The number ϱ is arbitrary, subject to the limitation indicated.

Theorem 10.1. *Let B be a Banach space of type $\mathfrak{A}_4$ such that $T_r f$ converges weakly to f as $r \to 1$, for each $f \epsilon B$. Then every linear functional $\gamma \epsilon B^*$ is representable in the form*

$$\gamma(f) = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} f(\varrho e^{i\vartheta}) F\left(\frac{r}{\varrho} e^{-i\vartheta}\right) d\vartheta, \quad f \epsilon B, \tag{10.2}$$

where $r < \varrho < 1$ and $F \epsilon B^0$, The element F of B^0 uniquely determines and is uniquely determined by γ. Furthermore,

$$\|\gamma\| \leqslant \|F\|' \leqslant A_4(B)\|\gamma\|. \tag{10.3}$$

Under the stronger hypothesis that $\lim\limits_{r\to 1} \|T_r f - f\| = 0$ for each $f \epsilon B$ we have the same representation (10.2). *In this case, however, F may be any element of B', and $\|\gamma\| = \|F\|'$.*

The theorem is merely a restatement of Theorems 8.4 and 9.3.

There will be circumstances under which it is legitimate to make $\varrho \to 1$ under the integral sign in (10.2). Sometimes we may even take the limit with respect to r under the integral sign. The possibility of carrying out these processes depends upon the character of the functions f and F at the boundary of the unit circle.

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Independent fields and cartesian products

by

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For every $\tau \epsilon T (\overline{\overline{T}} \geqslant 2)$ let $\boldsymbol{X}_\tau$ be a σ-field [1]) of subsets of a fixed set $\mathcal{X}$. We shall say that the fields $\boldsymbol{X}_\tau$ $(\tau \epsilon T)$ are *σ-independent* [2]) if

$$\prod_n X_n \neq 0 \tag{*}$$

for every σ-sequence [3]) of non-empty sets $X_n \epsilon \boldsymbol{X}_{\tau_n}$, where $\tau_k \neq \tau_l$ for $k \neq l$.

Suppose μ_τ is a σ-measure [4]) on $\boldsymbol{X}_\tau$. We shall say that the σ-fields X_τ $(\tau \epsilon T)$ are *stochastically σ-independent* (with respect to the σ-measures μ_τ), if there is a σ-measure μ (called the *stochastic σ-extension* of all μ_τ) on the least σ-field $\boldsymbol{X}$ containing all the σ-fields $\boldsymbol{X}_\tau$ such that μ is a common extension of all μ_τ $(\tau \epsilon T)$ and

$$\mu\Big(\prod_n X_n\Big) = \prod_n \mu(X_n) \tag{**}$$

for every sequence [5]) of sets $X_n \epsilon \boldsymbol{X}_{\tau_n}$, where $\tau_k \neq \tau_l$ for $k \neq l$.

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[1]) A non-void class $\boldsymbol{P}$ of subsets of a set $\mathcal{P}$ is called a *field*, if $P, Q \epsilon \boldsymbol{P}$ implies $\mathcal{P} - P \epsilon \boldsymbol{P}$ and $P + Q \epsilon \boldsymbol{P}$. A field $\boldsymbol{P}$ is called a *σ-field*, if $P_n \epsilon \boldsymbol{P}$ $(n = 1, 2, 3, \ldots)$ implies $P_1 + P_2 + P_3 + \ldots \epsilon \boldsymbol{P}$.

[2]) The concept of the independence of fields has been introduced by Marczewski in paper [7].

[3]) In this paper we shall write, for convenience, *a sequence* instead of *a finite sequence*, and *a σ-sequence* instead of *a finite or enumerable sequence*.

[4]) A *σ-measure* μ on a σ-field $\boldsymbol{P}$ of subsets of a set $\mathcal{P}$ is a non-negative function such that $\mu(\mathcal{P}) = 1$ and $\mu(\sum\limits_n P_n) = \sum\limits_n \mu(P_n)$ for every σ-sequence of disjoint sets $P_n \epsilon \boldsymbol{P}$. By omitting the letter σ in the above definition we obtain the analogous definition of a *measure* on a field.

[5]) Consequently for every σ-sequence also.

By omitting the letter σ in the above definitions we obtain the analogous definitions of *independent fields* and of *stochastically independent fields* (with respect to measures μ_τ).

In this paper I shall prove several theorems which explain the structure of families of σ-independent σ-fields and of stochastically σ-independent σ-fields. Known examples of σ-independent σ-fields and of stochastically σ-independent σ-fields are some σ-fields defined in cartesian products (see lemmas 3(i), 3(ii), 6(i), and 6(iii)). I shall show that, roughly speaking, every family of σ-independent σ-fields or of stochastically σ-independent σ-fields is of this product type [6]) (Theorems III-IV and VI-VIII). Therefore the study of families of σ-independent σ-fields and of stochastically σ-independent σ-fields can be reduced to the study of some σ-fields in cartesian products. For instance, Banach's theorem [7]) on the extension of σ-measures defined on σ-independent σ-fields can be deduced from an analogous theorem on σ-measures in cartesian products (see 6(ii)).

The same results hold for independent fields and stochastically independent fields. Since all theorems (and their proofs [8])) on the finite independence are completely analogous to those on the σ-independence, I shall formulate only the theorems on σ-independence. In order to obtain analogous theorems (and their proofs) on the finite independence it is sufficient to omit everywhere the letter σ.

