f_p which are assumed to map the sets $\sum_{n<p} A_n$ $(p = 0, 1, 2, \ldots)$ onto subsets of B. In fact, by transforming $\mathcal{P}_2$ in the way just mentioned, we obtain the following proposition:

$\mathcal{P}'_2$. *Given an infinite sequence of sets $A_0, A_1, \ldots, A_n, \ldots$, a set B, and an infinite sequence of biunique functions $f_0, f_1, \ldots, f_n, \ldots$, if, for every natural number p, the function f_p maps the set $\sum_{n<p} A_n$ onto a subset of B, then there is a biunique function f which maps the set $\sum_{n<\infty} A_n$ onto a subset of B.*

Now, by analyzing and slightly modifying the proof of Theorem 2, we arrive at

Theorem 3. *Proposition $\mathcal{P}'_2$ can be established without the help of the axiom of choice.*

Moreover, the proof of $\mathcal{P}'_2$ is effective; for the function f whose existence is claimed in the conclusion of $\mathcal{P}'_2$ is effectively definable in terms of sets and functions involved in the hypothesis. Hence we can say that the proof of the original proposition $\mathcal{P}_2$ is „almost" effective in its character.

§ 2. Algebraic aspect of the problem.

The essential differences between Propositions $\mathcal{P}_1$ and $\mathcal{P}_2$ discussed in the preceding section by no means reduce to questions of an axiomatic or methodological nature. The algebraic part of our discussion will throw more light on this subject [6]).

[5]) See Mostowski [1].

[6]) The discussion in § 2 will have a much more sketchy character than that in § 1. Most of the material contained in § 2 is discussed in greater detail in Tarski [1], especially in Sections 15-17; in particular, Theorems 4 and 5 of the present article are corollaries of more general results established in Section 2 of that work.

The sets involved in $\mathcal{P}_1$ and $\mathcal{P}_2$ can be regarded as elements of the Boolean algebra formed by all subsets of a certain set U, with set-theoretical addition and multiplication as the fundamental operations. Hence we can try to generalize $\mathcal{P}_1$ and $\mathcal{P}_2$ by considering, instead of the sets involved, elements of an *abstract Boolean algebra* formed by a set $\mathfrak{B}$ of arbitrary elements $a, b, c, \ldots$ and by the fundamental operations $+$ and $\cdot$ assumed to satisfy certain well-known postulates; the relation $\sim$ of set-theoretical equivalence is replaced by a binary relation R between elements of $\mathfrak{B}$. In terms of the fundamental operations of our algebra we define, in a familiar way, the elements 0 and 1, the inclusion relation $\leqslant$ between elements of $\mathfrak{B}$, the sum (the least upper bound) $\sum_{i\in I} a_i$ and the product (the greatest lower bound) of an arbitrary system of elements a_i; the operations Σ and Π are not supposed to be always performable. Propositions $\mathcal{P}_1$ and $\mathcal{P}_2$ assume now the following abstract form:

$\mathcal{Q}_1$. *Let $a_0, a_1, \ldots, a_n, \ldots$ and b be elements of a Boolean algebra $\mathfrak{B}$ such that the sum $\sum_{n<\infty} a_n$ exists; and let R be a binary relation between elements of $\mathfrak{B}$. If there is an element d in $\mathfrak{B}$ such that*

$$bRd \quad \textit{and} \quad d \leqslant \sum_{n<\infty} a_n,$$

then either $bR\sum_{n<\infty} a_n$, or else there exist a natural number p and an element c in $\mathfrak{B}$ for which

$$bRc \quad \textit{and} \quad c \leqslant \sum_{n<p} a_n.$$

$\mathcal{Q}_2$. *Let $a_0, a_1, \ldots, a_n, \ldots, b$, and R have the same meaning as in $\mathcal{Q}_1$. If, for every natural number p, there is an element c in $\mathfrak{B}$ such that*

