We define a map G such that $G\colon T \to A$ by

$$G(x) = F'(x,1), \quad \text{for} \quad x \in T.$$

Now G retracts T onto A because for any $a \in A$ we have

$$G(a) = F'(a,1) = F(a,1) = f(a,1) = a.$$

Hence by (3.6) A is an AR.

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What paths have length?

By

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In the classical theory, the length of the curve $y = f(x)$ $(a \leqslant x \leqslant b)$ is determined by computing the integral $\int_a^b \sqrt{1+f'^2(x)}\,dx$. Geometrically, this means that in determining the length of an arc we really compute the area of a plane domain. The length of the circular arc $y = \sqrt{1-x^2}$ $(0 \leqslant x \leqslant b)$ is the area of the plane domain $(0 \leqslant x \leqslant b, 0 \leqslant y \leqslant 1/\sqrt{1-x^2})$. If the arc happens to be a quarter of a circle, the domain is not even bounded.

In a series of previous papers[1]), the author has developed a more geometric approach to the problem based on the definition of the length of a path as the limit of the lengths of inscribed polygons which get indefinitely dense in the path. This length was studied in spaces of increasing generality. For instance, when applied to vector spaces our results comprise not only Finsler spaces but spaces with locally Minkowskian metrics in which the indicatrices (or unit spheres) are positive in some directions and negative or zero in others. On each stage we formulated sufficient conditions

[1]) [1] Mathematische Annalen **103** (1930), especially pp. 492-501. — [2] Fundamenta Mathematicae **25** (1935), p. 441. — [3] Three notes in the C. R. Paris **201** (1936), p. 705; **202** (1936), p. 1007; **202** (1936), p. 1648. — [4] Ergebnisse eines mathematischen Kolloquiums **8** (1937), p. 1-37. — [5] Proc. Nat. Acad. Sc., **23** (1937), p. 244. — [6] Ibid., **25** (1939), p. 474. — [7] Rice Institute Pamphlets **27** (1940), p. 1-40. — Cf. Pauc, *Les méthodes directes en calcul des variations et en géométrie différentielle*, Hermann, Paris 1941. — In [7], metric methods are also used for the formulation of necessary and sufficient conditions for a line integral to be independent of the path. We add a bibliography of more recent results along these lines: Menger, Proc. Nat. Acad. Sc., **25** (1939), p. 621. — Fubini, ibid., **26** (1940), p. 190. — Menger, ibid. **26** (1940), p. 660. — Artin, ibid., **27** (1941), p. 489. — Menger, Reports of a Mathematical Colloquium, 2-nd ser., **2** (1939), p. 45. — Milgram, ibid., **3** (1940), p. 28. — de Pazzi Rochford, ibid., **4** (1940), p. 6.

for the existence as well as the lower semi-continuity of the length. Since lower semi-continuity in a well known way implies the existence of minimizing paths we thus derived existence theorems not only for positively definite but also for semi-definite and indefinite problems of the calculus of variations.

In the course of these studies it became clear that such topological concepts as neighboring paths should not be based on the distance in terms of which the length of polygons and paths is defined. The topological concepts should rather be introduced in terms of a basic topology of the space. Only for the sake of simplicity we described, and continue to describe, this topology in terms of a metric space but we might as well describe it in terms of any topologically equivalent metric [2]).

In the present paper we shall formulate conditions which are not only sufficient but at the same time necessary for the existence of a finite length. On the other hand, we shall not compare the path with neighboring paths. Hence we shall treat the path intrinsically, that is, as a closed interval of real numbers [3]). We shall choose the interval $\mathcal{J}=[0,1]$ and shall describe its topology in terms of the euclidean distance $|x-y|$ of the points x and y.

The lengths of polygons and the length of $\mathcal{J}$ will be derived from a distance $\delta(x,y)$. More precisely, we assume that with every ordered pair of numbers x,y of $\mathcal{J}$ a real number $\delta(x,y)$ is associated, called the *distance from x to y*, which is connected with the underlying topology only by the following one-sided continuity condition [4]):

[2]) In [4] and [7] l. c. [1]) we prove that in theorems of the calculus of variations which refer to rectifiable paths or to paths of uniformly bounded lengths, the ordinary length may be replaced by one derived from a more general distance of comparison so that altogether we distinguish *three metrics in the calculus of variation*: the metric which describes the topology and is metrically insignificant; the metric for which we wish to minimize the length; the metric of comparison. In the classical calculus of variations, including Tonelli's theory, one studies only the euclidean metric as the metric describing the topology of the underlying vector space and at the same time as the metric of comparison while the length which we wish to minimize is obtained by a multiplicative distortion of this metric — the integrand being the distorting factor.

