On sequences of operations (II)

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In this part [1]) we deal with the sequences of linear operations in spaces which are more general than the Banach spaces.

Terminology and notation.

$X, Y, \ldots$ will denote linear spaces.

If a space X is a limit space of Fréchet, i. e. a notion α of *limit* (called also notion of *convergence*) is defined in it, satisfying the usual postulates of Fréchet, we shall denote the convergence of a sequence $\{x_n\}$ to x_0, according to the notion of limit α, by writing $x_n \overset{\alpha}{\to} x_0$ or $(\alpha)\lim\limits_n x_n = x_0$; then the sequence $\{x_n\}$ will be called *α-convergent* to x_0.

The linear space X provided with the notion α of limit will be denoted by X_α.

It may happen that in the same space X several notions of limit $\alpha, \beta, \ldots$ will be distinguished.

The convergences α and β are called *equivalent* (in symbols: $\alpha = \beta$) if $x_n \overset{\alpha}{\to} x_0$ implies $x_n \overset{\beta}{\to} x_0$ and conversely. The convergence α will be called *non-wider* than the convergence β if $x_n \overset{\alpha}{\to} x_0$ implies $x_n \overset{\beta}{\to} x_0$; if the convergence α is non-wider than the convergence β, and if $\alpha \neq \beta$, then α will be called *narrower* than β.

We say that the space X_α has the property P (e. g. satisfies a postulate) instead of saying that the convergence α in X has this property.

[1]) For the first part see this volume, p. 1-30.

The sequence $\{x_n\}$ will be termed *α-bounded* if, given any sequence $\{\vartheta_n\}$ of real numbers, $\vartheta_n \to 0$ implies $\vartheta_n x_n \overset{\alpha}{\to} 0$.

A sequence which is not α-convergent or not α-bounded will be said to be *α-divergent* or *α-unbounded* respectively.

A set D will be called *dense in X_α* if for every element $x \in X$ there exists a sequence $\{x_n\}$ of elements of D which is α-convergent to x.

In the sequel we suppose that all the notions of limit have the property that both the addition and the multiplication by real numbers are continuous in both variables. All the spaces with such a notion of limit will be called *Λ-spaces*.

An operation $U(x)$ from a Λ-space X_α to a Λ-space Y_β will be called *(X_α, Y_β)-continuous* [2]) *at* x_0 if $x_n \overset{\alpha}{\to} x_0$ implies $U(x_n) \overset{\beta}{\to} U(x_0)$. If this continuity holds at any point x of the space, $U(x)$ will be simply said to be *(X_α, Y_β)-continuous*.

Any operation $U(x)$ satisfying the equation

$$U(ax + by) = aU(x) + bU(y)$$

for real a and b will be called *additive*.

An additive and (X_α, Y_β)-continuous operation will be called *(X_α, Y_β)-linear*. If Y_β is the space of real numbers with the usual notion of limit, any (X_α, Y_β)-linear operation will be termed a *(X_α)-linear functional*.

In the case when the *α-convergence* is strong (i. e. equivalent with the convergence according to the norm) in a F^*-space X, we denote the space X_α by X, and omit the symbol α (except in section 2.2).

Contents.

This part is concerned with the problem under what conditions the following statements hold:

I^1. *Let $U(x)$ be the limit of a sequence $\{U_n(x)\}$ of (X_α, Y_β)-linear operations β-convergent everywhere. Then $U(x)$ is (X_α, Y_β)-linear.*

II^1. *Let $\{U_n(x)\}$ be a sequence of (X_α, Y_β)-linear operations β-bounded for any x, and β-convergent in a set D dense in X_α. Then this sequence is β-convergent everywhere.*

III$^1_1$. *Let $\{U_{pq}(x)\}_{q=1,2,\ldots}$ be a sequence of (X_α, Y_β)-linear operations, and suppose that, given any p, there exists an element x_p*

[2]) This notation has been introduced by Orlicz ([14], p. 61).

such that the sequence $\{U_{pq}(x_p)\}_{q=1,2,\dots}$ *is* β*-divergent. Then there exists an element* x_0 *such that the sequences* $\{U_{pq}(x_0)\}_{q=1,2,\dots}$ *are* β*-divergent for* $p=1,2,\dots$

