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On differentiation of vector-valued functions

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In recent years a series of papers appeared, dealing with the problem of differentiation of vector-valued functions. The most interesting problem was perhaps to inquire under what hypotheses the weak differentiability implies the strong one. The most complete results in this direction obtained Pettis [7].

In the present paper [1]) further remarks on this subject will be added, generalizing [2]) some results of my paper [1] and of the paper of Pettis.

In § 1 preliminary definitions are given, and the main result of this paper is formulated. In § 2 and § 3 the lemmas are grouped, upon which the principal theorems contained in § 4 are based. Finally, in § 5 some applications to Analysis are given.

§ 1. Preliminary considerations. X denotes a Banach space, $\|x\|$ — the norm of the element x of X, Ξ — the space conjugate to X, and $\xi(x)$ — the elements of Ξ.

By *functions* I mean in this paper the vector-valued functions, i. e. functions from an arbitrary fixed interval J or from a set E of reals to the space X; for these functions the symbols $x(t)$, $y(t)$ and $z(t)$ are reserved. Real-valued functions will be denoted by $f(t)$.

The limit of $\varphi(t)$ as t tends to t_0 by values of the set P will be denoted by $\lim_{P,\, t\to t_0}\varphi(t)$.

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[1]) whose results were in part presented September 22th, 1948, to the VI Polish Mathematical Congress in Warsaw.

[2]) The author is indebted to Professor W. Orlicz for having called his attention to the possibility of such a generalization.

The symbol $|E|$ will denote, as usually, the Lebesgue measure of the set E, and $|E|_e$ will denote the outer measure of the same set.

A subset Ξ_0 of the set Ξ will be called *fundamental for* X if, given any $\varepsilon>0$ and $x\in X$, there exist elements $\xi_1,\xi_2,\ldots,\xi_n\in\Xi_0$ and real numbers $a_1,a_2,\ldots,a_n$ such that

$$\|\xi\|=1, \quad\text{and}\quad |\xi(x)|\geqslant\|x\|-\varepsilon \quad\text{for}\quad \xi=a_1\xi_1+a_2\xi_2+\ldots+a_n\xi_n.$$

In § 1, § 2, § 3 and § 4 Ξ_0 will stand for an arbitrary but fixed set fundamental for X; in § 5 this set will be specialized to concrete cases.

We will deal with the following notions of differentiability of vector-valued functions:

A function $x(t)$ will be said to be *strongly differentiable* at t_0 to x_0, if the expression

$$\left\|\frac{x(t_0+h)-x(t_0)}{h}-x_0\right\| \tag{1}$$

tends to 0 when $h\to 0$; the element x_0 will be called the *strong derivative* of $x(t)$ at t_0, and denoted by $x'(t_0)$.

A function $x(t)$ will be said to be Ξ_0*-weakly differentiable* at t_0 to x_0, if for every $\xi\in\Xi_0$ the expression

$$\xi\left(\frac{x(t_0+h)-x(t_0)}{h}\right) \tag{2}$$

tends to $\xi(x_0)$ when $h\to 0$; the element x_0 will be termed the Ξ_0*-weak derivative* of $x(t)$ at t_0, and denoted by $x'_{\mathrm{w}}(t_0)$.

The function $x(t)$ will be said to be *approximately strongly differentiable* at t_0 to x_0, if the expression (1) tends approximately to 0 when $h\to 0$; in this case the element x_0 will be termed the *strong approximate derivative* of $x(t)$ at t_0, and written $x'_{\mathrm{ap}}(t_0)$.

It is obvious that the elements $x'(t_0)$, $x'_{\mathrm{w}}(t_0)$ and $x'_{\mathrm{ap}}(t_0)$ are uniquely determined, if existing. If the function $x(t)$ is differentiable at any point of a set E to the element $y(t)$ according to any one of the above definitions, I shall say that $x(t)$ is *differentiable* in the respective sense *in* E to $y(t)$.

The definition of the *differentiability a.e.* (almost everywhere) *in* E is obvious.

I shall also consider another notion of differentiability, which is not so closely related to the behaviour of the considered functions at particular points.