I. Independent fields.

§ 1. Definitions. A mapping h of a σ-field $\boldsymbol{P}$ (of subsets of a set $\mathscr{P}$) in another σ-field [9]) $\boldsymbol{Q}$ (of subsets of a set $\mathscr{Q}$) is called a *σ-homomorphism*, if P^0 denoting the complement of P, i. e. the set $\mathscr{P}-P$, we have

$$h(P^0)=h(P)^0 \quad \text{for each } P\epsilon\boldsymbol{P}$$

and

$$h\Big(\prod_n P_n\Big)=\prod_n h(P_n) \quad \text{for each } \sigma\text{-sequence of } P_n\epsilon\boldsymbol{P}.$$

[6]) A similar theorem on independent functions has been proved by van Kampen [11], p. 434.

[7]) Banach [2], Theorem 1.

[8]) The only exception is Theorem V, the proof of which is simpler in the case of the independence than in the case of the σ-independence.

[9]) More generally: of a σ-complete Boolean algebra $\boldsymbol{P}$ in another σ-complete Boolean algebra $\boldsymbol{Q}$.

A one-one mapping h of $\boldsymbol{P}$ on $\boldsymbol{Q}$ [10]) is said to be an *isomorphism* [11]) provided that $h(P_1)\subset h(P_2)$ if and only if $P_1\subset P_2$ for arbitrary sets $P_1, P_2\epsilon\boldsymbol{P}$. If it exists, $\boldsymbol{P}$ and $\boldsymbol{Q}$ are said to be *isomorphic*.

Every isomorphism is a one-one σ-homomorphism [12]) and conversely.

An isomorphism h of $\boldsymbol{P}$ on $\boldsymbol{Q}$ is called an *equivalence* [11]) if there exists a one-one mapping φ of $\mathscr{P}$ on $\mathscr{Q}$ such that $h(P)=\varphi(P)$ for every $P\epsilon\boldsymbol{P}$.

A σ-field $\boldsymbol{P}$ (of subsets of $\mathscr{P}$) is said to be *reduced* provided that for every pair $p_1, p_2\epsilon\mathscr{P}$ there is a set $P\epsilon\boldsymbol{P}$ such that $p_1\epsilon P$ and p_2 non ϵP.

$\boldsymbol{P}$ is said to be *σ-perfect* provided that every two-valued [13]) σ-measure μ on $\boldsymbol{P}$ is trivial, i. e. there is a point $p_0\epsilon\mathscr{P}$ such that $\mu(P)=1$ if and only if $p_0\epsilon P$.

If $S\subset\mathscr{P}$, the symbol $S\boldsymbol{P}$ will denote the class of all sets SP, where $P\epsilon\boldsymbol{P}$. $S\boldsymbol{P}$ is a σ-field of subsets of S.

A subset S of $\mathscr{P}$ is called *dense in* $\boldsymbol{P}$ if $SP\neq 0$ for every non-void set $P\epsilon\boldsymbol{P}$.

Obviously

(i) *S is dense in $\boldsymbol{P}$ if and only if the σ-homomorphism*

$$g(P)=SP \quad \textit{for} \quad P\epsilon\boldsymbol{P}$$

is an isomorphism of $\boldsymbol{P}$ on $S\boldsymbol{P}$.

In this paper $\{\boldsymbol{X}_\tau\}_{\tau\epsilon T}$ will denote a given family of σ-fields $\boldsymbol{X}_\tau$ (distinct or not) of subsets of a fixed sed $\mathscr{X}$. The symbol $\boldsymbol{X}$ will always denote the least σ-field containing all fields $\boldsymbol{X}_\tau (\tau\epsilon T)$.

The following lemmas are obvious:

(ii) *If the σ-fields $\boldsymbol{X}_\tau (\tau\epsilon T)$ are σ-independent and if h is an isomorphism of $\boldsymbol{X}$ on a σ-field $\boldsymbol{Z}$, the σ-fields* [14]) *$h(\boldsymbol{X}_\tau)$ $(\tau\epsilon T)$ are also σ-independent.*

(iii) *If the σ-fields $\boldsymbol{X}_\tau$ are σ-independent and if S is dense in $\boldsymbol{X}$, then the σ-fields $S\boldsymbol{X}_\tau (\tau\epsilon T)$ are also σ-independent.*

[10]) More generally: of a Boolean algebra $\boldsymbol{P}$ in another Boolean algebra $\boldsymbol{Q}$.

[11]) See Marczewski [6], p. 136.

[12]) A σ-homomorphism h is one-one if and only if $h(P)=0$ implies $P=0$.

[13]) A measure is *two-valued* if it assumes only the numbers 0 and 1.

[14]) $h(\boldsymbol{X}_\tau)$ is the class of all sets $h(X)$ where $X\epsilon\boldsymbol{X}_\tau$.

§ 2. The extension of homomorphisms and isomorphisms. Let $\boldsymbol{P}_0$ and $\boldsymbol{Q}_0$ be two classes of subsets of sets $\mathscr{P}$ and $\mathscr{Q}$ respectively, and let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be the least σ-fields (of subsets of the same sets containing $\boldsymbol{P}_0$ and $\boldsymbol{Q}_0$ respectively. Suppose f is a mapping of $\boldsymbol{P}_0$ on $\boldsymbol{Q}_0$.

I have proved in my paper [10] (Theorems V and II) that

(i) *The mapping f can be extended to a σ-homomorphism h of $\boldsymbol{P}$ on $\boldsymbol{Q}$, if and only if*

$$\prod_n P_n^{i_n}=0 \quad \textit{implies} \quad \prod_n f(P_n)^{i_n}=0$$

for every σ-sequence of $P_n \epsilon \boldsymbol{P}_0$ and for every σ-sequence $\{i_n\}$ of numbers 0 *and* 1[15]).