$$\sum_{n<p} a_n Rc \quad \textit{and} \quad c \leqslant b,$$

then there is an element d in $\mathfrak{B}$ for which

$$\sum_{n<\infty} a_n Rd \quad \textit{and} \quad d \leqslant b.$$

It is obvious that Propositions $\mathcal{Q}_1$ and $\mathcal{Q}_2$ do not apply to arbitrary Boolean algebras and arbitrary relations between elements of these algebras. Hence the question arises under what assumptions regarding the algebra $\mathfrak{B}$ and the relation R these propositions hold; since we are aiming at a generalization of the original propositions $\mathcal{P}_1$ and $\mathcal{P}_2$, we are interested only in those assumptions which are satisfied in case we take for $\mathfrak{B}$ the algebra of all subsets of a given set and for R the relation of set-theoretical equivalence.

The study of this question with regard to $\mathcal{Q}_1$ does not bring any interesting results. By analyzing the proof of $\mathcal{P}_1$ we are led to impose conditions on $\mathfrak{B}$ and R enabling us to construct an isomorphism which maps the algebra $\mathfrak{B}$ onto the algebra of all subsets of a set, and the relation R onto the relation of set-theoretical equivalence. The systems $(\mathfrak{B}, R)$ thus obtained are certainly not the only ones for which $\mathcal{Q}_1$ holds; the attempts, however, of extending $\mathcal{Q}_1$ to other, interesting and „naturally defined", classes of such systems prove unsuccessful. On the contrary, we shall see that $\mathcal{Q}_2$ has a wide range of applications.

A Boolean algebra $\mathfrak{B}$ is called *countably complete* if the sum $\sum_{n<\infty} a_n$ exists for any infinite sequence of elements a_n in $\mathfrak{B}$. By an *equivalence relation* in $\mathfrak{B}$ we understand a relation R which is reflexive in $\mathfrak{B}$ ($a\,R\,a$ for every a in $\mathfrak{B}$), symmetric, and transitive. The relation R is called *countably additive*, or *(finitely) refining*, if it satisfies the following condition (i), or (ii), respectively:

(i) If $a_0, a_1, \ldots, a_n, \ldots$ and $b_0, b_1, \ldots, b_n, \ldots$ are elements in $\mathfrak{B}$ such that

$$a_n \cdot a_p = 0 = b_n \cdot b_p \quad \text{for} \quad n < p < \infty, \quad \text{and} \quad a_n R b_n \quad \text{for } n < \infty,$$

then

$$\sum_{n<\infty} a_n \, R \sum_{n<\infty} b_n$$

(provided the sums in question exist).

(ii) If a, a_1, a_2, and b are elements in $\mathfrak{B}$ such that

$$a = a_1 + a_2, \quad a_1 \cdot a_2 = 0, \quad \text{and} \quad aRb,$$

then there are elements b_1 and b_2 for which

$$b = b_1 + b_2, \quad b_1 \cdot b_2 = 0, \quad a_1 R b_1, \quad \text{and} \quad a_2 R b_2.$$

Using this terminology, we can now formulate and establish the following

Theorem 4. *Proposition $\mathcal{Q}_2$ applies to every countably complete Boolean algebra $\mathfrak{B}$ and to every countably additive and refining equivalence relation R in this algebra.*

A detailed proof of Theorem 4 will not be given here; it can be obtained by a close analysis of that of Theorem 2. As can easily be guessed, instead of cardinals we use the equivalence classes under R, i. e., the subsets of $\mathfrak{B}$ consisting of all elements which are in the relation R to some element in $\mathfrak{B}$. The proofs of Lemmas $\mathcal{A}$ and $\mathcal{B}$ require a rather essential modification; since we cannot operate with functions mapping some sets onto others, we have to make an extensive use of the refining property of R.