The function $x(t)$ will be said to be Ξ_0*-pseudodifferentiable* to $y(t)$ in the set E, if for every $\xi\in\Xi_0$ there exists a set H_ξ, depending on ξ, such that

(i) $|E-H_\xi|=0$,

(ii) $\dfrac{d}{dt}\xi(x(t))=\xi(y(t))$ at any point of H_ξ.

In this case the function $y(t)$ will be termed the Ξ_0*-weak pseudoderivative* of $x(t)$ in E, and denoted by $x'_{\mathrm{p}}(t)$.

If, given any $\xi\in\Xi_0$, there exists for the function $x(t)$ a set H_ξ satisfying (i) and such that $\xi(y(t))$ is the approximate derivative of $\xi(x(t))$ at any point of H_ξ, the function $x(t)$ will be said to be Ξ_0*-approximately pseudodifferentiable* to $y(t)$ in E; the function $y(t)$ will be termed the Ξ_0*-approximate pseudoderivative* of $x(t)$ in E, and written $x'_{\mathrm{pap}}(t)$.

The above definitions are due essentially to Pettis [7].

A function $x(t)$ will be said to be *essentially separably valued*, or briefly *e.s.v.*, in E, if there exists a set H such that $|H|=0$, the set $\underset{y}{E}(y=x(t),\, t\in E-H)$ [3]) being separable.

The main result of this paper is included in the following.

Theorem 1. *Let the function $x(t)$ be Ξ_0-weakly differentiable in a set E, and let $x'_{\mathrm{w}}(t)$ be e.s.v. in E. If*

$$\varlimsup_{h\to 0}\left\|\frac{x(t+h)-x(t)}{h}\right\|<\infty$$

at any point of E, then $x(t)$ is strongly differentiable a.e. *in E to $x'_{\mathrm{w}}(t)$.*

Corollary 1. *Under the hypotheses of Theorem 1 there exists a set $H\subset E$ such that $|E-H|=0$ and*

$$\frac{d}{dt}\xi(x(t))=\xi(x'_{\mathrm{w}}(t))$$

for every $t\in H$ and every functional $\xi(x)$, linear on X.

[3]) i. e. the set of the y's satisfying the conditions in ( ).

The hypothesis of $x'_w(t)$ being e.s.v. cannot be removed in Theorem 1. Gelfand ([4], p. 265) has given an example of a function sAC, (see § 2) everywhere Ξ_0-weakly differentiable (Ξ_0 being a fundamental set), and nowhere strongly differentiable, but the Ξ_0-weak derivative of this function is not e.s.v., as may be easily verified.

§ 2. Lemmas. A function $x(t)$ will be said to be *sAC* (*strongly absolutely continuous*) on E, if to every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that, given any finite sequence $\{(a_i, b_i)\}$ of non-overlapping intervals the endpoints of which belong to the set E,

$$\sum_{i=1}^{n} |a_i - b_i| < \delta \quad \text{implies} \quad \sum_{i=1}^{n} \|x(a_i) - x(b_i)\| < \varepsilon.$$

Lemma 1. *If for a function $x(t)$ the inequality*

$$\overline{\lim_{h\to+0}} \operatorname{ap} \left\| \frac{x(t+h) - x(t)}{h} \right\| < \infty$$

holds at any point of a set E, then E can be decomposed in a sequence of sets on each of which the function $x(t)$ is sAC.

The well-known proof of this Lemma in case of X being the set of real numbers [4]) can be easily applied to the case of X being a Banach space.

Let E be a closed set with the bounds α and β, and let $\{(a_n, b_n)\}$ be the sequence of open intervals contiguous to E, contained in the closed interval $[\alpha, \beta]$. Then, $x(t)$ being any function defined in a set $H \supset E$, I shall denote by $\hat{x}(t)$ or, if necessary, by $\hat{x}(t; E)$ the function coinciding with the function $x(t)$ on the set E and linear on the intervals (a_n, b_n), i. e. the function defined by the formula

$$\hat{x}(t) = \begin{cases} x(t) & \text{for} \quad t \in E, \\ x(a_n) + \dfrac{x(b_n) - x(a_n)}{b_n - a_n}(t - a_n) & \text{for} \quad t \in (a_n, b_n). \end{cases}$$

Lemma 2. *If the function $x(t)$ is sAC on a closed set E, so is the function $\hat{x}(t; E)$.*

The easy proof is left to the reader.