III_2^l. *Let* $\{U_{pq}(x)\}_{q=1,2,\dots}$ *be a sequence of* (X_α, Y_β)*-linear operations, and suppose that, given any* p, *there exists an element* x_p *such that the sequence* $\{U_{pq}(x_p)\}_{q=1,2,\dots}$ *is* β*-unbounded for* $p=1,2,\dots$ *Then there exists an element* x_0 *such that the sequences* $\{U_{pq}(x_0)\}_{q=1,2,\dots}$ *are* β*-unbounded for* $p=1,2,\dots$

In order to point out what spaces are referred to in I^l, II^l, III_1^l and III_2^l we shall sometimes denote these statements also by $\mathrm{I}^l(X_\alpha, Y_\beta)$, $\mathrm{II}^l(X_\alpha, Y_\beta)$, $\mathrm{III}_1^l(X_\alpha, Y_\beta)$, and $\mathrm{III}_2^l(X_\alpha, Y_\beta)$ respectively.

This problem was investigated in 1933 by Mazur and Orlicz [12] in connection with the spaces conjugate to B_0-spaces. Kantorovitch [9] has proved the truth of $\mathrm{I}^l(X_\alpha, Y_\beta)$ and of $\mathrm{II}^l(X_\alpha, Y_\beta)$ in the case of X_α and Y_β being Kantorovitch spaces. Finally, Fichtenholz [7] has shown that I^l holds in a concrete Λ-space which is not a Banach space.

In this paper first two groups of postulates concerning the notion of limit will be analysed, and several examples of spaces satisfying some of them will be given. Then it will be shown that, contrarily to the spaces considered by Banach, in general Λ-spaces theorems I^l, II^l and III_1^l are independent of one another.

Finally, it will be shown what sets of previously considered postulates are sufficient for I^l, II^l, III_1^l, or III_2^l to be true. The theorems contain the results of the authors mentioned above.

1. Postulates. Let X_α be a Λ-space. We will need the following two groups of postulates (x_n and x_0 denoting the elements of X, λ_n and ϑ_n — real numbers, and $\{n_k\}, \{q_n\}, \{p_n\}, \{q_n\}$ and $\{r_p\}$ — sequences of indices, i. e. increasing sequences of positive integers):

(a_1) *If* $x_n \xrightarrow{\alpha} 0$, *then there exist sequences* $\lambda_k \to \infty$ *and* $\{n_k\}$ *such that* $\lambda_k x_{n_k} \xrightarrow{\alpha} 0$ [3]).

(a_2) *If* $x_n \xrightarrow{\alpha} 0$ *and* $\lambda_n \to 0$, *then there exists a subsequence* $\{x_{n_k}\}$ *such that the series* $\sum_{k=1}^{\infty} \lambda_{n_k} x_{n_k}$ *is* α*-convergent.*

(a_2') *If* $x_n \xrightarrow{\alpha} 0$, *then there exists a subsequence* $\{x_{n_k}\}$ *such that the series* $\sum_{k=1}^{\infty} x_{n_k}$ *is* α*-convergent* [4]).

(a_3) $\{x_n\}$ *being an arbitrary sequence there exists a sequence* $\{\vartheta_n\}$ *of numbers, all different from* 0, *such that* $\sum_{n=1}^{\infty} |\lambda_n| < \infty$ *implies the* α*-convergence of the series* $\sum_{n=1}^{\infty} \lambda_n \vartheta_n x_n$.

(b_1) *If the sequence* $\{x_n\}$ *is not* α*-convergent to* 0, *then there exists a subsequence* $\{x_{n_k}\}$ *such that every subsequence of* $\{x_{n_k}\}$ *is not* α*-convergent to* 0 [5]).

(b_2) *If* $(\alpha)\lim_q x_{pq} = x_p$ *for* $p=1,2,\dots$ *and, given any sequence* $\{q_p\}$ *of indices the sequence* $\{x_{pq_p}\}$ *is* α*-bounded, then the sequence* $\{x_p\}$ *is* α*-bounded.*

(b_2') *If* $(\alpha)\lim_q x_{pq} = x_p$ *for* $p=1,2,\dots$ *and, given any sequence* $\{q_p\}$ *of indices,* $x_{pq_p} \xrightarrow{\alpha} 0$, *then* $x_p \xrightarrow{\alpha} 0$.