The proof of Theorem 4 is based upon the axiom of choice. The particular case of this axiom which is involved in the proof can be referred to as the *principle of an infinite sequence of successive (dependent) choices*, and can be formulated as follows:

If A is a non-empty set and S a relation such that, for every element $x \in A$ there is an element $y \in A$ with $x\,S\,y$, then there exists an infinite sequence of elements $x_0, x_1, \ldots, x_n, \ldots$ in A such that $x_n\,S\,x_{n+1}$ for $n=0,1,2,\ldots$

It would be interesting to clear up the problem of the logical relations between this principle (which is often applied in mathematical arguments) and the axiom of choice. It can easily be shown that our principle implies the restricted axiom of choice; we see, however, no way of establishing the implication in the opposite direction. On the other hand, it has recently been shown that the principle in question does not imply the general axiom of choice [5]).

$\mathcal{P}_1$ and $\mathcal{P}_2$ can be regarded as representatives of two very different kinds of theorems concerning addition of set-theoretically equivalent sets (or addition of cardinals). The two kinds of theorems exhibit the axiomatic and methodological differences which were discussed in § 1; the proofs of the theorems of the first kind essentially involve the general axiom of choice and are of an essentially non-effective character, while the proofs of those of the second kind require at most an application of the restricted axiom of choice and are „almost" effective (in the sense described at the end of the preceding section). What is perhaps more important, the theorems of the first kind do not seem to be susceptible of any interesting generalizations in the direction of abstract algebra, while those of the second kind can be extended to a wide class of algebraic systems [7]).

[7]) The difference between the two kinds of theorems was pointed out in earlier papers of the author; cf. Lindenbaum-Tarski [1] and Tarski [3]. However, the idea of extending the results of the second kind in the direction of abstract algebra has been fully realized and developed only in Tarski [1].

As examples of theorems of the first kind we may mention, in addition to $\mathcal{P}_1$, the trichotomy law (see § 1) or the theorem by which the sum of any two infinite sets has the same power as one of these sets. The theorems of the second kind are usually of a more special nature; we find, however, among them a number of interesting statements, as is seen from the following examples:

$\mathcal{P}_3$ *(Equivalence Theorem). If A, B, and C are any sets such that*

$$A \subset B \subset C \quad \text{and} \quad A \sim C,$$

then

$$A \sim B \sim C.$$

$\mathcal{P}_4$ *(Mean-value Theorem). If A, B, C, A', and C' are any sets such that*

$$A \subset B \subset C, \quad A' \subset C', \quad A \sim A' \quad \text{and} \quad C \sim C',$$

then there is a set B' for which

$$A' \subset B' \subset C' \quad \text{and} \quad B \sim B'.$$

$\mathcal{P}_5$ *(Subtraction Theorem). If A, A', A'', B, and C are any sets such that*

$$A+B \subset A'+A''+C, \quad A \cdot B = 0, \quad \text{and} \quad A \sim A' \sim A'',$$

then there is a set B' for which

$$B \sim B' \subset A'+C.$$

$\mathcal{P}_6$ *(Division Theorem). If $A_0, A_1, \ldots, A_{p-1}$ and $B_0, B_1, \ldots, B_{p-1}$ $(p>0)$, are any two finite sequences of pairwise disjoint sets such that*

$$A_0 \sim A_n \quad \text{and} \quad B_0 \sim B_n \quad \text{for} \quad n=0,1,\ldots,p-1, \quad \text{and} \quad \sum_{n<p} A_n \sim \sum_{n<p} B_n,$$

then

$$A_0 \sim B_0.$$

$\mathcal{P}_7$ *(Interpolation Theorem). Given any two sequences of sets A_n and B_p, and a double sequence of sets $C_{n,p}$, if*

$$A_n \sim C_{n,p} \subset B_p \quad \text{for} \quad n,p=0,1,2,\ldots,$$

then there are sets D, A'_n, and B'_p such that

$$A_n \sim A'_n \subset D \quad \text{and} \quad D \sim B'_p \subset B_p \quad \text{for} \quad n,p=0,1,2,\ldots$$

The content of some of these statements would be clearer if they were formulated in terms of cardinals.

