ON FITE'S OSCILLATION THEOREMS

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The general solution of the differential equation

(1) $$y^{(n)}+y=0$$

is

$$\sum_{\nu=1}^{\frac{n}{2}} a_\nu e^{\alpha_\nu x}\sin(\beta_\nu x+b_\nu) \qquad \text{if } n \text{ is even,}$$

and

$$a_0e^{-x}+\sum_{\nu=1}^{\frac{n-1}{2}} a_\nu e^{\alpha_\nu x}\sin(\beta_\nu x+b_\nu) \qquad \text{if } n \text{ is odd,}$$

where $\alpha_\nu=\cos\frac{\nu\pi}{n}$, $\beta_\nu=\sin\frac{\nu\pi}{n}$ and a_ν, b_ν are arbitrary constants. It is easy to see that n being even, any particular (non trivial) solution $y(x)$ of (1) oscillates, i. e. changes its sign an infinite number of times as $x\to\infty$. If n is odd, it is so except when $a_1=\ldots=a_{\frac{n-1}{2}}=0$; in the last case $y(x)$ is everywhere different from 0 and approaches 0 as $x\to\infty$.

This property of solutions holds for the more general equation

(2) $$y^{(n)}+A(x)y=0$$

if convenient hypotheses are made on $A(x)$.

Sturm showed in 1836 [1]) that if $n=2$, $A(x)$ is continuous and

(3) $$A(x)>m>0,$$

then every solution of (2) oscillates.

Kneser proved in 1893 [2]) that $A(x)$ being a continuous function the condition (3) suffices that any solution of (2) oscillates, in case n is even, and either oscillates or approaches 0 monotonically if n is odd.

Now, Fite stated in 1918 [3]) two more general theorems by replacing the condition (3) by the following ones respectively:

(4) $A(x)>0$ and $\int_a^\infty A(x)dx=\infty$ (in his theorem V),

(5) $A(x)>x^{-n+\varepsilon}$ for some $\varepsilon>0$ and great values of x (in his theorem VI).

We shall show that a more general theorem holds, both Fite's theorems being its immediate conclusions.

Theorem. Let $A(x)$ be a continuous function for $x\geqslant a$.

If n is odd, and

(6) $$A(x)\geqslant 0 \quad \text{and} \quad \int_a^\infty x^{n-1}A(x)\,dx=\infty,$$

then each solution of (2) either oscillates or approaches 0 monotonically as $x\to\infty$.

If n is even, and

(7) $$A(x)\geqslant 0 \quad \text{and} \quad \int_a^\infty x^{n-1-\varepsilon}A(x)\,dx=\infty \qquad (0<\varepsilon<n-1),$$

then each solution of (2) oscillates as $x\to\infty$.

The following example, given by Fite, shows that in the second part of the theorem ε cannot be omitted. In fact, let $n=4$ and $A(x)=15/16x^4$; then the equation (2) has a non-oscillating solution $y=x^{5/2}$.

The proof of the theorem is essentially the same as in the case of Fite's theorem VI, but some difficulty lies in dealing with iterated integrals. We first establish the following

Lemma. If $0\leqslant\alpha<1<m$, $A(x)\geqslant 0$ for $x\geqslant a$, and

$$\int_a^\infty x^{m-2+\alpha}A(x)\,dx<\infty \quad \text{and} \quad \int_a^\infty x^{m-1}A(x)\,dx=\infty,$$

[1]) E. Sturm, Journal de Mathématiques pures et appliquées 1 (1836), p. 106-186.

[2]) A. Kneser, *Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen*, Mathematische Annalen 42 (1893), p. 409-435.