[4]) See, for instance, Saks [8], p. 239.

The function $x(t)$ is said to *fulfil the condition* (l) at t_0, if there exists a constant M such that for any $t \in J$

$$(3) \qquad \|x(t) - x(t_0)\| \leqslant M |t - t_0|.$$

Lemma 3. *If the function $x(t)$ fulfils at any point of the set E the condition (l) and is strongly approximately differentiable in E, then $x(t)$ is strongly differentiable a.e. in E.*

Proof. Denote by T_n the set of the elements t_0 at which (3) holds with $M \leqslant n$. Each of these sets is closed, and $E \subset \sum_{n=1}^{\infty} T_n$. Denote by D_n the set of the points of density of the set T_n. By Density Theorem it is sufficient to prove that $x'(t)$ exists at any point of the set ED_n. Let $t_0 \in ED_n$, and write $x_0 = x'_{ap}(t_0)$. There exists a set P for which t_0 is a point of outer density such that

$$\lim_{t\to t_0}{}_P \left\| \frac{x(t) - x(t_0)}{t - t_0} - x_0 \right\| = 0.$$

Hence

$$\lim_{t\to t_0}{}_{PD_n} \left\| \frac{x(t) - x(t_0)}{t - t_0} - x_0 \right\| = 0,$$

and we easily see that t_0 is a point of outer density of the set PD_n. Let $t_i \to t_0$, and, to fix ideas, suppose that $t_i < t_0$. Then

$$(4) \qquad \frac{|(t_i, t_0) PD_n|_e}{t_0 - t_i} = \delta_i \to 1.$$

In the interval (t_i, t_0) there must exist at least one point τ_i such that $\tau_i \in PD_n$ and

$$\frac{\tau_i - t_i}{t_0 - t_i} \leqslant \varepsilon_i = 1 - \delta_i\left(1 - \frac{1}{i}\right),$$

for in the contrary case the interval $(t_i, t_i + \varepsilon_i(t_0 - t_i))$ would be contained in the set $(t_i, t_0) - PD_n$, and hence

$$\begin{aligned} |(t_i, t_0) PD_n|_e &\leqslant (t_0 - t_i) - \varepsilon_i (t_0 - t_i) \\ &\leqslant (t_0 - t_i)(1 - \varepsilon_i) = (t_0 - t_i)\,\delta_i \left(1 - \frac{1}{i}\right), \end{aligned}$$

contrarily to (4). Since

$$\frac{x(t_i) - x(t_0)}{t_i - t_0} = \frac{x(t_i) - x(\tau_i)}{t_i - t_0} + \frac{x(\tau_i) - x(t_0)}{t_i - t_0}, \qquad \left\| \frac{x(\tau_i) - x(t_0)}{\tau_i - t_0} - x_0 \right\| \to 0,$$

and

$$\left\|\frac{x(t_i)-x(\tau_i)}{t_i-t_0}\right\| = \left\|\frac{x(t_i)-x(\tau_i)}{t_i-\tau_i}\right\| \cdot \left|\frac{t_i-\tau_i}{t_i-t_0}\right| \leqslant n\varepsilon_i,$$

we get

$$\frac{x(\tau_i)-x(t_0)}{t_i-t_0} = \frac{x(\tau_i)-x(t_0)}{\tau_i-t_0}\,\frac{\tau_i-t_0}{t_i-t_0} = \left(1-\frac{t_i-\tau_i}{t_i-t_0}\right)\frac{x(\tau_i)-x(t_0)}{\tau_i-t_0} \to x_0$$

as $i\to\infty$. Hence

$$\left\|\frac{x(t_i)-x(t_0)}{t_i-t_0} - x_0\right\| \to 0.$$

Lemma 4. *Let $x(t)$ be sAC on a closed interval J, let $x(t)$ be Ξ_0-approximately pseudodifferentiable in J, and let $x'_{\mathrm{pap}}(t)$ be e.s.v. in J. Then $x(t)$ is strongly differentiable to $x'_{\mathrm{pap}}(t)$ a.e. in J.*