Propositions $\mathcal{P}_3$—$\mathcal{P}_6$ can be derived from axioms of set theory without the help of the axiom of choice. The same applies to Proposition $\mathcal{P}_7$ restricted to finite sequences; in the general case the proof of $\mathcal{P}_7$ requires an application of the restricted axiom of choice [8]).

From $\mathcal{P}_3$—$\mathcal{P}_7$ we now can obtain the abstract formulations $\mathfrak{Q}_3$—$\mathfrak{Q}_7$ in exactly the same way in which $\mathfrak{Q}_2$ has been obtained from $\mathcal{P}_2$; and by analyzing the proofs of $\mathcal{P}_3$—$\mathcal{P}_7$ we arrive at

Theorem 5. *Propositions $\mathfrak{Q}_3$—$\mathfrak{Q}_7$ apply to every countably complete Boolean algebra $\mathfrak{B}$ and to every countably additive and refining equivalence relation R in this algebra.*

The proof of this theorem rests again on the principle of an infinite sequence of successive choices.

We can thus develop a whole theory of countably additive and refining equivalence relations in countably complete Boolean algebras. To complete this paper, we want to give various examples of systems to which this theory applies.

In constructing and discussing these examples we shall use some further notions from the theory of Boolean algebras — such as the notions of an ideal I in a Boolean algebra $\mathfrak{B}$, the congruence of two elements modulo I, the quotient algebra $\mathfrak{B}/I$, and the principal ideal $I(a)$ generated by an element a. All these notions are familiar from the literature.

The simplest instances of countably complete Boolean algebras are found among fields of sets. A family of subsets of a set U is here referred to as a field of sets if it contains the set U as an element and is closed under finite set-theoretical addition and subtraction; a field of sets is called countably complete if it is closed under addition of infinite sequences.

[8]) $\mathcal{P}_3$ is of course the famous Cantor-Bernstein theorem. Regarding the origin of $\mathcal{P}_4$—$\mathcal{P}_6$, compare remarks and notes in Lindenbaum-Tarski [1], pp. 302-305. $\mathcal{P}_7$ is a new result. The direct proofs of all these theorems — and of other related results — can be obtained by analyzing the proofs of corresponding algebraic theorems in Tarski [1], Section 2; cf. remarks in Section 17 of the same work.

Every field of sets is obviously a Boolean algebra (under set-theoretical addition and multiplication), and every countably complete field is a countably complete Boolean algebra. Conversely, every Boolean algebra is, as is well known, isomorphic with a field of sets. There are, however, countably complete Boolean algebras which are not isomorphic with any countably complete field of sets. For instance, consider the Boolean algebra $\mathfrak{B}$ constituted by all subsets of a set U with $\overline{\overline{U}}=2^{\aleph_0}$, and the ideal I in $\mathfrak{B}$ consisting of all at most denumerable subsets of U. Since both $\mathfrak{B}$ and I are countably complete, the Boolean algebra $\mathfrak{B}/I$ is also countably complete. However, $\mathfrak{B}/I$ lacks another property which is common to all algebras isomorphic with countably complete fields of sets. In fact, $\mathfrak{B}/I$ is not countably distributive, i. e., it does not satisfy the following condition:

(i) For every double sequence of elements $a_{n,p}$ in $\mathfrak{B}$,

$$\prod_{n<\infty} \sum_{p<\infty} a_{n,p} = \sum_{q\in Q} \prod_{n<\infty} a_{n,q_n}$$

where Q is the set of all infinite sequences $q=(q_0, q_1, \ldots, q_n, \ldots)$ of natural numbers.