This theorem implies the following:

(ii)[16]) *In order that the mapping f can be extended to an isomorphism h of $\boldsymbol{P}$ on $\boldsymbol{Q}$ it is necessary and sufficient that for every σ-sequence of $P_n \epsilon \boldsymbol{P}_0$ and for every σ-sequence $\{i_n\}$ of numbers* 0 *and* 1

(a) $$\prod_n P_n^{i_n}=0 \ \textit{if and only if} \ \prod_n f(P_n)^{i_n}=0.$$

The necessity is obvious.

In order to prove the sufficiency suppose the condition (a) is satisfied. If $f(P_1)=f(P_2)$, then $f(P_1)\cdot f(P_2)^0=0$ and $f(P_1)^0\cdot f(P_2)=0$. Consequently, by (a), $P_1\cdot P_2^0=0$ and $P_1^0\cdot P_2=0$, that is, $P_1=P_2$. This proves that the mapping f is one-one. By (i) and (a) the mappings f and f^{-1} can be extended to σ-homomorphisms h (of $\boldsymbol{P}$ in $\boldsymbol{Q}$) and g (of $\boldsymbol{Q}$ in $\boldsymbol{P}$) respectively. By definition,

(b) $$gh(P)=P \quad \text{and} \quad hg(Q)=Q$$

for every $P \epsilon \boldsymbol{P}_0$ and for every $Q \epsilon \boldsymbol{Q}_0$. Hence we infer that the formulae (b) hold for every $P \epsilon \boldsymbol{P}$ and $Q \epsilon \boldsymbol{Q}$. This proves that $h=g^{-1}$, i. e. that the σ-homomorphism h is a one-one mapping of $\boldsymbol{P}$ on $\boldsymbol{Q}$. Thus h is an isomorphism of $\boldsymbol{P}$ on $\boldsymbol{Q}$, q. e. d.

[15]) For convenience we assume that $P^1=P$, and P^0 is the complement of P, as in § 1; analogously for Q.

[16]) For the case of finite additivity this theorem has been proved by Kuratowski and Posament [4], p. 282-283.

Theorem I[17]). *For every $\tau \epsilon T$ let h_τ be a σ-homomorphism of $\boldsymbol{X}_\tau$ in a σ-field $\boldsymbol{Q}$. If the σ-fields $\boldsymbol{X}_\tau$ are σ-independent, then the σ-homomorphisms h_τ can be extended to a σ-homomorphism h of $\boldsymbol{X}$ in $\boldsymbol{Q}$.*

This follows immediately from (i) and (*).

Theorem II. *Suppose the fields $\boldsymbol{X}_\tau$ $(\tau \epsilon T)$ are σ-independent, and, for every $\tau \epsilon T$, let h_τ be an isomorphism of $\boldsymbol{X}_\tau$ on a σ-field $\boldsymbol{Z}_\tau$ of subsets of a set $\mathscr{Z}$. If the σ-fields $\boldsymbol{Z}_\tau$ $(\tau \epsilon T)$ are also σ-independent, then the isomorphisms h_τ can be extended to an isomorphism h of $\boldsymbol{X}$ on the least σ-field $\boldsymbol{Z}$ containing all the fields $\boldsymbol{Z}_\tau$.*

If $\boldsymbol{Z}$ is σ-perfect and $\boldsymbol{X}$ is reduced, there exists a set $S \subset \mathscr{Z}$ dense in $\boldsymbol{Z}$ such that the mapping

$$g(X)=S\cdot h(X) \quad \textit{for} \quad X \epsilon \boldsymbol{X}$$

is an equivalence of $\boldsymbol{X}$ on $S\boldsymbol{Z}$.

If both $\boldsymbol{X}$ and $\boldsymbol{Z}$ are σ-perfect and reduced, then $S=\mathscr{Z}$, and $h=g$ is an equivalence of $\boldsymbol{X}$ on $S\boldsymbol{Z}$.

The first part follows immediately from (ii) and (*).

Suppose $\boldsymbol{Z}$ is σ-perfect and $\boldsymbol{X}$ is reduced. Then the converse isomorphism h^{-1} is induced[18]) by a point mapping φ of $\mathscr{X}$ in $\mathscr{Z}$, i. e.

$$h^{-1}(Z)=\varphi^{-1}(Z) \quad \text{for every} \quad Z \epsilon \boldsymbol{Z}.$$

Let $S=\varphi(\mathscr{X})$. Since $\boldsymbol{X}$ is reduced, φ is one-one. We have

$$X=h^{-1}h(X)=\varphi^{-1}(h(X))=\varphi^{-1}(g(X)) \quad \text{for every} \quad X \epsilon \boldsymbol{X};$$

hence $g(X)=\varphi(X)$ for every $X \epsilon \boldsymbol{X}$, that is, g is an equivalence of $\boldsymbol{X}$ on $S\boldsymbol{Z}$. Since the mapping $gh^{-1}(Z)=SZ$ is an isomorphism of $\boldsymbol{Z}$ on $S\boldsymbol{Z}$, the set S is dense in $\boldsymbol{Z}$ on account of § 1 (i).

Now suppose $\boldsymbol{X}$ and $\boldsymbol{Z}$ are σ-perfect and reduced. Then, by symmetry, there is also another mapping ψ of $\mathscr{Z}$ in $\mathscr{X}$ such that

$$h(X)=\psi^{-1}(X) \quad \text{for every} \quad X \epsilon \boldsymbol{X}.$$

Consequently $\varphi^{-1}\psi^{-1}(X)=X$ and $\varphi^{-1}\psi^{-1}(Z)=Z$ for every $X \epsilon \boldsymbol{X}$ and for every $Z \epsilon \boldsymbol{Z}$. Since $\boldsymbol{X}$ and $\boldsymbol{Z}$ are reduced, we infer that $\varphi=\psi^{-1}$. Thus $S=\varphi(\mathscr{X})=\psi^{-1}(\mathscr{X})=\mathscr{Z}$, q. e. d.