(b_3) *If* $(\alpha)\lim_q x_{pq} = 0$ *for* $p=1,2,\dots$ *and, given any sequence* $\{q_p\}$ *of indices, the sequence* $\{x_{pq_p}\}$ *is* α*-bounded, then there exists a sequence* $\{r_p\}$ *of indices such that* $x_{pr_p} \xrightarrow{\alpha} 0$.

(b_4) *If* $(\alpha)\lim_q x_{pq} = 0$ *for* $p=1,2,\dots$ *and* $\lambda_q \to 0$, *then there exists a sequence of indices* $\{q_n\}$ *such that*

$$(\alpha)\lim_p \sum_{i=p}^{\omega_p} \varepsilon_i \lambda_{q_i} x_{pq_i} = 0,$$

where $\{\varepsilon_i\}$ *is an arbitrary sequence of zeros and ones, and* $\omega_p \geqslant p$.

(b_5) *If, given any pair* $\{p_n\}$ *and* $\{q_n\}$ *of sequences of indices,* $(x_{p_n} - x_{q_n}) \xrightarrow{\alpha} 0$, *then the sequence* $\{x_n\}$ *is* α*-convergent.*

It is obvious that (a_2') implies (a_2), and (b_2') implies (b_2). The pair of postulates (a_1) and (a_2) implies (a_2').

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[3]) This postulate constitutes a slight modification of a postulate of Fichtenholz ([6], p. 196).

[4]) Introduced by Mazur and Orlicz [12].

[5]) Introduced by Fichtenholz ([6], p. 195).

Given a convergence α, it is customary to introduce a new convergence α^*, called the *star-convergence*, as follows: $(\alpha^*)\lim\limits_n x_n = x_0$ means that any subsequence $\{x_{n_k}\}$ contains a subsequence which is α-convergent to x_0.

The star-convergence satisfies the postulate (b_1); moreover, if the convergence α does not satisfy (b_1), and if the convergence α_1 satisfies (b_1) and is wider than α, then α^* is non-wider than α_1, i. e. α^* is the most narrow convergence satisfying (b_1) and wider than α.

2. Examples. In this section several examples of Λ-spaces satisfying some of the postulates considered in section 1 will be given; the proof that these spaces are Λ-spaces is omitted, and some obvious properties of these spaces will be also left without proof.

If the space X is composed of real-valued functions defined in a set E, the addition and the multiplication by real numbers may be understood in the usual manner.

2.1. Normed spaces. Suppose that in the linear space X a functional $\|x\|$ is defined satisfying the following postulates:

1^0 $\|x\| \geqslant 0$,
2^0 $\|x\| = 0$, *if and only if* $x = 0$,
3^0 $\|x+y\| \leqslant \|x\| + \|y\|$,
4^0 $\lambda_n \to 0$ *implies* $\|\lambda_n x\| \to 0$,
5^0 $\|x_n\| \to 0$ *implies* $\|\lambda x_n\| \to 0$.

The space X will be called *F^*-space*, and the functional $\|x\|$ — *F^*-norm*.

If the postulates 4^0 and 5^0 are replaced by the stronger one

6^0 $\|\lambda x\| = |\lambda|\,\|x\|$,

the space X will be called *B^*-space*, and the norm $\|x\|$ — *B^*-norm*.

In any F^*- and B^*-space the strong convergence (called later on simply *convergence* or *convergence generated by the norm*) is defined in the usual way: $\{x_n\}$ converges to x_0 if $\|x_n - x_0\| \to 0$. If this convergence satisfies the postulate (b_5), the space X is said to be a F- or B-space (called also *Banach space*) respectively.

It is obvious that the F^*-spaces satisfy the postulates (a_1), (b_1), (b_2), (b_2'), (b_3), (b_4), and that the F-spaces satisfy all the postulates (a_1)-(b_5).

2.2. The two-norms convergence [6]**.** Let in a linear space X two F^*-norms $\|x\|$ and $\|x\|^*$ be defined, satisfying the following condition:

(n_1) $\|x_n\| \to 0$ implies $\|x_n\|^* \to 0$.

Denote by ν and ν^* respectively the convergences generated by the norms $\|x\|$ and $\|x\|^*$. A sequence $\{x_n\}$ is said to be *γ-convergent* if it is ν-bounded and $\|x_n - x_0\|^* \to 0$.

The convergence γ will be termed the *two-norms convergence*.