[3]) We have formulated an intrinsic definition of the concept of path even if the path is imbedded in a space. Cf. C. R. Paris **221** (1945), p. 739.

[4]) In subsequent papers we shall also admit *complex* distances, in fact, distances belonging to a *normed group*. We shall furthermore weaken the one-sided continuity condition mentioned above.

For each $\delta>0$ there exists an $\eta(\delta)>0$ such that $|x-x'|<\eta(\delta)$ implies $|\delta(x,x')|<\delta$.

We shall not assume that, conversely, for points x,x' whose distance $\delta(x,x')$ is small, the number $|x-x'|$ is small. Even if both $\delta(x,x')$ and $\delta(x',x)$ are small or $=0$, we shall not be able to conclude that $|x-x'|$ is small.

Under this assumption we shall formulate conditions which are both necessary and sufficient in order that $\mathcal{J}$ have a finite length.

With regard to the relation between length and area, our theory reverses the classical point of view. Instead of representing the length as an integral or an area, we may represent integrals and areas as lengths. For instance, we obtain the *integral* $\int_a^b f(x)\,dx$ as the length of the interval $[a,b]$ derived from the distance $\delta(x,y)=f(x)\cdot(y-x)$. In order to obtain the variation of the function $f(x)$ as a length we set $\delta(x,y)=|f(y)-f(x)|$. What we may call the *left-side Stieltjes integral* [5]) $\int_a^b f(x)\,dg(x)$ is derived from the distance $\delta(x,y)=f(x)\cdot(g(y)-g(x))$. The *Weierstrass line integral* [6]) $\int_a^b f(x,\mathring{x})\,dt$ where x is a vector in any vector space, and f is positively homogeneous of degree 1 in $\mathring{x}$, is obtained by setting $\delta(t,t')=f(x(t),x(t')-x(t))$.

We begin by defining the necessary auxiliary concepts. By a *polygon* we mean an ordered set $P=\{x_0,x_1,\dots,x_{n-1},x_n\}$ of elements of $\mathcal{J}$ such that $n\geqslant 1$ and $x_0<x_1<\dots<x_{n-1}<x_n$. We call $x_1,\dots,x_{n-1}$ the *inner* points of P. The polygons P and $Q=\{y_0,\dots,y_m\}$ are said to be *exclusive* if the closed intervals $[x_0,x_n]$ and $[y_0,y_m]$ have at most one point in common. The polygon P is called an *end-to-end* polygon if $x_0=0$ and $x_n=1$.

We set

$$\nu P=\mathrm{Max.}\,(x_{i+1}-x_i),\quad \sigma P=x_n-x_0,$$

[5]) cf. the author's paper *The Stieltjes integrals considered as lengths*, Ann. de la Soc. Pol. de Math. t. XXI (1948).

[6]) Concerning Weierstrass integrals cf. [7] l. c. [1]) and, in particular, Pauc's comprehensive presentation in his booklet *Les méthodes directes*.

and call these numbers the *norm* of P and the *span* of P, respectively. By the *length* of P we mean the number

$$\lambda P = \Sigma \delta(x_i, x_{i+1}),$$

by the *absolute length* of P the number

$$\Lambda P = \Sigma |\delta(x_i, x_{i+1})|.$$

We call

$$\chi P = \delta(x_0, x_n)$$

the *chord* of P. Of importance for our purpose is the following number

$$\varkappa P = \frac{\chi P - \lambda P}{|\chi P|} \quad \text{if} \quad \chi P \neq 0.$$

We call $\varkappa P$ the *contraction* of P and complete the definition by setting

$$\varkappa P = +\infty \quad \text{if} \quad \chi P = 0 > \lambda P,$$
$$\varkappa P = -\infty \quad \text{if} \quad \chi P = 0 < \lambda P.$$

$\varkappa P = 0$ is equivalent with $\lambda P = \chi P$ which, in a euclidean space, holds if and only if P is an ordered linear polygon. In case that $\chi P > 0$, the condition $\varkappa P = 1$ is equivalent with $\lambda P = 0$.