[17]) See Sikorski [10], Theorems VII and III.

[18]) See Sikorski [9], p. 7, p. 12 (theorem 2.1), and p. 10 (theorem 1.2).

§ 3. Independent fields in cartesian products. Besides the family $\{\boldsymbol{X}_\tau\}_{\tau\in T}$ we shall also consider another family $\{\boldsymbol{Y}_\tau\}_{\tau\in T}$ of σ-fields. Every $\boldsymbol{Y}_\tau$ is a σ-field of subsets of a set $\mathcal{Y}_\tau$ (the sets $\mathcal{Y}_\tau$ may be distinct or not).

$\mathcal{Y}$ will always denote the cartesian product of all sets $\mathcal{Y}_\tau$ $(\tau\in T)$.

If $Y\subset\mathcal{Y}_\tau$, the symbol $\mathfrak{c}_\tau(Y)$ will denote the set of all points of $\mathcal{Y}$ whose τ-th co-ordinate belongs to Y.

The symbol $\boldsymbol{Y}_\tau^*$ will denote always the class of all sets $\mathfrak{c}_\tau(Y)$, where $Y\in\boldsymbol{Y}_\tau$. $\boldsymbol{Y}_\tau^*$ is a σ-field of subsets of $\mathcal{Y}$, and the mapping $\mathfrak{c}_\tau$ is an isomorphism of $\boldsymbol{Y}_\tau$ on $\boldsymbol{Y}_\tau^*$.

The least σ-field (of subsets of $\mathcal{Y}$) containing all the fields $\boldsymbol{Y}_\tau^*$ $(\tau\in T)$ will be denoted by $\boldsymbol{Y}^*$.

(i) *The σ-fields $\boldsymbol{Y}_\tau^*$ $(\tau\in T)$ are σ-independent.*

More generally:

(ii) *If a set $S\subset\mathcal{Y}$ is dense in $\boldsymbol{Y}^*$, then the σ-fields $S\boldsymbol{Y}_\tau^*$ are σ-independent.*

(i) is obvious. (ii) follows from § 1 (iii).

Theorem III. *Suppose the σ-fields $\boldsymbol{X}_\tau$ are σ-independent and, for every $\tau\in T$, $\boldsymbol{X}_\tau$ is isomorphic to $\boldsymbol{Y}_\tau$, thus to $\boldsymbol{Y}_\tau^*$ also. Let h_τ be an isomorphism of $\boldsymbol{X}_\tau$ on $\boldsymbol{Y}_\tau^*$. Then the isomorphisms h_τ can be extended to an isomorphism h of $\boldsymbol{X}$ on $\boldsymbol{Y}^*$.*

If $\boldsymbol{X}$ is reduced and $\boldsymbol{Y}^*$ is σ-perfect[19]*, then there is a set $S\subset\mathcal{Y}$ dense in $\boldsymbol{Y}^*$ such that the mapping*

$$g(X)=S\cdot h(X)\quad \textit{for}\quad X\in\boldsymbol{X}$$

is an equivalence of $\boldsymbol{X}$ on $S\boldsymbol{Y}$[20]*.*

If both $\boldsymbol{X}$ and $\boldsymbol{Y}^*$ are reduced and σ-perfect, then $S=\mathcal{Y}$, and $h=g$ is an equivalence of $\boldsymbol{X}$ on $\boldsymbol{Y}^*$.

Theorem III follows directly from (i) and Theorem II. It shows that every family of σ-independent σ-fields may be considered as such a family of the type mentioned in (i) or (ii).

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[19]) These conditions are little restrictive (see Lemmas (iii)-(v)).

[20]) If $\boldsymbol{X}_\tau$ $(\tau\in T)$ are not σ-independent, then there is also an equivalence of $\boldsymbol{X}$ on a σ-fields $S\boldsymbol{Y}^*$, but the set S is not dense in $\boldsymbol{Y}^*$.

We note that

(iii) *$\boldsymbol{Y}^*$ is reduced if and only if every σ-field $\boldsymbol{Y}_\tau$ is reduced.*

(iv) *$\boldsymbol{Y}^*$ is σ-perfect if and only if every σ-field $\boldsymbol{Y}_\tau$ is σ-perfect.*

(v) *Every σ-field is isomorphic to a σ-perfect reduced σ-field.*

The easy proofs are omitted.

In the first part of Theorem III we may assume e. g. $\mathcal{Y}_\tau=\mathcal{X}$ and $\boldsymbol{Y}_\tau=\boldsymbol{X}_\tau$. In this case Theorem III can be expressed more precisely as follows:

Theorem IV. *Suppose $\mathcal{Y}_\tau=\mathcal{X}$ and $\boldsymbol{Y}_\tau=\boldsymbol{X}_\tau$ for every $\tau\in T$, and let S be the diagonal of $\mathcal{Y}$. Then there is an equivalence g of $\boldsymbol{X}$ on $S\boldsymbol{Y}^*$. The σ-fields $\boldsymbol{X}_\tau$ $(\tau\in T)$ are σ-independent if and only if S is dense in $\boldsymbol{Y}^*$.*

The required equivalence is induced by the mapping φ which transforms a point $x\in\mathcal{X}$ in the point of $\mathcal{Y}$, all co-ordinates of which are equal to x.