2.2.1. *The γ-convergence satisfies the postulates* (a_2), (a_3), (b_1), (b_3) *and* (b_4).

Proof. We only prove that the condition (b_4) is satisfied. Suppose $(\gamma)\lim\limits_q x_{pq} = 0$ for $p = 1, 2, \ldots$ and $\lambda_q \to 0$; hence

$$\lim_{q\to\infty} \|\lambda_q x_{pq}\| = 0 \qquad \text{for } p = 1, 2, \ldots$$

We easily construct by the diagonal method a sequence $\{q_n\}$ of indices such that

$$\|\lambda_{q_n} x_{pq_n}\| < 1/2^n \quad \text{for } p = 1, 2, \ldots \text{ and } n = p, p+1, \ldots;$$

ε_i being zeros or ones, it follows

$$\Big\|\sum_{i=p}^{\omega_p} \varepsilon_i \lambda_{q_i} x_{pq_i}\Big\| \leqslant \sum_{i=p}^{\omega_p} \|\lambda_{q_i} x_{pq_i}\| < \frac{1}{2^p} + \frac{1}{2^{p+1}} + \ldots + \frac{1}{2^{\omega_p}},$$

and by (n_1)

$$(\gamma)\lim_p \sum_{i=p}^{\omega_p} \varepsilon_i \lambda_{q_i} x_{pq_i} = 0.$$

The γ-convergence does not in general satisfy the postulate (a_1). A more precise result is the following:

2.2.2. *If the γ-convergence satisfies the postulate* (a_1), *then it is equivalent to the ν-convergence* [7].

Proof. It is sufficient to prove that $x_n \xrightarrow{\gamma} x_0$ implies $\|x_n - x_0\| \to 0$. Suppose it is not the case; then there exists an $\varepsilon > 0$ and a sequence of indices $\{k_n\}$ such that $\|x_{k_n} - x_0\| \geqslant \varepsilon$;

[6] This convergence is a generalization of the notion of limit considered by Fichtenholz [6] in some concrete spaces.

[7] This theorem has been formulated in a slightly different form by Fichtenholz ([6], p. 203).

by (a_1) there exists a sequence $\{l_n\}$ extracted from $\{k_n\}$ and a sequence of numbers $\lambda_n\to\infty$ such that $\lambda_n(x_{l_n}-x_0)\xrightarrow{\gamma}0$; hence

$$\|\lambda_n^{-1}\lambda_n(x_{l_n}-x_0)\|=\|x_{l_n}-x_0\|\to 0,$$

which is impossible.

It is obvious that γ-boundedness is equivalent to ν-boundedness.

2.2.3. *If $\|x\|$ is a F-norm, the γ-convergence satisfies the postulates* (a_2) *and* (a_3).

Proof. We prove only that (a_2) is satisfied. Let $x_n\xrightarrow{\gamma}0$ and $\lambda_n\to 0$; it follows $\|\lambda_n x_n\|\to 0$. It is sufficient to choose n_k so that $\|\lambda_{n_k}x_{n_k}\|<1/2^k$.

If the norms $\|x\|$ and $\|x\|^*$ are both of F-type, the condition (n_1) implies by a theorem of Banach ([3], p. 41) that the convergences ν and ν^* (and hence γ also) are equivalent.

The following particular case of two-norms convergence is important. Let $\|x\|$ be a F-norm. Consider the space X with the norm $\|x\|^*$; it can be completed (by addition of new elements) to a F-space X^* without altering the norm; let $\|x\|^*$ denote this norm in X^*. Let us formulate the following conditions:

(n_2) *If $x_n\in X$, $x_0\in X^*$, $\|x_n-x_0\|^*\to 0$, and the sequence $\{x_n\}$ is ν-bounded, then $x_0\in X$.*

(n_3) *If $x_0\in X$, $x_n\in X$, and $\|x_n-x_0\|^*\to 0$, then $\varliminf_{n\to\infty}\|x_n\|\geqslant\|x_0\|$.*

The conditions (n_1), (n_2) and (n_3) being satisfied the two-norms convergence will be termed *strong* and will be denoted by γ'.