We call a sequence of polygons $\{P_n\}$ *distinguished* if each P_n is an end-to-end polygon and $\lim \nu P_n = 0$. We set

$$\lambda^*\{P_n\} = \limsup \lambda P_n \quad \text{and} \quad \lambda_*\{P_n\} = \liminf \lambda P_n.$$

We call the least upper bound of the numbers $\lambda^*\{P_n\}$ for all distinguished sequences the *upper length* of $\mathcal{J}$, the greatest lower bound of the numbers $\lambda_*\{P_n\}$ the *lower length* of $\mathcal{J}$. We say that $\mathcal{J}$ has a *length* if upper and lower lengths are equal.

We say that $\mathcal{J}$ is *of bounded absolute length* if the least upper bound, Λ, of the numbers ΛP for all end-to-end polygons is finite. Clearly, if $\mathcal{J}$ is of bounded absolute length, and has a length $\lambda\mathcal{J}$, then

$$|\Lambda \mathcal{J}| \leqslant \Lambda.$$

In terms of these concepts we shall prove the following

Theorem. *If $\mathcal{J}$ is of bounded absolute length, then in order that $\mathcal{J}$ have a finite length it is necessary and sufficient that for each $\varepsilon > 0$ and $\varkappa > 0$ there exists a $\nu = \nu(\varepsilon, \varkappa) > 0$ such that for each finite set of exclusive polygons $Q_1, Q_2, \ldots, Q_k$ whose spans are $< \nu$ and whose contractions are $> \varkappa$ we have*

(1) $$\Sigma |\chi Q_i| < \varepsilon,$$
(2) $$\Sigma \lambda Q_i > -\varepsilon.$$

Necessity of 1). If Condition 1) is not satisfied, then there exist two numbers ε_0 and $\varkappa_0$, both > 0, such that for each n there exists a finite set $\mathfrak{Q} = \{Q_1^n, Q_2^n, \ldots, Q_{k_n}^n\}$ of exclusive polygons of spans $< 1/n$ and contractions $\geqslant \varkappa_0$ for which

$$\Sigma |\chi Q_i^n| \geqslant \varepsilon_0.$$

For each n, since the span of each Q_i^n is $< 1/n$ we can complete the polygons of $\mathfrak{Q}$ to one polygon Q^n of a norm $< 1/n$. The polygon P^n obtained from Q^n by omitting all the inner points of the k_n polygons Q_i^n likewise has a norm $< 1/n$. If λ_n is the sum of the lengths of the polygons completing the k_n polygons Q_i^n to Q^n, then

$$\lambda P^n = \lambda_n + \Sigma \chi Q_i^n \quad \text{and} \quad \lambda Q^n = \lambda_n + \Sigma \lambda Q_i^n.$$

Since the contraction of each Q_i^n is $\geqslant \varkappa_0$ we have

$$\lambda Q_i^n < \chi Q_i^n - \varkappa_0 |\chi Q_i^n| \qquad (i = 1, 2, \ldots, k_n)$$

and thus

$$\lambda Q^n < \lambda_n + \Sigma \chi Q_i^n - \varkappa_0 \Sigma |\chi Q_i^n|.$$

Hence

$$\lambda Q^n < \lambda P^n - \varkappa_0 \Sigma |\chi Q_i^n| < \lambda P^n - \varkappa_0 \cdot \varepsilon_0.$$

We see for each n there exist two polygons P^n and Q^n of norms $< 1/n$ whose lengths differ by more than $\varkappa_0 \varepsilon_0$. Thus $\mathcal{J}$ has no finite length.

Necessity of 2). If Condition 2) is not satisfied, then there exist two numbers ε_0 and $\varkappa_0$, both > 0, such that for each n there exists a finite set $\mathfrak{Q}$ of exclusive polygons $Q_1^n, \ldots, Q_{k_n}^n$ whose spans are $< 1/n$, whose contractions are $\geqslant \varkappa_0$ and for which

(*) $$\Sigma \lambda Q_i^n < -\varepsilon_0.$$

Since the necessity of Condition 1) has been established we can assume that

(**) $$\lim_{n \to \infty} \Sigma |\chi Q_i^n| = 0.$$

For each n, we form Q^n and P^n as before and have

$$\lambda P^n = \lambda_n + \Sigma \chi Q_i^n \quad \text{and} \quad \lambda Q^n = \lambda_n + \Sigma \lambda Q_i^n.$$

Thus

$$\lambda Q^n - \lambda P^n = \Sigma \lambda Q_i^n - \Sigma \chi Q_i^n.$$

From (*) and (**) it follows that

$$\lambda Q^n - \lambda P^n < -\varepsilon_0/2 \quad \text{for all large } n,$$

and again $\mathcal{J}$ has no finite length.