If S is dense in $\boldsymbol{Y}^*$, then the σ-fields $S\boldsymbol{Y}_\tau^*$ $(\tau\in T)$ are σ-independent on account of (ii). The σ-fields $\boldsymbol{X}_\tau$ $(\tau\in T)$ are also independent on account of § 1 (ii).

Conversely, if the σ-fields $\boldsymbol{X}_\tau$ are σ-independent, then, on account of Theorem III, there is an isomorphism h of $\boldsymbol{Y}^*$ on $\boldsymbol{X}$ such that $X=h(\mathfrak{c}_\tau(X))$ for every $X\in\boldsymbol{X}_\tau$. Hence

$$gh(\mathfrak{c}_\tau(X))=g(X)=S\,\mathfrak{c}_\tau(X)\quad\text{for every } X\in\boldsymbol{X}_\tau,$$

and consequently

$$gh(Y)=SY\quad\text{for every}\quad Y\in\boldsymbol{Y}^*.$$

The mapping gh being an isomorphism we infer from 1(i) that S is dense in $\boldsymbol{Y}^*$.

II. Stochastic independence.

§ 4. Definitions. If μ is a σ-measure[4]) on a σ-field $\boldsymbol{P}$ of subsets of a set $\mathcal{P}$, the symbol $\boldsymbol{I}(\mu)$ will denote the ideal of all sets P such that $\mu(P)=0$. For $P\in\boldsymbol{P}$ the symbol $[P]_\mu$ will denote the element A of the Boolean algebra $\boldsymbol{P}/\boldsymbol{I}(\mu)$ such that $P\in A$. Analogously, if $\boldsymbol{K}\subset\boldsymbol{P}$, the symbol $|\boldsymbol{K}]_\mu$ will denote the class of all elements $[P]_\mu$ where $P\in\boldsymbol{K}$. The symbol μ° will denote the σ-measure on $\boldsymbol{P}/\boldsymbol{I}(\mu)$ defined by the formula

$$\mu^\circ([P]_\mu)=\mu(P).$$

For any set $Q\subset\mathscr{P}$, $\mu_e(Q)$ will denote the lower bound of numbers $\mu(P)$, where $Q\subset P\epsilon\boldsymbol{P}$. Let $S\subset\mathscr{P}$ and $\mu_e(S)=1$. The function $\mu_e(Q)$ restricted to $Q\epsilon S\boldsymbol{P}$ will be denoted by μ^S. It is a σ-measure on $S\boldsymbol{P}$.

Let ν be another σ-measure defined on a σ-field $\boldsymbol{Q}$. The measures μ and ν are said to be *isomorphic (equivalent)* provided there is an isomorphism (equivalence) h of $\boldsymbol{P}$ on $\boldsymbol{Q}$ such that $\mu(P)=\nu(h(P))$ for every $P\epsilon\boldsymbol{P}$.

The measures μ and ν are said to be *almost isomorphic* provided there is an isomorphism h of $\boldsymbol{P}/\boldsymbol{I}(\mu)$ on $\boldsymbol{Q}/\boldsymbol{I}(\nu)$ such that $\mu^\circ(A)=\nu^\circ(h(A))$ for every $A\epsilon\boldsymbol{P}/\boldsymbol{I}(\mu)$.

The symbols $\mathscr{X}$, $\boldsymbol{X}_\tau$, $\boldsymbol{X}$, $\mathscr{Y}_\tau$, $\mathscr{Y}$, $\boldsymbol{Y}_\tau$, $\boldsymbol{Y}_\tau^*$, $\boldsymbol{Y}^*$ will have the same meaning as before. μ_τ and ν_τ will always denote σ-measures on $\boldsymbol{X}_\tau$ or $\boldsymbol{Y}_\tau$ respectively. The symbol ν_τ^* will denote the σ-measure

$$\nu_\tau^*(\mathfrak{c}_\tau(Y))=\nu_\tau(Y) \quad \text{for} \quad Y\epsilon\boldsymbol{Y}_\tau$$

induced on $\boldsymbol{Y}_\tau^*$ by ν_τ.

μ and ν will denote the stochastic σ-extensions of all μ_τ and ν_τ^* respectively whenever they exist.

We suppose always $\overline{\overline{T}}\geqslant 2$.

The following lemma is obvious:

(i) *If the σ-fields $\boldsymbol{X}_\tau$ are stochastically σ-independent with respect to the σ-measures μ_τ, and if $\mu_e(S)=1$, then the σ-fields $S\boldsymbol{X}_\tau$ $(\tau\epsilon T)$ are stochastically σ-independent with respect to μ_τ^S.*

§ 5. The extension of isomorphisms. We shall now prove the following

Theorem V. *For every $\tau\epsilon T$ let λ_τ be a σ-measure on a σ-field $\boldsymbol{Z}_\tau$ of subsets of a set $\mathscr{Z}$. Suppose the σ-fields $\boldsymbol{X}_\tau$ and $\boldsymbol{Z}_\tau$ $(\tau\epsilon T)$ are stochastically σ-independent with respect to the σ-measures μ_τ and λ_τ respectively. If μ_τ is almost isomorphic to λ_τ for every $\tau\epsilon T$, then the stochastic σ-extensions μ and λ (of all μ_τ and λ_τ respectively) are also almost isomorphic.*

The Boolean algebras $\boldsymbol{X}_\tau/\boldsymbol{I}(\mu_\tau)$ and $\boldsymbol{Z}_\tau/\boldsymbol{I}(\lambda_\tau)$ being isomorphic to the Boolean algebras $[\boldsymbol{X}_\tau]_\mu$ and $[\boldsymbol{Z}_\tau]_\lambda$ respectively, there is an isomorphism h_τ of $[\boldsymbol{X}_\tau]_\mu$ on $[\boldsymbol{Z}_\tau]_\lambda$ such that

$$\mu^\circ(A)=\lambda^\circ(h_\tau(A)) \quad \text{for} \quad A\epsilon[\boldsymbol{X}_\tau]_\mu.$$