Proof. It is obvious that $x(t)$ being continuous on J is e.s.v. in J. Since $y(t)=x'_{\mathrm{pap}}(t)$ is also e.s.v. in J, we may suppose that the space X is separable. By a theorem of Banach ([3], p. 124) there exists a sequence $\{\xi_n\}$ of elements of Ξ_0, weakly dense in Ξ_0. Denoting by $\{\xi_n^*\}$ the sequence of linear combinations with rational coefficients of the ξ_n's of norm less than 1, we have

$$\sup_n |\xi_n^*(x)| = \|x\| \quad \text{for every } x\in X.$$

Thus the set Ξ_1 composed of the ξ_n^*'s is fundamental for X, and $x(t)$ is Ξ_1-approximately pseudodifferentiable to $y(t)$ in J. The real-valued functions $\xi_n^*(x(t))$ are absolutely continuous on J. Hence $x(t)$ is Ξ_1-pseudodifferentiable to $y(t)$ in J. Thus Lemma 4 results from a theorem of Pettis ([7], theorem 2.6).

§ 3. Measurability. A function constant on each of a finite number of measurable sets is termed *simple*.

Each function which is the limit of an a.e. convergent sequence of simple functions is said to be *measurable* (Bochner, [3]).

Any function coinciding in the set E with a measurable function is said to be *measurable in E*.

The function $x(t)$ is said to be *Ξ_0-weakly measurable*, if for each $\xi\in\Xi_0$ the real-valued function $\xi(x(t))$ is measurable.

If, given any $\xi\in\Xi_0$, the real-valued function $\xi(x(t))$ coincides in the set E (measurable or not) with a measurable function $\varphi_\xi(t)$, the function $x(t)$ is qualified *Ξ_0-weakly measurable in E*.

Lemma 6. *If the function $x(t)$ is Ξ_0-weakly measurable in a set E and e.s.v. in E, then $x(t)$ is measurable in E.*

The proof runs quite similarly as in a paper by Pettis ([6], proof of theorem 1.1).

§ 4. Principal theorems. The following theorem is analogous to Theorem 1:

Theorem 2. *Let the function $x(t)$ be measurable in E and Ξ_0-approximately pseudodifferentiable to $y(t)$ in E. If the function $y(t)$ is e.s.v. in E, and*

$$\overline{\lim_{h\to 0}}\,\mathrm{ap}\left\|\frac{x(t+h)-x(t)}{h}\right\| < \infty \tag{5}$$

at any point of E, then the function $x(t)$ is approximately strongly differentiable to $y(t)$ a.e. in E.

Proof. We can suppose without loss of generality that the space X is separable. By Lemma 6 there exists a sequence $\{y_n(t)\}$ of simple functions converging to $y(t)$ a.e. in E. The set E_1 of the points at which $y^*(t)=\lim\limits_{n\to\infty} y_n(t)$ exists and (5) holds is, as may be easily seen, measurable, and $|E-E_1|=0$. By a theorem of Banach ([2], p. 124) there exists a sequence $\{\xi_n\}$ of elements of Ξ_0, weakly dense in Ξ_0. The set Ξ_1 of the linear combinations with rational coefficients of the ξ_n's is fundamental for X and we easily observe that $x(t)$ is Ξ_1-approximately pseudodifferentiable to $y^*(t)$ a.e. in E_1.

Let E_2 be the set of the points at which

$$\lim_{h\to 0}\mathrm{ap}\,\xi\left(\frac{x(t+h)-x(t)}{h} - y^*(t)\right) = 0 \quad \text{for any } \xi\in\Xi_1.$$

The set E_2 is measurable, and $|E-E_2|=0$. We shall prove that $x'_{\mathrm{ap}}(t)=y^*(t)$ exists a.e. in $E_3=E_1E_2$.