We need the following lemma

2.2.4. *Let X be a F^*-space. A necessary and sufficient condition for the sequence $\{x_n\}$ to be bounded is the boundedness of the sequence $\{x_{p_n}-x_{q_n}\}$, where $\{p_n\}$ and $\{q_n\}$ are any sequences of indices.*

Proof. The necessity is obvious. Suppose now the condition satisfied. It follows that, given any $\varepsilon>0$, there exists a $\delta>0$ and a $M>0$ such that $|\lambda|<\delta$, and that $p,q\geqslant M$ implies $\|\lambda(x_p-x_q)\|<\varepsilon$.

Suppose the sequence $\{x_n\}$ unbounded; then there exists a sequence $\lambda_p\to 0$ such that $\|\lambda_{n_k}x_{n_k}\|\geqslant\varepsilon$, where $n_k\to\infty$. Choose $\delta>0$ and M such that $|\lambda|<\delta$ and $p,q>M$ imply $\|\lambda(x_p-x_q)\|<\varepsilon/3$.

There exists a P such that $\|\lambda_n x_M\|<\varepsilon/3$ for $n>P$. Hence k being sufficiently large we have

$$\|\lambda_{n_k}x_{n_k}\|\leqslant\|\lambda_{n_k}(x_{n_k}-x_M)\|+\|\lambda_{n_k}x_M\|<{}^2/_3\,\varepsilon,$$

which is impossible.

2.2.5. *The γ'-convergence satisfies the postulates* (b_2), (b_2') *and* (b_5).

Proof. We first prove that (b_2') is satisfied. Let $(\gamma')\lim\limits_q x_{pq}=x_p$ for $p=1,2,\ldots$ Suppose that $q_p\to\infty$ implies $(\gamma')\lim\limits_p x_{pq_p}=0$, and let $\lambda_p\to 0$. It follows that, given any $\varepsilon>0$, we have $\|\lambda_p x_{pq}\|<\varepsilon$ for p,q sufficiently large. Since

$$\lim_{q\to\infty}\|\lambda_p x_{pq}-\lambda_p x_p\|^*=0 \qquad \text{for } p=1,2,\ldots,$$

we get by (n_3)

$$\|\lambda_p x_p\|<\varepsilon \qquad \text{for } p\geqslant P(\varepsilon);$$

hence $\|\lambda_p x_p\|\to 0$. It is obvious that $\|x_p\|^*\to 0$; it follows $x_p\xrightarrow{\gamma'}0$.

We prove (b_5) only. Suppose that $p_n\to\infty, q_n\to\infty$ imply $(\gamma')\lim\limits_n(x_{p_n}-x_{q_n})=0$. By 2.2.4. the sequence $\{x_n\}$ is ν-bounded. Since $\lim\limits_{p,q\to\infty}\|x_p-x_q\|^*=0$, there exists an element $x_0\in X^*$ such that $\|x_p-x_0\|^*\to 0$. By (n_2) $x_0\in X$.

2.3. Kantorovitch spaces. We recall here the definition of an important class of spaces.

Let X be a linear *semi-ordered space*, i. e. one in which an asymmetric and transitive relation $x_1\leqslant x_2$ is defined for certain pairs of elements. Suppose that for these inequalities the usual arithmetical laws hold. Suppose further that X is a *conditional σ-lattice*, i. e. given any sequence $\{x_n\}$ such that $x_n\leqslant x_0$ (or $x_0\leqslant x_n$ respectively), there exists an element x, denoted by $\sup\limits_{n=1,2,\ldots}x_n$ (or $\inf\limits_{n=1,2,\ldots}x_n$ respectively), such that

1⁰ $x_n\leqslant x$ (or $x\leqslant x_n$ respectively) for $n=1,2,\ldots$

2⁰ $x_n\leqslant x^*$ for $n=1,2,\ldots$ implies $x\leqslant x^*$ (or $x^*\leqslant x_n$ for $n=1,2,\ldots$ implies $x^*\leqslant x$).

Adding the ideal elements $-\infty$ and $+\infty$ as usual we can define the lower and the upper limit by the formulae:

$$\varliminf_n x_n=\sup_{n=1,2,\ldots}\ \inf_{m=n,n+1,\ldots}x_m,\qquad \varlimsup_n x_n=\inf_{n=1,2,\ldots}\ \sup_{m=n,n+1,\ldots}x_m.$$

The sequence $\{x_n\}$ is called $\varkappa$-convergent if

$$\overline{\lim_n}\, x_n = \underline{\lim_n}\, x_n \neq \pm\infty;$$

the common value of these elements will be denoted by $(\varkappa)\lim\limits_n x_n$.