Sufficiency. We begin by proving: If Conditions 1) and 2) hold, and thus for every $\varepsilon > 0$ and $\varkappa > 0$ a number $\nu(\varepsilon, \varkappa)$ with the specified properties exists, then for every $\varkappa > 0$ and $\zeta > 0$ the following condition holds:

Condition $C_{\varkappa\zeta}$. For each polygon whose norm is sufficiently small, namely, $< \nu = \nu(\zeta/5, \varkappa)$, there exists a number $\nu' > 0$ with the property that each polygon Q whose norm is $< \nu'$, satisfies the inequality

$$\lambda Q > \lambda P - \varkappa \cdot \Lambda P - \zeta.$$

By assumption, for every finite set of exclusive polygons $Q_1, \ldots, Q_k$ whose spans are $< \nu = \nu(\zeta/5, \varkappa)$ and whose contractions are $> \varkappa$, we have

(1′) $$\Sigma |\chi Q_i| < \zeta/5,$$
(2′) $$\Sigma \lambda Q_i > -\zeta/5.$$

Let P be any polygon whose norm is $< \nu$. Let P be the polygon $\{x_0, \ldots, x_n\}$ so that $n+1$ is the number of points of P. For the number $\zeta/5n$ we form the number $\eta(\zeta/5n)$ mentioned in the basic continuity postulate, that is to say, the number for which

($\dagger$) $$|x - x'| < \eta \quad \text{implies} \quad |\delta(x, x')| < \zeta/5n.$$

If ν' denotes the smaller of the two numbers

$$\eta \quad \text{and} \quad \tfrac{1}{2}\,\mathrm{Min.}\,(x_{i+1} - x_i),$$

then we shall prove that ν' has the property claimed in Condition $C_{\varkappa\zeta}$.

In order to prove this contention, let Q be any polygon whose norm is $< \nu'$. Hence

(3) $$\nu Q < \tfrac{1}{2}\,\mathrm{Min.}\,(x_{i+1} - x_i)$$

and

(4) $$\nu Q < \eta = \eta(\zeta/5n).$$

For each point x_i of P we denote

by $y_{j(i)}$ the first point of Q which is $> x_i$,
by $y_{k(i)}$ the last point of Q which is $\leqslant x_{i+1}$.

We shall set

$$Q_i = \{y_{j(i)}, y_{j(i)+1}, \ldots, y_{k(i)}\}$$

and

$$Q_i^* = \{x_i, y_{j(i)}, y_{j(i)+1}, \ldots, y_{k(i)}, x_{i+1}\}.$$

From (3) it follows that

$$x_i < y_{j(i)} < y_{k(i)} \leqslant x_{i+1} \qquad (i = 0, 1, \ldots, n-1).$$

Hence each Q_i contains at least two points and thus is a polygon, while each Q_i^* is a polygon containing at least three points. The points $y_{k(i)}$ and $y_{j(i+1)}$ of Q are consecutive and

$$y_{k(i)} \leqslant x_{i+1} < y_{j(i+1)} \qquad (i = 0, 1, \ldots, n-1).$$

From (4) it thus follows that

(5) $$|\delta(x_i, y_{j(i)})| < \zeta/5n \quad \text{and} \quad |\delta(y_{k(i)}, y_{j(i+1)})| < \zeta/5n.$$

The spans of Q_i and Q_i^* are $\leqslant x_{i+1} - x_i < \nu$. The polygons $Q_0^*, \ldots, Q_{n-1}^*$ are mutually exclusive. Also the polygons obtained by replacing some (or all) of the Q_i^* by the corresponding Q_i are mutually exclusive. We have

$$\lambda Q_i^* = \delta(x_i, y_{j(i)}) + \lambda Q_i + \delta(y_{k(i)}, x_{i+1}).$$

From (5) it thus follows that

(6) $$\lambda Q_i > \lambda Q_i^* - 2\zeta/5n.$$

If we set $y_{k(0)} = x_0$, then

$$\lambda Q = \Sigma \lambda Q_i + \Sigma \delta(y_{k(i)}, y_{j(i+1)}).$$

Hence by (5)

(7) $$\lambda Q > \Sigma \lambda Q_i - \zeta/5.$$

We shall say that a given polygon Q_i^* is *of the first kind* if $\varkappa Q_i^* > \varkappa$, *of the second kind* if $\varkappa Q_i^* \leqslant \varkappa$. We shall denote summations restricted to polygons of the first kind or the second kind by Σ' and by Σ'', respectively. We have