By Lemma 1 the set E_3 can be represented as a sum $\sum\limits_{n=1}^{\infty} H_n$ of sets on each of which $x(t)$ is sAC. Since the sets K_n of all points of outer density of the set H_n are measurable, the sets E_3K_n are measurable; moreover $|H_n-K_n|=0$. Hence

$$\left|E_3 - \sum_{n=1}^{\infty} E_3K_n\right| = 0.$$

The function $x(t)$ being measurable, it can be proved similarly as for real-valued functions that for almost every $t \epsilon E_3$ there exists a set E_t for which t is a point of density, and

$$\lim_{\tau \to t} {}_{E_t}\, x(\tau) = x(t).$$

Thus, any point of K_n being a point of outer density for H_n, and $x(t)$ being sAC on H_n, we can easily prove that there exists a set R_n of measure 0 such that $x(t)$ is sAC on $L_n = E_3 K_n - R_n$.

It is sufficient to prove that $x(t)$ is approximately strongly differentiable to $y^*(t)$ at almost every point of L_n. Let $\varepsilon > 0$ be arbitrary. F being any closed set such that $F \subset L_n$ and $|L_n - F| < \varepsilon$, put $z(t) = \hat{x}(t; F)$. The function $z(t)$ is evidently e.s.v. in J and is strongly differentiable to $c_i = \text{const.}$ in any interval J_i contiguous to F. Since for every $\xi \epsilon \Xi_1$ the real-valued function $\xi(z(t))$ is sAC on J, the derivative $\frac{d}{dt}\xi(z(t))$ exists a.e. in J; moreover, $\frac{d}{dt}\xi(z(t)) = \xi(y^*(t))$ a.e. in F. Thus $x(t)$ is Ξ_1-approximately pseudodifferentiable to $y^*(t)$ in F. By Lemma 4 $z(t)$ is strongly differentiable a.e. in F; it follows that at almost any point $t \epsilon F$

$$z'_{\mathrm{ap}}(t) = x'_{\mathrm{ap}}(t) = y^*(t) = y(t).$$

The number $\varepsilon > 0$ being arbitrary, the above relation holds a.e. in L_n.

Theorem 1 is an immediate consequence of the following

Theorem 3. *Let the function $x(t)$ be Ξ_0-approximately pseudodifferentiable in E to a function $y(t)$ e.s.v. in E, and let*

$$\text{(6)} \qquad \varlimsup_{h \to 0} \left\| \frac{x(t+h) - x(t)}{h} \right\| < \infty$$

at any point t of E. Then $x(t)$ is strongly differentiable to $y(t)$ a.e. in E.

Proof. It is sufficient to prove that, given any point $t_0 \epsilon E$, there exists an interval $I = (\alpha, \beta)$ including the point t_0, in which $x'(t)$ exists a.e. in E. By (6) there exist for any $t \epsilon E$ two numbers $M(t)$ and $\delta(t)$ such that

$$|\tau - t| < \delta(t) \quad \text{implies} \quad \|x(\tau) - x(t)\| \leqslant M(t)|\tau - t|.$$

Write $I = (t_0 - \delta(t_0), t_0 + \delta(t_0))$; the function $x(t)$ is then bounded on I, i. e. $\|x(t)\| \leqslant A$.

Let t' be any point of the set IE. We easily observe that

$$\tau \epsilon I \quad \text{implies} \quad \|x(\tau) - x(t')\| \leqslant \left[M(t') + \frac{2A}{\delta(t)} \right] |\tau - t'|.$$

Thus the function $x(t)$ fulfils the condition (l) on IE. Hence $x(t)$ is continuous in IE. We can suppose that $x(t)$ is measurable in IE. Applying Theorem 2, we see that $x'_{\mathrm{ap}}(t)$ exists a.e. in IE, and by Lemma 3 also $x'(t)$ exists a.e. in IE.

Any function Ξ-weakly differentiable will be now simply said to be *weakly differentiable*.

Theorem 4. *Let $x(t)$ be weakly differentiable in a set E to $y(t)$. Then $x(t)$ is strongly differentiable a.e. in E to $y(t)$* [5].