If $0\neq A_n\epsilon[\boldsymbol{X}_{\tau_n}]_\mu$ and $0\neq B_n\epsilon[\boldsymbol{Z}_{\tau_n}]_\lambda$ $(0<n\leqslant m<\infty)$, and $\tau_k\neq\tau_l$ for $k\neq l$, then

$$\mu^\circ\Big(\prod_{n=1}^m A_n\Big)=\prod_{n=1}^m\mu^\circ(A_n)\neq 0 \quad \text{and} \quad \lambda^\circ\Big(\prod_{n=1}^m B_n\Big)=\prod_{n=1}^m\lambda^\circ(B_n)\neq 0;$$

hence

$$\prod_{n=1}^m A_n\neq 0 \quad \text{and} \quad \prod_{n=1}^m B_n\neq 0.$$

Consequently the isomorphisms h_τ can be extended[21] to an isomorphism h_0 of $[\boldsymbol{X}_0]_\mu$ on $[\boldsymbol{Z}_0]_\lambda$, where $\boldsymbol{X}_0$ and $\boldsymbol{Z}_0$ denote the least fields containing all σ-fields $\boldsymbol{X}_\tau$ and $\boldsymbol{Z}_\tau$ $(\tau\epsilon T)$ respectively. Obviously

$$\text{(c)} \qquad \mu^\circ(A)=\lambda^\circ(h_0(A)) \quad \text{for} \quad A\epsilon[\boldsymbol{X}_0]_\mu.$$

Consider the Boolean algebras $\boldsymbol{A}=\boldsymbol{X}/\boldsymbol{I}(\mu)$ and $\boldsymbol{B}=\boldsymbol{Z}/\boldsymbol{I}(\lambda)$[22] as metric spaces with Nikodym's metrics[23]

$$\delta(A_1,A_2)=\mu^\circ(A_1-A_2)+\mu^\circ(A_2-A_1) \qquad \text{for } A_1,A_2\epsilon\boldsymbol{A},$$

$$\vartheta(B_1,B_2)=\lambda^\circ(B_1-B_2)+\lambda^\circ(B_2-B_1) \qquad \text{for } B_1,B_2\epsilon\boldsymbol{B}$$

respectively.

The set $[\boldsymbol{X}_0]_\mu$ is dense in the space $\boldsymbol{A}$. In fact, the construction of the common extension μ is the following: first extend the σ-measures μ_τ to a measure μ_0 on $\boldsymbol{X}_0$ on account of $(\ast)$; further apply Carathéodory's exterior measure method. Therefore for any $X\epsilon\boldsymbol{X}$ and $\varepsilon>0$ there is a set $X_0\epsilon\boldsymbol{X}_0$ such that

$$\delta([X]_\mu,[X_0]_\mu)<\varepsilon.$$

By symmetry, the set $[\boldsymbol{Z}_0]_\lambda$ is dense in $\boldsymbol{B}$. By (c), h_0 is an isometric mapping of $[\boldsymbol{X}_0]_\mu$ on $[\boldsymbol{Z}_0]_\lambda$. The spaces $\boldsymbol{A}$ and $\boldsymbol{B}$ being complete[23], the mapping h_0 can be extended to an isometric mapping h of $\boldsymbol{A}$ on $\boldsymbol{B}$. Since the addition and the complementation are continuous operations in $\boldsymbol{A}$ and $\boldsymbol{B}$, the continuous extension h is also additive and complementative, i. e. h is an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{B}$.

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[21]) This follows from Theorem III in my paper [10]. The exact proof is similar to that of the first part of § 2 (ii).

[22]) $\boldsymbol{Z}$ denotes here the least σ-field containing all the σ-fields $\boldsymbol{Z}_\tau$ $(\tau\epsilon T)$.

[23]) Nikodym [8], p. 137-139. See also Hahn and Rosenthal [3], p. 31-32.

We have

$$\mu^\circ(A)=\delta(A,0)=\vartheta(h(A),0)=\lambda^\circ(h(A)) \quad \text{for} \quad A\in \boldsymbol{A}.$$

This proves that μ and λ are almost isomorphic.

§ 6. Stochastically independent fields in cartesian products. It is well known[24]) that

(i) *The σ-fields $\boldsymbol{Y}_\tau^*$ in the cartesian product $\mathscr{Y}$ are stochastically σ-independent with respect to arbitrary σ-measures ν_τ^*.*

Theorems (i) and III imply the following theorem of Banach[25]):

(ii) *If the σ-fields $\boldsymbol{X}_\tau$ are σ-independent, they are also stochastically σ-independent with respect to arbitrary σ-measures μ_τ.*

Theorem (i) and lemma 4(i) imply that

(iii) *If $S\subset\mathscr{Y}$ and $\nu_e^*(S)=1$, then the σ-fields $S\boldsymbol{Y}_\tau^*$ $(\tau\in T)$ are stochastically σ-independent with respect to ν_τ^{*S}.*

The following two theorems show that, conversely, every family of stochastically σ-independent σ-fields may be considered as such a family of the type mentioned in (i) or (iii).

Theorem VI. *If the σ-fields $\boldsymbol{X}_\tau$ are stochastically σ-independent with respect to μ_τ $(\tau\in T)$, and if for every $\tau\in T$ the σ-measure μ_τ is almost isomorphic to ν_τ, then the stochastic σ-extensions μ and ν^* are also almost isomorphic.*

This follows immediately from (i) and Theorem V.