(8) $$\Sigma \lambda Q_i = \Sigma' \lambda Q_i + \Sigma'' \lambda Q_i.$$

Now polygons of the first kind have spans $< \nu$ and contractions $> \varkappa$. Thus by (1′),

$$\Sigma' |\chi Q_i| < \zeta/5,$$

that is to say,

$$\Sigma'|\delta(x_i,x_{i+1})<\zeta/5 \tag{9}$$

and by (2')

$$\Sigma'\lambda Q_i > -\zeta/5. \tag{10}$$

From (6) it follows that

$$\Sigma''\lambda Q_i > \Sigma''\lambda Q_i^* - 2\zeta/5. \tag{11}$$

Since for each polygon of the second kind $\varkappa Q_i^* \leqslant \varkappa$ we have

$$\lambda Q_i^* \geqslant \chi Q_i^* - \varkappa\cdot|\chi Q_i^*|.$$

Thus

$$\Sigma''\lambda Q_i^* \geqslant \Sigma''\chi Q_i^* - \varkappa\Sigma|\chi Q_i^*| \geqslant \Sigma''\chi Q_i^* - \varkappa\cdot\Delta P = \Sigma''\delta(x_i,x_{i-1}) - \varkappa\cdot\Delta P =$$
$$= \lambda P - \Sigma'\delta(x_i,x_{i+1}) - \varkappa\cdot\Delta P.$$

By (9)

$$\Sigma''\lambda Q_i^* > \lambda P - \zeta/5 - \varkappa\cdot\Delta P.$$

Combining the last inequality with (7), (8), (10), (11) we conclude

$$\lambda Q > P - \varkappa\cdot\Delta P - \zeta.$$

Thus conditions $C_{\varkappa\zeta}$ is satisfied.

We conclude the demonstration of the sufficiency of the Conditions (1) and (2) by proving:

If $\mathcal{J}$ is of bounded absolute length and Condition $C_{\varkappa\zeta}$ holds for every $\varkappa$ and ζ, then $\mathcal{J}$ has a finite length.

Let Δ be the finite least upper bound of the numbers ΔP for all polygons. Clearly, the upper length λ^* of $\mathcal{J}$ is finite and there exists a *maximal* sequence of polygons $\{P_n\}$, that is to say, a sequence such that

$$\lim \lambda P_n = \lambda^*.$$

We claim: for every $\varepsilon>0$ there exists a $\nu'>0$ such that for each polygon P whose norm is $<\nu'$ we have

$$\lambda P > \lambda^* - \varepsilon.$$

We choose n so large that $\lambda P_n > \lambda^* - \varepsilon/3$. We further choose $\varkappa$ and ζ so that

$$\zeta < \varepsilon/3 \quad \text{and} \quad \varkappa < \varepsilon/3\Delta.$$

Under these circumstances, Condition $C_{\varkappa\zeta}$ yields our contention.

In subsequent papers we shall formulate conditions which are both necessary and sufficient for $\mathcal{J}$ to have the length ∞, in which case the interval will, of course, be of unbounded absolute length. On the other hand, in connection with the theorem which we have proved above, the question arises as to whether an interval of finite length is not eo ipso of bounded absolute length.

We conclude this paper with an example for the fact *that finite length is compatible with unbounded absolute length.* One is reminded of infinite series which converge without converging absolutely — except that there are more polygons in $\mathcal{J}$ than terms or partial sums in an infinite series.

Our example is based on the following auxiliary function f which is defined for all integers:

$$f(2n)=\tfrac{1}{2} \quad \text{and} \quad (2n+1)=-\tfrac{1}{4} \qquad (n=0,1,\ldots).$$

We divide $\mathcal{J}$ into 8 equal segments and define the distances of the end points as follows:

$$\delta\left(\frac{i_1}{8},\frac{i_1+1}{8}\right)=f(i_1)$$

We further set

$$\delta\left(\frac{i}{8},\frac{j}{8}\right)=\delta\left(\frac{i}{8},\frac{i+1}{8}\right)+\delta\left(\frac{i+1}{8},\frac{i+2}{8}\right)+\ldots+\delta\left(\frac{j-1}{8},\frac{j}{8}\right) \quad \text{for } i<j.$$

In particular,

$$\delta(0,1)=\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{2}-\tfrac{1}{4}=1.$$

