the void subset being not counted. It is therefore evident that the inequality

$$2^z - 1 > nz$$

would contradict to the condition (A), consequently:

$$2^z - 1 \leqslant nz.$$

It may be seen from (B) that

when	$n=1$	then	$z\leqslant 1$	when	$n=4$	then	$z\leqslant 4$
	$n=2$		$z\leqslant 2$		$n\geqslant 5$		$z<n$
	$n=3$		$z\leqslant 3$		$n=\aleph_0$		$z=$ finite.

Problem II. Let E be a countably infinite set. Denote by H the set of all finite subsets of E. Denote further by E^* the subset of E consisting of the elements $x \in E$ which are in the relation rRx with countably many $r \in H$.

The question is: what is the power of E^*?

Theorem II. *The power of E^* is $\aleph_0$.*

Proof. Denote by E_1 the subset of E consisting of those $x \in E$, for which there are only a finite number of $r \in H$ such that xRr. By Theorem I E_1 is finite. The power of E^* cannot be finite, because E is countably infinite and by condition (A) each element of E is in the relation with at least one element of H. The theorem is proved.

Let E be again an arbitrary set. Let H be the set of all subsets of E and n a cardinal number less than the power of E. Denote by E_1 the set of those $x \in E$ which are in the relation xRr with only such subsets $r \in H$ for which $\overline{\overline{r}} \leqslant n$.

Theorem III. *The power of E_1 is at most n.*

Proof: Suppose the contrary, i. e. that the power of E_1 is greater than n. Then by (A) there is an $x \in E_1$ such that xRE_1 holds. But, by the definition of E_1, this element x cannot belong to E_1, which is a contradiction.

References.

Sierpiński W., [1] *Sur un problème de la théorie des relations*, Fund. Math. **28**, pp. 71-74.

Piccard S., [1] *Solution d'un problème de la théorie des relations*, Fund. Math. **28**, pp. 197-202.

— [2] *Sur un problème de M. Ruziewicz de la théorie des relations*, Fund. Math. **29**, pp. 5-8.

Remark on an Invariance Theorem.

By

Casimir Kuratowski (Warszawa).

Borsuk gave a set theoretic proof of the following theorem [1]), which was previously proved by algebraic topology [2]).

Let $\mathfrak{X}$ denote the n-dimensional sphere; then:

(i) *For any closed set $F \subset \mathfrak{X}$ the number of components of the set $\mathfrak{X}-F$ is a topological invariant of the set F.*

Call (i*) the more general statement obtainable from (i) by omitting the assumption that F is closed [3]).

Theorem (i*) has been shown by Eilenberg using algebraic topology [4]).

In this note I shall deduce (i*) from (i) using set theoretic method.

In fact, I shall show that:

(ii) *Theorem* (i*) *holds in every locally connected continuum $\mathfrak{X}$ satisfying* (i).

The proof of (ii) will be based on the following theorem [5]):

(iii) *Let E be a subset of a locally connected space $\mathfrak{X}$. In order that the set $\mathfrak{X}-E$ be decomposable into n separated non void subsets, it is necessary and sufficient that E contain a closed set F such that for each closed set H satisfying the condition $F \subset H \subset E$ the set $\mathfrak{X}-H$ is decomposable into n separated non void subsets.*

Theorem (iii) may be established as follows.

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[1]) See *A Set Theoretical Approach to the Disconnection Theory of the Euclidean Space*, this volume, p. 217-241.

[2]) Cf. J. W. Alexander, Trans. Amer. Math. Soc. **23** (1922), p. 333, P. Alexandroff, Annals of Math. **30** (1928), p. 163.

[3]) If $\mathfrak{X}-F$ consists of an infinity of components, the „number of components" is to be understood as being equal to ∞ (not to be confused with the cardinal number of the set of components).

[4]) Bull. Amer. Math. Soc. **47** (1941), p. 73. See also P. Alexandroff, Doklady Akad. Nauk SSSR **57** (1947), p. 110.

[5]) See my *Topologie II* (1950), p. 174.

The condition is necessary. For, suppose

$$\mathfrak{X}-E=A_1+\ldots+A_n,\quad A_i\neq 0,\quad A_i\cdot\bar{A}_j=0 \quad\text{for}\quad i\neq j.$$

Let $G_1,\ldots,G_n$ be a system of open sets such that [6]):

$$A_i\subset G_i,\quad G_i\cdot G_j=0 \quad\text{for}\quad i\neq j.$$

Put $F=\mathfrak{X}-(G_1+\ldots+G_n)$. Hence $F\subset E$.