A linear space with a $\varkappa$-convergence is called a *Kantorovitch space.*

Kantorovitch has shown ([8], p. 134) that

2.3.1. *The $\varkappa$-convergence satisfies the postulate* (b$_5$).

Write $|x| = \sup(x, -x, -x, \ldots)$ [8]; then $(\varkappa)\lim\limits_n x_n = x_0$ implies $(\varkappa)\lim\limits_n |x_n - x_0| = 0$, and conversely. We have also $|\overline{\lim\limits_n}\, x_n| \leqslant \overline{\lim\limits_n}\, |x_n|$.

Adding three supplementary postulates ([8], p. 138) [9] we get special Kantorovitch spaces called *regular* in the sense of Kantorovitch. It follows easily from Kantorovitch's results that

2.3.2. *The $\varkappa$-convergence in regular Kantorovitch spaces satisfies the postulates* (a$_1$), (a$_2$), (a$_2'$), (a$_3$), (b$_2$), (b$_2'$), (b$_3$), (b$_4$) *and* (b$_5$).

The regular Kantorovitch spaces have also the following properties:

(I) A sequence $\{x_n\}$ is $\varkappa$-convergent to x_0 if and only if there exists an element $x \neq \pm\infty$ such that, given any $\varepsilon > 0$, the inequality $|x_n - x_0| < \varepsilon x$ holds for n sufficiently large;

(II) A necessary and sufficient condition for the sequence $\{x_n\}$ to be $\varkappa$-bounded is the existence of an element x such that $|x_n| \leqslant x$ for $n = 1, 2, \ldots$

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[8]) This absolute value has similar properties to those of the reals: $|x_1 + x_2| \leqslant |x_1| + |x_2|$, and $|ax| = |a|\,|x|$ for real a.

[9]) The set E is said to be *bounded from above (from below)* if there exists an element x_0 such that $x \leqslant x_0$ $(x_0 \leqslant x)$ for each $x \in E$; if the set E is not bounded from above (below), we write $\sup E = +\infty$ $(\inf E = -\infty)$. The supplementary postulates are:

(a) *Every set bounded from above (below) has a supremum (infimum),* i. e. *X is a complete lattice.*

(b) *If given a sequence of sets $\{E_n\}$ we have* $x_0 = (\varkappa)\lim\limits_n \sup E_n$, *there exist finite sets $H_n \subset E_n$ such that* $a_0 = (\varkappa)\lim\limits_n \sup H_n$.

(c) *If* $\sup E_n = \infty$ *for* $n = 1, 2, \ldots$, *then there exist finite sets $H_n \subset E_n$ such that* $\sup\limits_{n=1,2,\ldots} \sup H_n = \infty$.

From the results of Kantorovitch it follows that the star-convergence corresponding to the $\varkappa$-convergence has the following property:

2.3.3. *The $\varkappa^*$-convergence in regular Kantorovitch spaces satisfies the postulates* (a$_1$), (a$_2$), (a$_2'$), (a$_3$), (b$_1$), (b$_2$), (b$_2'$), (b$_3$) *and* (b$_4$).

2.4. Weak convergences. Let X be a Banach space. A set Ω of linear functionals over X is said to be *fundamental* (Orlicz, [14], p. 66), if there exist positive numbers C and ε such that

$$\sup_{\xi \in \Omega} \|\xi\| \leqslant C, \qquad \sup_{\xi \in \Omega} |\xi(x)| \geqslant \varepsilon \|x\|.$$

A sequence $\{x_n\}$ will be termed *ω-convergent* if $\xi(x_n) \to \xi(x_0)$ for each $\xi \in \Omega$. The ω-convergence will be called also the *Ω-weak-convergence.* If Ω is identical with the set of all linear functionals over X, the ω-convergence will be denoted by σ and termed the *weak-convergence.*

2.4.1. *The ω-convergence satisfies the postulates* (b$_1$), (b$_2$) *and* (b$_2'$).