Hence, $\mathfrak{B}/I$ is not isomorphic with any countably complete field of sets. The same remarks apply if $\mathfrak{B}$ is, e. g., the family of all sets of real numbers which are measurable in the sense of Lebesgue, and I is the ideal of all sets of measure 0. An even more interesting example is provided by the Boolean algebra $\mathfrak{B}'$ of all subsets of a set U' with the power $2^{2^{\aleph_0}}$ and by the ideal I' of all those subsets of U whose power is at most $2^{\aleph_0}$. In this case $\mathfrak{B}'/I'$ proves to be both countably complete and countably distributive, but nevertheless it is not isomorphic with any countably complete field of sets. For it can be shown that no prime ideal in $\mathfrak{B}'/I'$ is countably complete; while for a countably complete Boolean algebra to be isomorphic with a countably complete field of sets, it is necessary and sufficient that every element different from 1 of such an algebra belongs to a countably complete prime ideal [9]). The

[9]) While the proof of the necessity of this condition is rather obvious, the proof of the sufficiency is analogous to that of the representation theorem for Boolean algebras in Stone [1], Chapter IV. For the proof that the algebra $\mathfrak{B}'/I'$ has no countably complete prime ideals see Tarski [2], p. 58. The remaining properties of the algebras $\mathfrak{B}/I$ and $\mathfrak{B}'/I'$ mentioned above can be established without difficulties.

countably complete fields of sets constitute a rather special class of countably complete Boolean algebras — though undoubtedly a class which is especially important from the point of view of applications.

As the first example of countably additive and refining equivalence relations in countably complete Boolean algebras we may take the relation of homogeneity, two elements a and b of an algebra $\mathfrak{B}$ being called homogeneous if the principal ideals $I(a)$ and $I(b)$ are isomorphic; in case $\mathfrak{B}$ is the field of all subsets of a set U, the relation of homogeneity coincides with that of set-theoretical equivalence. Theorems 4 and 5 (and other analogous results) can now be applied to the relation of homogeneity. We obtain in this way a series of results which are interesting for the reason that they imply as immediate corollaries various conclusions concerning direct products of countably complete Boolean algebras and of isomorphism types of such algebras. (The notions of the isomorphism type β of an algebra $\mathfrak{B}$ and of the direct product $\alpha\times\beta$ of two isomorphism types are defined by abstraction in terms of the isomorphism relation). Thus, for instance, from the fact that Q_3 and Q_6 apply to the relation of homogeneity we immediately derive the following conclusions:

If α, β, and γ are isomorphism types of countably complete Boolean algebras, then

(i) *the formula* $\alpha\times\beta\times\gamma=\gamma$ *implies* $\alpha\times\gamma=\beta\times\gamma=\gamma$,

(ii) *the formula* $\alpha\times\alpha=\beta\times\beta$ *implies* $\alpha=\beta$.

It would be interesting to show by means of examples that neither of these conclusions can be extended to isomorphism types of arbitrary Boolean algebras.

The relation of homogeneity is a particular instance of a comprehensive class of relations which can be described in the following way. By a partial automorphism in a Boolean algebra $\mathfrak{B}$ we understand every function f whose domain is a principal ideal $I(a)$ in $\mathfrak{B}$ and which maps $I(a)$ isomorphically onto another principal ideal, $I(b)$. A set G of partial automorphisms in $\mathfrak{B}$ is called a quasi-group or simply a group if it contains the identity function over $\mathfrak{B}$ and is closed under the operations of composition and inversion (taking inverses). G is said to be countably additive if it satisfies the following

condition: let $f_0, f_1, \ldots, f_n, \ldots$ be any functions in G and let $a, a_0, a_1, \ldots, a_n, \ldots$ and $b, b_0, b_1, \ldots, b_n, \ldots$ be any elements in $\mathfrak{B}$ such that

(i) $$a=\sum_{n<\infty} a_n,\quad b=\sum_{n<\infty} b_n,$$

(ii) $a_n\cdot a_p=0=b_n\cdot b_p$ for $n<p<\infty$, and $f_n(a_n)=b_n$ for $n<\infty$;