Proof. Since, given any $t \epsilon E$,

$$\xi \epsilon \Xi \quad \text{implies} \quad \lim_{h \to 0} \xi\left(\frac{x(t+h) - x(t)}{h} \right) = \xi(y(t)),$$

we see, applying a theorem of Banach ([2], p. 80), that the condition (6) is satisfied at any point of E. The function $y(t)$ being e.s.v. by a theorem of Pettis ([6], theorem 1.2), we can apply Theorem 3 to get the conclusion.

A function $x(t)$ is said to be *Lipschitzian*, if

$$\|x(t_1) - x(t_2)\| \leqslant M|t_1 - t_2|$$

with M non depending on t_1 and on t_2.

A Banach space X will be said to have the *property* (D), if every Lipschitzian function from J to X is strongly differentiable a.e. in J [6].

Examples of spaces with the property (D) are furnished by the uniformly convex, the reflexive, and the locally weakly compact spaces (Pettis, [7], p. 262).

[5]) This theorem is a slight generalization of theorem 2.9 of Pettis ([7], p. 262).
[6]) This condition has been introduced by Pettis ([5], p. 427).

Theorem 5. *Let the space X have the property* (D). *If for a function $x(t)$ the inequality*

(7) $$\varlimsup_{h\to 0}\left\|\frac{x(t+h)-x(t)}{h}\right\|<\infty$$

is satisfied at any point of a set E, then the strong derivative $x'(t)$ exists a.e. in E [7]).

Proof. It can be easily shown that the set of the points at which (7) holds is measurable. Hence we can suppose that the set E is so. By Lemma 1 there exists a sequence $\{E_n\}$ of sets on each of which $x(t)$ is sAC, and such that $E=\sum_{n=1}^{\infty}E_n$. Since $x(t)$ is continuous at any point of E, the function $x(t)$ is sAC on $E\overline{E}_n$. It follows that the sets E_n may be supposed to be measurable.

Let n be fixed. Given an arbitrary $\varepsilon>0$, denote by F a closed set for which $F\subset E_n$ and $|E_n-F|<\varepsilon$. Write $z(t)=\hat{x}(t,F)$. This function is sAC by Lemma 2. Hence $z'(t)=x'_{\text{ap}}(t)$ exists a.e. in F by a theorem of Pettis ([5], theorem 7).

§ 5. Applications. Consider first as the space X the space $\boldsymbol{c}$ composed of the convergent sequences $x=\{a_n\}$ with the norm $\|x\|=\sup_{n=1,2,\dots}|a_n|$. This space is separable. Any convergent sequence of real-valued functions $\{f_n(t)\}$ defined in J may be considered as a function $x(t)$ from J to $\boldsymbol{c}$.

The functions $f_n(t)$ are said to be *equidifferentiable* at t_0, if the derivatives $f'_n(t_0)$ exist, and if for every $\varepsilon>0$ there exists an $\eta>0$ such that

$$|h|<\eta \quad\text{implies}\quad |f_n(t_0+h)-f_n(t_0)-hf'_n(t_0)|<|h|\,\varepsilon \quad \text{for } n=1,2,\dots$$

It is easy to see that the strong differentiability of $x(t)$ at t_0 is then equivalent to the convergence of the sequence $\{f'_n(t_0)\}$ together with its equidifferentiability at t_0.

Consider the set Ξ_0 composed of the functionals

(8) $$\xi_1(x)=a_1,\quad \xi_2(x)=a_2,\quad \dots$$

[7]) This result may be considered as a generalization of Denjoy's relations to vector-valued functions.

The set Ξ_0 is fundamental for the space $\boldsymbol{c}$. The Ξ_0-weak differentiability at t_0 is equivalent to the convergence of the sequence $\{f'_n(t_0)\}$. Since the functional $\xi(x)=\lim_{n\to\infty}a_n$ is linear in $\boldsymbol{c}$, we get, applying Theorem 1 and Corollary 1, the following