Theorem VII. *Suppose the σ-fields $\boldsymbol{X}_\tau$ are stochastically σ-independent with respect to μ_τ $(\tau\in T)$, and for every $\tau\in T$ the σ-measure μ_τ is isomorphic to ν_τ. If $\boldsymbol{X}$ is reduced and $\boldsymbol{Y}^*$ is σ-perfect[19]), then there is a set $S\subset\mathscr{Y}$ such that $\nu_e^*(S)=1$ and μ is equivalent to ν^{*S}.*

Let h_τ be an isomorphism of $\boldsymbol{Y}_\tau^*$ on $\boldsymbol{X}_\tau$ such that

$$\text{(d)} \qquad \nu_\tau(Y)=\nu_\tau^*(\mathfrak{c}_\tau(Y))=\mu_\tau(h_\tau(\mathfrak{c}_\tau(Y))) \quad \text{for} \quad Y\in \boldsymbol{Y}_\tau.$$

[24]) Łomnicki and Ulam [5], p. 245 and 252; Andersen and Jessen [1], p. 22.

[25]) Banach [2], p. 160, Theorem 1. Analogously, (i) and Theorem III, formulated for the finite independence, imply immediately Marczewski's Theorem II of his paper [7], p. 126.

By Theorem I and 3(i) the isomorphisms h_τ can be extended to a σ-homomorphism h of $\boldsymbol{Y}^*$ on $\boldsymbol{X}$. The σ-field $\boldsymbol{Y}^*$ being σ-perfect, there is a mapping φ which induces[18]) h, i. e.

$$h(Y)=\varphi^{-1}(Y) \quad \text{for every} \quad Y\in \boldsymbol{Y}.$$

Let $S=\varphi(\mathscr{X})$. Since $\boldsymbol{X}$ is reduced, φ is one-one. By (d)

$$\mu(\varphi^{-1}(Y))=\nu^*(Y) \quad \text{for} \quad Y\in \boldsymbol{Y}^*.$$

Consequently $\nu_e^*(S)=1$ and the mapping

$$g(X)=\varphi(X) \quad \text{for} \quad X\in \boldsymbol{X}$$

is an equivalence between μ and ν^{*S}, q. e. d.

In Theorem VI we may assume e. g. $\mathscr{Y}_\tau=\mathscr{X}$, $\boldsymbol{Y}_\tau=\boldsymbol{X}_\tau$, and $\nu_\tau=\mu_\tau$. In this case Theorem VI can be formulated in the following more precise form:

Theorem VIII. *Suppose $\mathscr{Y}_\tau=\mathscr{X}$, $\boldsymbol{Y}_\tau=\boldsymbol{X}_\tau$, $\nu_\tau=\mu_\tau$, and let S be the diagonal of $\mathscr{Y}$. The fields $\boldsymbol{X}_\tau$ are stochastically σ-independent if and only if $\nu^*(S)=1$. If this equality is true, then μ is equivalent to ν_e^{*S}.*

By Theorem I and 3(i) there is a σ-homomorphism h of $\boldsymbol{Y}^*$ on $\boldsymbol{X}$ such that

$$\text{(e)} \qquad h(\mathfrak{c}_\tau(X))=X \quad \text{for} \quad X\in \boldsymbol{X}_\tau=\boldsymbol{Y}_\tau.$$

By Theorem IV there is an equivalence g of $\boldsymbol{X}$ on $S\boldsymbol{Y}^*$ such that

$$g(X)=S\cdot\mathfrak{c}_\tau(X) \quad \text{for} \quad X\in \boldsymbol{X}_\tau.$$

Consequently

$$\text{(f)} \qquad gh(Y)=SY \quad \text{for every} \quad Y\in \boldsymbol{Y}^*.$$

Suppose $\boldsymbol{X}_\tau$ are stochastically σ-independent. Then (e) implies

$$\text{(g)} \qquad \nu^*(Y)=\mu(h(Y)) \quad \text{for every} \quad Y\in \boldsymbol{Y}^*.$$

If $Y\in \boldsymbol{Y}^*$ and $S\cdot Y=0$, then, by (f), $gh(Y)=0$; hence $h(Y)=0$. By (g), $\nu^*(Y)=0$. This proves that $\nu_e^*(S)=1$.

Now suppose $\nu_e^*(S)=1$. By 6(iii) there is a stochastic σ-extension ν^{*S} of all ν_τ^{*S} $(\tau\in T)$ on $S\boldsymbol{Y}^*$.

We have

$$\mu_\tau(X)=\nu_\tau(X)=\nu^*_\tau(\mathfrak{c}_\tau(X))=\nu^*(\mathfrak{c}_\tau(X))=\nu^{*S}(S\cdot\mathfrak{c}_\tau(X))=\nu^{*S}(g(X))=\nu^{*S}_\tau(g(X))$$

for every $X\epsilon\boldsymbol{X}_\tau$. Thus the formula

$$\mu(X)=\nu^{*S}(g(X)) \quad \text{for} \quad X\epsilon\boldsymbol{X}$$

defines a stochastic σ-extension of all μ_τ $(\tau\epsilon T)$ on $\boldsymbol{X}$, that is, the σ-fields $\boldsymbol{X}_\tau$ are stochastic σ-independent with respect to μ_τ.

Appendix.

Another proof of Theorem I.