[3]) W. B. Fite, *Concerning the zeros of the solutions of certain differential equations*, Transactions of the American Mathematical Society, 19 (1918), p. 341-352.

then given any positive $x_1 > a$ *and* $K > 0$ *there exists an* $x_2 > x_1$ *such that*

(8) $$\int_{x_1}^{x} dt \int_{t}^{\infty} (s-t)^{m-2+a} A(s)\, ds \geqslant K(x-x_2)^a \qquad \textit{for } x \geqslant x_2.$$

Indeed, from $\Gamma(a)\Gamma(1-a) = \pi/\sin a\pi$ we easily get for $0 < a < 1$ the general formula

$$\int_{x_1}^{x} f(t)\, dt = \frac{\sin a\pi}{\pi} \int_{x_1}^{x} (x-t)^{a-1} dt \int_{x_1}^{t} (t-u)^{-a} f(u)\, du \quad (f \text{ continuous}),$$

and the left member of (8) may be written

$$\frac{\sin a\pi}{\pi} \int_{x_1}^{x} (x-t)^{a-1} F(t)\, dt,$$

where

(9) $$F(t) = \int_{x_1}^{t} (t-u)^{-a} du \int_{u}^{\infty} (s-u)^{m-2+a} A(s)\, ds \geqslant$$
$$\geqslant \int_{x_1}^{t} (t-u)^{-a} du \int_{u}^{t} (s-u)^{m-2+a} A(s)\, ds =$$
$$= \int_{x_1}^{t} A(s)\, ds \int_{x_1}^{s} (t-u)^{-a} (s-u)^{m-2+a} du \geqslant$$
$$\geqslant \int_{x_1}^{t} A(s)\, ds \int_{x_1}^{s} (s-u)^{m-2} du \geqslant \frac{1}{m-1} \int_{x_1}^{t} (s-x_1)^{m-1} A(s)\, ds.$$

By hypothesis, the integral $\int_a^t s^{m-1} A(s)\, ds$ diverges as t increases and it is easily seen that so does the last integral in (9), because it is greater than

$$\int_b^t \left(1 - \frac{x_1}{b}\right)^{m-1} s^{m-1} A(s)\, ds \qquad \text{for } t \geqslant b > x_1.$$

Thus for properly chosen $x_2 > b$ we shall have $F(t) > \dfrac{K\pi}{\sin a\pi}$, and the left member of (8) will be greater, for $x \geqslant x_2$, than

$$\frac{\sin a\pi}{\pi} \int_{x_2}^{x} (x-t)^{a-1} \frac{K\pi}{\sin a\pi} dt > K(x-x_2)^a.$$

If $a = 0$, the left member of (8) is greater than

$$\int_{x_1}^{x} dx \int_{t}^{x} (s-t)^{m-2} A(s)\, ds = \frac{1}{m-1} \int_{x_1}^{x} (s-x_1)^{m-1} A(s)\, ds$$

and becomes consequently greater than any given number K. This completes the proof of the lemma.

Now, suppose $y(x)$ to be a non-oscillating solution of (2). Then it is of constant sign for sufficiently great values of x, and consequently monotonic, because of (2) and of the hypothesis that $A(x) \geqslant 0$. To prove the theorem it is sufficient to show that the assumption

$$y(x) \geqslant k > 0 \quad \text{for} \quad x \geqslant x_1 \geqslant a$$

leads to absurd.

In the following we shall reproduce the argument of Fite, but modified by introducing convenient integral inequalities. The parts in quotation marks are taken literally from his paper mentioned above.

"We take x_1 sufficiently great so that $y(x) > k$ for $x \geqslant x_1$ and consider the consequences of the inequality"

(10) $$y^{(n)}(x) \leqslant -kA(x).$$

"From (10) we get"

$$y^{(n-1)}(x) \leqslant y^{(n-1)}(x_1) - k\int_{x_1}^{x} A(t)\, dt.$$

If $\int_{x_1}^{x} A(t)\, dt$ diverges for $x \to \infty$, "$y^{(n-1)}(x)$ would be negative for sufficiently great values of x, and y would become negative. We assume therefore that, if $n > 1$", the value of $\int_{x_1}^{\infty} A(t)\, dt$ is finite and

$$y^{(n-1)}(x_1) - k\int_{x_1}^{\infty} A(t)\, dt \geqslant 0.$$

"Moreover, if the expression in the left member of this inequality should become negative as x_1 increases, we would apply the preceding argument for a properly chosen x_1. There remains then to be considered the case in which"

$$y^{(n-1)}(x) - k\int_{x}^{\infty} A(t)\, dt \geqslant 0.$$

"From this inequality we get"

$$y^{(n-2)}(x) \geqslant y^{(n-2)}(x_1) + k\int_{x_1}^{x} dt \int_t^{\infty} A(s)\,ds.$$