Proof. We only prove that (b$_2'$) is satisfied. Suppose that

$$(\omega)\lim_q x_{pq} = x_p \qquad \text{for } p = 1, 2, \ldots,$$

and that $q_p \to \infty$ implies $x_{pq_p} \xrightarrow{\omega} 0$. Suppose that the sequence $\{x_p\}$ does not converge to 0. Then there exists a functional $\xi_0 \in \Omega$ such that $\overline{\lim\limits_{p\to\infty}}\, |\xi_0(x_p)| = \varepsilon > 0$. Given any p, choose q_p so that

$$q_p > p \quad \text{and} \quad |\xi_0(x_p) - \xi_0(x_{pq_p})| < \varepsilon/2;$$

we get

$$|\xi_0(x_{pq_p})| \geqslant |\xi_0(x_p)| - |\xi_0(x_{pq_p}) - \xi_0(x_p)|,$$

hence $\overline{\lim\limits_{p\to\infty}}\, |\xi_0(x_{pq_p})| \geqslant \varepsilon/2$. This is, however, impossible since $x_{pq_p} \xrightarrow{\omega} 0$.

2.4.2. *If the σ-convergence satisfies the postulate* (a$_1$), *it is equivalent to the convergence generated by the norm.*

Proof. It suffices to prove that $x_n \xrightarrow{\sigma} 0$ implies $\|x_n\| \to 0$. The σ-convergence implies the boundedness of the sequence of norms (Banach [3], p. 80). Let $\{x_n^*\}$ be any subsequence of the sequence $\{x^n\}$. By (a$_1$) there exist $\lambda^k \to \infty$ and a sequence $\{n_k\}$ of indices such that $\lambda_k x_{n_k}^* \xrightarrow{\sigma} 0$; hence $\overline{\lim\limits_{k\to\infty}}\, \|\lambda_k x_{n_k}^*\| < \infty$ and $\|x_{n_k}^*\| \to 0$. From this we easily infer that $\|x_n\| \to 0$.

It is obvious that

2.4.3. *The σ-convergence satisfies the postulates* (a_2), (a_3) *and* (b_1), (b_2), (b'_2), (b_4).

2.5. B_0-spaces and their conjugate spaces. Let X be a B_0-space (Mazur and Orlicz [11], p. 185), and let $\{|x|_n\}$ be the sequence of pseudonorms determining the metrics in this space. The space X may be considered as a F-space with the norm

$$\|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x|_n}{1+|x|_n}.$$

Hence

2.5.1. *The space X satisfies all of the postulates* (a_1)-(a_5).

The space conjugate to X is the set Ξ of all linear functionals $\xi(x)$ defined on X (the addition of elements and their multiplication being defined in the usual manner). Mazur and Orlicz [12] have introduced in Ξ the ν-convergence, called the *strong convergence*, as follows: $(\nu)\lim\limits_{n} \xi_n = \xi_0$ means that there exists a $r>0$ such that $\xi_0(x)$ tends to $\xi_0(x)$ uniformly in the sphere $\|x\| \leqslant r$.

Mazur and Orlicz [12] have shown that either the space X is isomorphic with a Banach space, or Ξ_ν is not (topologically) isomorphic with any complete metric space.

It is easy to show that

2.5.2. *The space Ξ_ν satisfies the postulates* (a_1), (a_2), (a'_2), (b_1), (b_2), *and* (b_3).

2.6. Convergence almost everywhere. Let X be a linear space of real measurable functions $x=x(t)$ defined on an interval $I=\langle a,b\rangle$. Let two functions equal almost everywhere be considered as one element of the space. Call *π-convergence* the convergence almost everywhere.

2.6.1. *The sequence $\{x_n\}$ is π-bounded if and only if* $\overline{\lim\limits_{n\to\infty}}|x_n(t)| < \infty$ *almost everywhere* [10]).

Proof. The necessity only requires proof. Writing

$$x_n^*(t) = \max(|x_1(t)|, |x_2(t)|, \ldots, |x_n(t)|),$$

it is sufficient to prove that $\overline{\lim\limits_{n\to\infty}} x_n^*(t) < \infty$. Suppose it is not

[10]) This too follows from a theorem of Kantorovitch ([8], p. 140).