Let $F\subset H\subset E$. Thus $\mathfrak{X}-E\subset\mathfrak{X}-H\subset\mathfrak{X}-F$ and consequently

$$\mathfrak{X}-H=(G_1-H)+\ldots+(G_n-H)\quad\text{and}\quad 0\neq A_i\subset G_i-H.$$

The condition is sufficient. For, suppose that $\mathfrak{X}-E$ is not decomposable into n separated non void sets. Let F be an arbitrary closed subset of E. Owing to the local connectedness of $\mathfrak{X}$, the components of $\mathfrak{X}-F$ are open.

Let $C_1,\ldots,C_k$ be the system of all components of $\mathfrak{X}-F$ such that $C_i-E\neq 0$. Hence $k<n$. For otherwise, as $\mathfrak{X}-E\subset\mathfrak{X}-F$, we would have

$$\mathfrak{X}-E=(C_1-E)+\ldots+(C_{n-1}-E)+[\mathfrak{X}-F-(C_1+\ldots+C_{n-1})-E],$$

a decomposition of $\mathfrak{X}-E$ into n non void separated sets.

Put $H=\mathfrak{X}-(C_1+\ldots+C_k)$. Thus

$$F\subset H=\bar{H}\subset E\quad\text{and}\quad \mathfrak{X}-H=C_1+\ldots+C_k.$$

The sets $C_1,\ldots,C_k$ being connected, $\mathfrak{X}-H$ is not decomposable into n separated non void subsets.

Thus, theorem (iii) is established. Let us now prove theorem (ii).

Let E be a subset of a locally connected continuum $\mathfrak{X}$ and let n be the number of components of $\mathfrak{X}-E$ $(n<\infty)$.

Let f be a homeomorphism of E onto $E^*=f(E)\subset\mathfrak{X}$. Obviously it will suffice to show that the number of components of $\mathfrak{X}-E^*$ is $\geqslant n$.

By theorem (iii) there exists a closed set $F\subset E$ such that for each closed set H satisfying the condition $F\subset H\subset E$ the set $\mathfrak{X}-H$ contains $\geqslant n$ components.

Put $F^*=f(F)$. Owing to the compactness of the space $\mathfrak{X}$, the set F^* is a closed subset of E^*. Let K be a closed set such that $F^*\subset K\subset E^*$. Put $H=f^{-1}(K)$. As $F\subset H=\bar{H}\subset E$, the number of components of $\mathfrak{X}-H$ is $\geqslant n$, and so is the number of components of $\mathfrak{X}-K$ (for $\mathfrak{X}$ satisfies by assumption condition (i)).

It follows by (iii) that the number of components of $\mathfrak{X}-E^*$ is $\geqslant n$.

[6]) See my *Topologie I* (1948), p. 122.

Sur les types d'ordre des ensembles linéaires.

Par

Wacław Sierpiński (Warszawa).

On peut généraliser, d'après M. Fraïssé [1]), aux types d'ordre la relation $<$ connue pour les nombres ordinaux, en convenant qu'on a, pour deux types d'ordre α et β, l'inégalité $\alpha<\beta$ dans ce cas et seulement dans ce cas si A et B étant deux ensembles ordonnés respectivement de types α et β, l'ensemble A est semblable à un sous-ensemble de B, mais l'ensemble B n'est semblable à aucun sous-ensemble de A.

On démontre sans peine que les types d'ordre deviennent ainsi partiellement ordonnés par la relation $<$.

Nous dirons que deux types d'ordre α et β sont *incomparables* et nous écrirons $\alpha\|\beta$, si, A et B étant deux ensembles ordonnés respectivement de types α et β, aucun de ces ensembles n'est semblable à un sous-ensemble de l'autre.

Théorème 1. *E étant un ensemble linéaire de puissance $2^{\aleph_0}$, il existe toujours un ensemble linéaire de puissance $2^{\aleph_0}$ dont le type d'ordre est plus petit que celui de l'ensemble E.* (Nous supposons naturellement les ensembles linéaires ordonnés d'après la grandeur des abcisses des points qui les constituent) [2]).

La démonstration de notre théorème sera basée sur le

Lemme 1. *E étant un ensemble linéaire de puissance $2^{\aleph_0}$, il existe un ensemble linéaire $H\subset E$ de puissance $2^{\aleph_0}$ tel que f étant une fonction croissante quelconque définie dans E, on a $f(E)-H\neq 0$.*

[1]) Comptes Rendus Acad. Sc. Paris **226**, p. 1330 (séance du 26 avril 1948).

[2]) Les théorèmes de 1 à 7 et le théorème 9 ont été énoncés sans démonstration dans ma Note du 13 mai 1950 parue dans les Rendiconti dell'Accademia Nazionale dei Lincei, ser. VIII, vol. VIII, p. 427-428.