"If $n>2$ and $y^{(n-2)}(x)$ is to remain negative as x increases", the integral $\int_{x_1}^{x} dt \int_t^{\infty} A(s)\,ds$ must converge for $x\to\infty$ and we have

$$y^{(n-2)}(x) + k\int_x^{\infty} dt \int_t^{\infty} A(s)\,ds \leqslant 0, \quad \text{i. e.} \quad y^{(n-2)}(x) + \frac{k}{1!}\int_x^{\infty} (t-x) A(t)\,dt \leqslant 0.$$

Hence

$$y^{(n-3)}(x) \leqslant y^{(n-3)} - \frac{k}{1!}\int_{x_1}^{x} dt \int_t^{\infty} (s-t) A(s)\,ds.$$

"Now if y does not change sign as x increases, no derivative of index $n-2i-1$ can become negative if all derivatives with indices of the form $n-2j$, where $j\leqslant i$, remain negative. There are therefore but two conceivable results of continuation of this argument:

(a) *The derivatives of y from the n-th to the first inclusive are of alternate signs for $x\geqslant x_1$.*

In case n is odd this supposition leads to the inequality"

$$y'(x) + \frac{k}{(n-2)!}\int_x^{\infty} (t-x)^{n-2} A(t)\,dt \leqslant 0,$$

and further

$$y(x) \leqslant y(x_1) - \frac{k}{(n-2)!}\int_{x_1}^{x} dt \int_t^{\infty} (s-t)^{n-2} A(s)\,ds.$$

From this we conclude by lemma (for $a=0$ and $m=n$) and hypothesis (6) that y must become negative.

If n is even, we have

$$y'(x) \geqslant \frac{k}{(n-2)!}\int_x^{\infty} (t-x)^{n-2} A(t)\,dt; \tag{11}$$

hence

$$y(x) \geqslant y(x_1) + \frac{k}{(n-2)!}\int_{x_1}^{x} dt \int_t^{\infty} (s-t)^{n-2} A(s)\,ds.$$

By lemma (for $a=\varepsilon$ and $m=n-\varepsilon$) and hypothesis (7) we have $y(x)\geqslant(x-x_2)^{\varepsilon}$ for $x\geqslant x_2$, and we can replace (10) by

$$y^{(n)}(x) \leqslant -(x-x_2)^{\varepsilon} A(x).$$

"This in turn gives us" $y(x)\geqslant(x-x_3)^{2\varepsilon}$ for $x\geqslant x_3$, "and we can accordingly replace (10) by"

$$y^{(n)}(x) \leqslant -(x-x_3)^{2\varepsilon} A(x).$$

"A continuation of this process leads finally to the relation"

$$y^{(n)}(x) \leqslant -(x-x_p)^{p\varepsilon} A(x),$$

where $1-\varepsilon\leqslant p\varepsilon<1$. This in turn gives us

$$y'(x) \leqslant y'(x_p) - \frac{1}{(n-3)!}\int_{x_p}^{x} dt \int_t^{\infty} (s-t)^{n-3+p\varepsilon} A(s)\,ds,$$

and by lemma (for $a=(p+1)\varepsilon-1$ and $m=n-\varepsilon$)

$$y'(x) \leqslant y'(x_p) - K(x-x_{p+1})^{(p+1)\varepsilon-1},$$

where K may by supposed greater then $y'(x_p)$.

"But it is contrary to (11).

(b) *The derivative of index $n-2i$ is positive for sufficiently great values of x, while all derivatives with indices of the form $n-2j$ $(0\leqslant j<i)$ are negative for $x\geqslant x_1$ and all the derivatives with indices of the form $n-2j+1$ $(1\leqslant j\leqslant i)$ are positive for these values of x.*

Since $y^{(n-2i)}(x)$ is positive and increasing, we should have" $y(x)>(x-x_1)^{n-2i}$ $(k>0)$ "for sufficiently great values of x. We could therefore replace (10) by"

$$y^{(n)}(x) \leqslant -k(x-x_1)^{n-2i} A(x).$$

"But a series of steps similar to those described under (a) shows that this inequality requires that $y^{(n-2i+1)}(x)$ be negative for sufficiently great values of x".