finally, let f be a partial automorphism whose domain is $I(a)$ and which agrees with f_n on $I(a_n)$ for $n=0,1,2,\ldots$; then f belongs to G. Two elements a and b in $\mathfrak{B}$ are called congruent under G if $f(a)=b$ for some $f\in G$, in symbols, $a \underset{G}{\simeq} b$; they are called equivalent by infinite decomposition under G if there exist a sequence of functions f_n in G and two sequences of elements a_n and b_n in $\mathfrak{B}$ which satisfy formulas (i) and (ii) just given. As is easily seen, for every group G of partial automorphisms in a countably complete Boolean algebra, we can construct a countably additive group H of partial automorphisms such that equivalence by infinite decomposition under G coincides with congruence under H; in fact, H is the smallest countably additive group including G. Thus the study of equivalence by infinite decomposition under arbitrary groups reduces to that of congruence under countably additive groups. Now, G being a countably additive group of partial automorphisms in a countably complete Boolean algebra $\mathfrak{B}$, the relation $\widetilde{G}$ is easily seen to be a countably additive and refining equivalence relation in $\mathfrak{B}$. Hence, propositions like Q_2—Q_7 apply to every relation of this sort (and, moreover, a direct proof of Q_2—Q_7 for the relations $\widetilde{G}$ requires at most the restricted axiom of choice and has an „almost" effective character) [10]). If we take for G the (countably additive) group of all partial automorphisms in $\mathfrak{B}$, congruence under G clearly coincides with homogeneity.

Various interesting instances of relations $\widetilde{G}$ can be found in Boolean algebras constituted by countably complete fields of sets. $\mathfrak{B}$ being an algebra of this type, let f be a function with the following properties: f is biunique and its domain is a set in $\mathfrak{B}$; if f maps a set X onto a set Y and one of these sets is in $\mathfrak{B}$, then the other is also in $\mathfrak{B}$. Consider now the correlated function $\bar{f}$ defined as at the beginning of § 1, but with the domain restricted to those sets

[10]) Propositions Q_3 and Q_4 apply to an even wider class of relations, in fact, to congruences under finitely additive groups of partial automorphisms. (The definition of a finitely additive group is entirely analogous to that of a countably additive group). See Tarski [1], Sections 11 and 15.

which belong to $\mathfrak{B}$ and which are included in the domain of f; $\bar{f}$ is clearly a partial automorphism in $\mathfrak{B}$. From every set G of functions f with the properties specified above we obtain in this way the set $\bar{G}$ of correlated partial automorphisms $\bar{f}$; and, under appropriate conditions imposed on G, the set $\bar{G}$ proves to be a group or even a countably additive group. For instance, let $\mathfrak{B}$ be the algebra of all sets in a metric space S; and let G be the set of all isometric functions (distance-preserving transformations) whose domain and counter-domain are sets in $\mathfrak{B}$. Then $\bar{G}$ is a group of partial automorphisms in $\mathfrak{B}$; and the relation of equivalence by infinite decomposition under $\bar{G}$ — or, what amounts to the same, the relation of congruence under the smallest countably additive group over $\bar{G}$ — is a countably additive and refining equivalence relation in $\mathfrak{B}$ which plays a certain role in the discussion of the problem of measure [11]). Or, let $\mathfrak{B}$ be the algebra of all Borel sets in a topological space T; and let G be the family of all biunique Baire functions whose domain and counter-domain are sets in $\mathfrak{B}$. Then $\bar{G}$ proves to be a countably additive group of partial automorphisms in $\mathfrak{B}$; congruence under $\bar{G}$ coincides with what is called the relation of generalized homeomorphism [12]); and by applying Theorems 4 and 5 we obtain a series of theorems on generalized homeomorphisms in arbitrary topological spaces. If, for instance, A, B, and C are Borel sets with $A \subset B \subset C$, and f is a biunique Baire function which maps A onto C, then there are two biunique Baire functions g and h such that g maps A onto B, and h maps B onto C; moreover, in consequence of the effective nature of the proof of this statement, we can determine the Baire classes of g and h if we know the Borel classes of A, B, and C, and the Baire class of f.