Theorem 7. *Let $\{f_n(t)\}$ be a convergent sequence of real-valued functions, and let the derivatives $f'_n(t)$ exist in a set E. If the sequence $\{f'_n(t)\}$ converges in E, and*

$$\varlimsup_{h\to 0}\sup_{n=1,2,\dots}\left|\frac{f_n(t+h)-f_n(t)}{h}\right|<\infty \text{ [8])}$$

at every point of E, then the functions $f_n(t)$ are equidifferentiable a.e. in E; moreover,

$$\frac{d}{dt}\left(\lim_{n\to\infty}f_n(t)\right)=\lim_{n\to\infty}f'_n(t)\quad \textit{a.e. in } E.$$

In a similar manner we can apply Theorem 1 and Corollary 1 to the space $\boldsymbol{l}^2$ of the sequences $x=\{a_n\}$ such that $\|x\|^2=\sum_{n=1}^{\infty}a_n^2<\infty$, considering as Ξ_0 the set of the functionals (8).

We easily get

Theorem 8. *If the real-valued functions $f_n(t)$ are differentiable at every point of the set E, and satisfy the conditions*

$$\sum_{n=1}^{\infty}f_n^2(t)<\infty,\qquad \sum_{n=1}^{\infty}f_n'^2(t)<\infty$$

and

$$\varlimsup_{h\to 0}\frac{1}{h^2}\sum_{n=1}^{\infty}[f_n(t+h)-f_n(t)]^2<\infty \quad \textit{at every } t \textit{ of } E,$$

then there exists a set H such that $|E-H|=0$, and

$$\lim_{h\to 0}\sum_{n=1}^{\infty}\left[\frac{f_n(t+h)-f_n(t)}{h}-f'_n(t)\right]^2=0 \quad \textit{in } H;$$

moreover, $\sum_{n=1}^{\infty}a_n^2<\infty$ *and* $t\in H$ *imply* $\frac{d}{dt}\left(\sum_{n=1}^{\infty}a_nf_n(t)\right)=\sum_{n=1}^{\infty}a_nf'_n(t)$.

[8]) This condition may be replaced by the following one:

$$h_n\to 0 \quad\text{implies}\quad \varlimsup_{n\to\infty}\left|\frac{f_n(x+h_n)-f_n(x)}{h_n}\right|<\infty.$$

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Remarque au travail „Sur les bases statistiques“

par

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Les termes et les notations employés dans la suite sont les mêmes que dans mon travail précédent [1]). Parmi les résultats de ce travail se trouve une estimation de la valeur de $L(F,\eta)$ pour un système F de deux fonctions continues périodiques $f_1(x)$ et $f_2(x)$ à périodes incommensurables, ce symbole désignant un nombre positif tel que tout intervalle de longueur $L(F,\eta)$ contient au moins une η-presque-période commune de $f_1(x)$ et $f_2(x)$. L'estimation en question fait l'objet du théorème II.

Le but de cette remarque est d'en donner une démonstration plus simple et qui permet même d'en améliorer la thèse [2]). En conséquence, la thèse du théorème III, qui donne une estimation de $L(F,\eta)$ pour un cas spécial et dont la démonstration est basée sur le théorème II, est susceptible d'une amélioration analogue.

Montrons d'abord un lemme concernant la répartition mod 1 de la suite $\{n\vartheta\}$, où ϑ est un nombre irrationnel fixé.

Lemme. *Soit I un sous-intervalle de longueur β de l'intervalle demi-ouvert $\langle 0,1)$. Soient q un nombre naturel et p un entier, tels que $|q\vartheta-p|<\beta$. Soit enfin $\{Q_i\}$ la suite croissante de tous les entiers non-négatifs tels que $R(\vartheta Q_i)\in I$. Alors*

$$|Q_{i+1}-Q_i|\leqslant \mathrm{E}\left(\frac{1}{|q\vartheta-p|}+1\right)q \qquad (i=1,2,\ldots). \tag{1}$$

Démonstration. La distance entre les points $R(kq\vartheta)$ et $R((k+1)q\vartheta)$ $(k=0,1,2,\ldots)$, prise le long du plus petit arc de la circonférence C de périmètre 1, est égale à $\min[R(q\vartheta),\ 1-R(q\vartheta)]$;

[1]) Voir Studia Mathematica 10 (1948), p. 120-139.

[2]) L'idée de cette simplification m'a été suggérée par K. Florek.