(A) *For every $\tau\epsilon T$ let h_τ be a σ-homomorphism of $\boldsymbol{Y}^*_\tau$ in a σ-field $\boldsymbol{Q}$ of subsets of $\mathscr{Q}$. If the σ-fields $\boldsymbol{Y}_\tau$ are σ-perfect, then the σ-homomorphisms h can be extended to a σ-homomorphism h of $\boldsymbol{Y}^*$ in $\boldsymbol{Q}$.*

Every σ-field $\boldsymbol{Y}_\tau$ being σ-perfect, every σ-homomorphism

$$g_\tau(Y)=h_\tau(\mathfrak{c}_\tau(Y)) \qquad \text{for } Y\epsilon\boldsymbol{Y}_\tau$$

is induced [18]) by a point mapping φ_τ of $\mathscr{Q}$ in $\mathscr{Y}_\tau$, i. e.

$$g_\tau(Y)=\varphi_\tau^{-1}(Y) \qquad \text{for } Y\epsilon\boldsymbol{Y}_\tau.$$

The mapping $\varphi(q)=\{\varphi_\tau(q)\}_{\tau\epsilon T}$ of $\mathscr{Q}$ in $\mathscr{Y}$ has the property

$$\varphi^{-1}(\mathfrak{c}_\tau(Y))=\varphi_\tau^{-1}(Y)=g_\tau(Y)=h_\tau(\mathfrak{c}_\tau(Y)) \qquad \text{for } Y\epsilon\boldsymbol{Y}_\tau.$$

Consequently, the σ-homomorphism h of $\boldsymbol{Y}^*$ in $\boldsymbol{Q}$, defined by the equality $h(Y)=\varphi^{-1}(Y)$ for $Y\epsilon\boldsymbol{Y}^*$, is a common extension of all h_τ.

(B) *Suppose $\boldsymbol{Y}_\tau$ is a σ-perfect σ-field isomorphic to $\boldsymbol{X}_\tau$ $(\tau\epsilon T)$. Let h_τ be an isomorphism of $\boldsymbol{Y}^*_\tau$ on $\boldsymbol{X}_\tau$. If the σ-fields $\boldsymbol{X}_\tau$ are σ-independent, then the isomorphisms h_τ can be extended to an isomorphism h of $\boldsymbol{Y}^*$ on $\boldsymbol{X}$.*

By (A) there is a σ-homomorphism h of $\boldsymbol{Y}^*$ on $\boldsymbol{X}$ which is a common extension of all h_τ. It is sufficient to prove that $h(Y)\neq 0$ for $0\neq Y\epsilon\boldsymbol{Y}^*$.

Let $\boldsymbol{K}$ denote the class of all cartesian products $Y_0=\underset{\tau\epsilon T}{P}Y_\tau$, where $0\neq Y_\tau\epsilon\boldsymbol{Y}_\tau$, and the inequality $Y_\tau\neq\mathscr{Y}_\tau$ holds for an at most enumerable set of elements τ only.

We have

$$h(Y_0)=\prod_{\tau\epsilon T}h_\tau(Y_\tau)\neq 0$$

since $0\neq h_\tau(Y_\tau)\epsilon\boldsymbol{X}_\tau$, and the inequality $h_\tau(Y_\tau)\neq\mathscr{X}$ holds for an at most enumerable set of elements τ.

Let $\boldsymbol{Y}^*_0$ be the class of all sets $Y\epsilon\boldsymbol{Y}^*$ with the property: if $y\epsilon Y$, then there is a set $Y_0\epsilon\boldsymbol{K}$ such that $y\epsilon Y_0\subset Y$.

It is easy to see that:

(a) the least field containing all fields $\boldsymbol{Y}^*_\tau$ is contained in $\boldsymbol{Y}^*_0$;

(b) if $Y_n\epsilon\boldsymbol{Y}^*_0$ $(n=1,2,\ldots)$, then $Y_1+Y_2+\ldots\epsilon\boldsymbol{Y}^*_0$ and $Y_1\cdot Y_2\cdot\ldots\epsilon\boldsymbol{Y}^*_0$.

Consequently, $\boldsymbol{Y}^*_0=\boldsymbol{Y}^*$. Therefore every set $0\neq Y\epsilon\boldsymbol{Y}^*$ contains a subset $Y_0\epsilon\boldsymbol{K}$ and $0\neq h(Y_0)\subset h(Y)$, q. e. d.

Theorem I follows immediately from (A), (B) and from every σ-field being isomorphic to a σ-perfect σ-field [26]).

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On differentiation of vector-valued functions

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In recent years a series of papers appeared, dealing with the problem of differentiation of vector-valued functions. The most interesting problem was perhaps to inquire under what hypotheses the weak differentiability implies the strong one. The most complete results in this direction obtained Pettis [7].

In the present paper [1]) further remarks on this subject will be added, generalizing [2]) some results of my paper [1] and of the paper of Pettis.

In § 1 preliminary definitions are given, and the main result of this paper is formulated. In § 2 and § 3 the lemmas are grouped, upon which the principal theorems contained in § 4 are based. Finally, in § 5 some applications to Analysis are given.

§ 1. Preliminary considerations. X denotes a Banach space, $\|x\|$ — the norm of the element x of X, Ξ — the space conjugate to X, and $\xi(x)$ — the elements of Ξ.

By *functions* I mean in this paper the vector-valued functions, i. e. functions from an arbitrary fixed interval J or from a set E of reals to the space X; for these functions the symbols $x(t)$, $y(t)$ and $z(t)$ are reserved. Real-valued functions will be denoted by $f(t)$.

The limit of $\varphi(t)$ as t tends to t_0 by values of the set P will be denoted by $\lim_{P,\, t\to t_0} \varphi(t)$.

[1]) whose results were in part presented September 22th, 1948, to the VI Polish Mathematical Congress in Warsaw.

[2]) The author is indebted to Professor W. Orlicz for having called his attention to the possibility of such a generalization.