Besides congruence relations under groups of partial automorphisms, other examples of countably additive and refining equivalence relations in Boolean algebras are also known. For instance, every relation of congruence modulo a countably complete ideal I in a countably complete Boolean algebra $\mathfrak{B}$ is of this type. However, the results obtained by applying Theorems 4 and 5 to such a congruence relation are rather trivial and can easily be obtained in a direct way, as elementary consequences of the fact that the quotient algebra $\mathfrak{B}/I$ is a countably complete Boolean algebra.

[11]) See Banach-Tarski [1], § 3.
[12]) See Kuratowski [1].

Finally, we can point out a more comprehensive class of countably additive and refining equivalence relations, which includes both classes previously discussed. G being a set of partial automorphisms in a Boolean algebra $\mathfrak{B}$, and I being an ideal in $\mathfrak{B}$, we call two elements a and b of $\mathfrak{B}$ congruent under G modulo I if there are elements a_1, a_2, b_1, and b_2 such that

$$a = a_1 + a_2, \quad b = b_1 + b_2, \quad a_1 \underset{G}{\cong} b_1, \quad a_2 \in I, \quad \text{and} \quad b_2 \in I.$$

If I consists of the unique element 0, the relation just defined coincides with congruence under G; if, on the other hand, G consists only of the identity function, this relation reduces to congruence modulo I. In case the algebra $\mathfrak{B}$ and the ideal I are countably complete, and the set G is a countably additive group, congruence under G modulo I proves to be a countably additive and refining equivalence relation in $\mathfrak{B}$. To give an example, consider the algebra $\mathfrak{B}$ of all subsets of an n-dimensional cube (in n-dimensional Euclidean space) which are measurable in the sense of Lebesgue. Let G be the set of all isometric functions whose domain and counter-domain are in $\mathfrak{B}$; let H be the smallest countably additive group over $\bar{G}$ of partial automorphisms in $\mathfrak{B}$; and let I be the ideal of all sets in $\mathfrak{B}$ of measure 0. The relation of congruence under H modulo I has been discussed in the literature; it has been shown that this relation holds between two sets if, and only if, they have the same measure [11]).

These examples give us an idea of the variety of systems to which Theorems 4 and 5 — and other results analogously obtained — can be applied.

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Sur la continuité et la classification de Baire des fonctions abstraites[1]).

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0. Nous désignons par $\underset{u}{E}\{w(u)\}$ l'ensemble des éléments u qui satisfont à la condition $w(u)$.

X désigne un espace de Banach, c. à d. un espace vectoriel normé et complet (Banach [1][2]), p. 52);

$\|x\|$ — la norme de l'élément $x \in X$;

Ξ — l'espace conjugué à X, c. à d. l'ensemble linéaire des fonctionnelles linéaires $\xi(x)$ définies dans X (la norme dans Ξ étant définie par la formule $\|\xi\| = \sup\limits_{\|x\| \leqslant 1} \xi(x)$, l'espace Ξ est un espace de Banach).

Ξ_0 — l'ensemble fondamental[3]) de fonctionnelles linéaires dans X c. à d. un sous-ensemble de Ξ satisfaisant à la condition suivante: pour tout $\varepsilon > 0$ et $x \in X$, il existe une combinaison linéaire $\xi = a_1\xi_1 + a_2\xi_2 + \ldots + a_n\xi_n$ d'éléments $\xi_1, \xi_2, \ldots, \xi_n$ de Ξ_0 telle que

$$(0.1) \qquad \|\xi\| \leqslant 1 \quad et \quad \xi(x) \geqslant \|x\| - \varepsilon;$$

Ξ_0^* — l'ensemble des combinaisons linéaires d'éléments de Ξ_0 qui satisfont à la condition (0.1).

[1]) Les résultats de cette Note ont été présentés à la séance du 14 décembre 1946 au IV Congrès Polonais de Mathématique à Wrocław.

[2]) Les nombres entre les crochets renvoient aux ouvrages cités à la fin de cette Note.

[3]) La notion d'un ensemble fondamental de fonctionnelles linéaires ne coïncide pas avec celle de l'ensemble fondamental d'éléments de l'espace conjugué, due à Banach ([1], p. 58).