Next we divide each of the intervals $\left[\frac{i_1}{8},\frac{i_1+1}{8}\right]$ into 8 equal segments and define

$$\delta\left(\frac{i_1}{8}+\frac{i_2}{8^2},\frac{i_1}{8}+\frac{i_2+1}{8^2}\right)=f(i_1)\cdot f(i_2).$$

Again, we set

$$\delta\left(\frac{i_1}{8}+\frac{i_2}{8^2},\frac{j_1}{8}+\frac{j_2}{8^2}\right)=\delta\left(\frac{i_1}{8}+\frac{i_2}{8^2},\frac{i_1}{8}+\frac{i_2+1}{8^2}\right)+\ldots+\delta\left(\frac{j_1}{8}+\frac{j_2-1}{8^2},\frac{j_1}{8}+\frac{j_2}{8^2}\right).$$

Proceeding in this way we set

$$\delta\left(\frac{i_1}{8}+\ldots+\frac{i_{n-1}}{8^{n-1}}+\frac{i_n}{8^n},\frac{i_1}{8}+\ldots+\frac{i_{n-1}}{8^{n-1}}+\frac{i_n+1}{8^n}\right)=f(i_1)\cdot\ldots\cdot f(i_{n-1})\cdot f(i_n)$$

and define a distance $\delta(x,y)$ for every two octogonally rational numbers x and y of $\mathcal{J}$ such that $x<y$. We set $\delta(y,x)=\delta(x,y)$ and $\delta(x,x)=0$. If two octogonally rational numbers differ by less than $1/8^n$, their distances differ by less than $1/2^n$. Hence it is easy to extend the definition of $\delta(x,y)$ to any two numbers x and y of $\mathcal{J}$. The length of each end-to-end polygon is 1. The absolute length of $\mathcal{J}$ is unbounded [7]).

[7]) A slight modification of the above construction leads to an arc having *the absolute length* ∞ *and the length* 0. We divide the interval $[0,1]$ into four instead of eight equal parts and define the distances from 0 to $\frac{1}{4}$, and from $\frac{1}{2}$ to $\frac{3}{4}$ to be $\frac{1}{2}$, and the distances from $\frac{1}{4}$ to $\frac{1}{2}$ and from $\frac{3}{4}$ to 1 to be $-\frac{1}{4}$. Iteration of this procedure leads to the indicated result.

Mr. Sheldon L. Levy pointed out that the original example (with divisions into eight parts) can be simplified. It is sufficient to divide the interval $[0,1]$ into three equal parts and to define the distance from 0 to $\frac{1}{3}$ as $\frac{2}{3}$, the distance from $\frac{1}{3}$ to $\frac{2}{3}$ as $-\frac{1}{3}$, and the distance from $\frac{2}{3}$ to 1 as $\frac{2}{3}$. Iteration of this procedure eads to an arc whose absolute length is ∞ and whose length is 1.

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Group invariant continua.

By

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1. We denote by X a compact ($=$bicompact) connected Hausdorff space. Let Z be a group which is also a topological space. It is not required that Z be a topological group. Let f be a map ($=$continuous transformation) of $Z\times X$ into X. Writing Z multiplicatively it is assumed that

$$f(e,x)=x \quad \text{for each } x\in X,\ e \text{ the neutral element}$$

and

$$f(z,f(z',x))=f(zz',x) \quad \text{for each } x\in X \text{ and } z,z'\in Z.$$

On setting $z(x)=f(z,x)$ it may easily be verified that z is a homeomorphism of X onto X and that z^{-1} is the inverse of z as a transformation. Accordingly we shall say (somewhat incorrectly) that Z *acts as a group of homeomorphisms on* X.

If A is any subset of X we define $Z(A)$ as the union of all the sets $z(A)$, $z\in Z$. It is an immediate consequence that

$$Z(A)=\bigcup_{z\in Z} z(A)=\bigcup_{a\in A} Z(a).$$

A subset A of X will be termed *Z-invariant* if $Z(A)=A$ or, equivalently $z(A)=A$ for each z in Z. Clearly X is Z-invariant.

In this note we prove among other things the

Theorem. *Let X be metric and locally connected. If Z is abelian then there exists a Z-invariant cyclic element.*

The first result of this character was proved by W. L. Ayres [1] who assumed that Z was generated by a single map, i. e., that Z was cyclic. For other results of this type see [6], Chap. XII, and [5] and the reference given here to G. E. Schweigert. In addition to extending this result from the case in which Z is cyclic to the case in which Z is merely abelian we remove the restrictions that X be metric and locally connected.