true. Since $x_1^*(t) \leqslant x_2^*(t) \leqslant \ldots$, we have $\lim\limits_{n\to\infty} x_n^*(t) = \infty$ in a set of positive measure, and by the theorem of Egoroff, there exists a set H of positive measure such that $\inf\limits_{t\in H} x_n^*(t) = a_n \to \infty$. Putting $\vartheta_n = a_n^{-1/2}$ we have $\vartheta_1 \geqslant \vartheta_2 \geqslant \ldots$ and $\vartheta_n \to 0$. The π-boundedness of the sequence $\{x_n\}$ implies the existence of a set $R \subset H$ such that $|H-R|=0$ and $\vartheta_n x_n(t) \to 0$ in R. Let $t_0 \epsilon R$; choose N to have $|\vartheta_n x_n(t_0)| < 1$ for $n>N$, and put

$$K = \max(|x_1(t_0)|, |x_2(t_0)|, \ldots, |x_N(t_0)|).$$

There exists a $k_n \leqslant n$ such that $x_n^*(t_0) = x_{k_n}(t_0)$; hence $|x_n(t_0)| < K$ if $k_n \leqslant N$; otherwise $|\vartheta_n x_n^*(t_0)| = |\vartheta_n x_{k_n}(t_0)| \leqslant |\vartheta_{k_n} x_{k_n}(t_0)| < 1$. In both cases $|x_n^*(t_0)| \leqslant \max(K, \vartheta_n^{-1}) = \max(K, a_n^{1/2})$. This leads to contradiction: $1 < a_n \leqslant a_n^{1/2}$ for n sufficiently large.

It is easy to prove that

2.6.2. *The π-convergence satisfies the postulates* (a_1), (b_2), (b'_2), (b_3) *and* (b_4).

If the space X is composed of all measurable functions, then (a_2), (a'_2), (a_3) *and* (b_5) *are also satisfied.*

Denote by $c_E(t)$ the characteristic function of the set E, and consider the following condition:

(p) Δ being any interval in I, and $x=x(t)$ being any element of X, the function $x(t)c_\Delta(t)$ belongs to X.

2.6.3. *The condition* (p) *being satisfied, every (X_π)-linear functional is identically equal to* 0.

Proof. Let $\xi(x)$ be a (X_π)-linear functional, and $\xi(x_0) \neq 0$. We can suppose $\xi(x_0) = \alpha > 0$. Dividing I into two intervals Δ_1, Δ_2 we have either $\xi(x_0 c_{\Delta_1}) \geqslant \alpha/2$ or $\xi(x_0 c_{\Delta_2}) \geqslant \alpha/2$; hence there exists an element $x_1 = x_1(t)$ such that $\xi(x_1) \geqslant \alpha/2$ and $x_1(t) = 0$ in a set of measure greater than $|I|(1-1/2)$. Continuing this process we obtain a sequence $\{x_n\}$ such that $\xi(x_n) \geqslant \alpha/2^n$ and $x_n(t) = 0$ in a set of measure greater than $|I|(1-1/2^n)$. Setting $x_n^* = 2^n x_n$ we have $x_n^* \xrightarrow{\pi} 0$ and $\xi(x_n^*) \geqslant \alpha$, in contradiction to the (X_π)-linearity of $\xi(x)$.

It is well known that *π^*-convergence is identical with asymptotic convergence.*

2.7. The space *M*. This space is composed of the measurable and essentially bounded functions $x=x(t)$ defined on an interval $\langle a,b\rangle$. Two equivalent functions are considered as one element of the space. This space with the norm $\|x\|=\operatorname{ess\,sup}_{a\leqslant t\leqslant b}|x(t)|$ [11]) is a Banach space.

Consider the following definitions of limit:

(i) $(\gamma')\lim_n x_n=x_0$ means that $\|x_n\|\leqslant K$ for $n=1,2,\dots$ and $\lim_{n\to\infty}\mathrm{as}\, x_n(t)=x_0(t)$ [12]),

(ii) $(\varkappa)\lim_n x_n=x_0$ means that $\|x_n\|\leqslant K$ for $n=1,2,\dots$ and $x_n(t)\to x_0(t)$ almost everywhere,

(iii) $(\pi)\lim_n x_n=x_0$ means that $x_n(t)\to x_0(t)$ almost everywhere.

Denoting by X^* the space of Lebesgue integrable functions in $\langle a,b\rangle$ and putting $\|x\|^*=\int_a^b|x(t)|\,dt$, we can easily prove that γ'-convergence is a strong two-norms convergence. Hence

2.7.1. *The space $M_{\gamma'}$ satisfies the postulates* (a_2), (a_3), (b_1), (b_2), (b_2'), (b_3), (b_4) *and* (b_5).

As may be easily proved, *the space $M_{\gamma'}$ does not satisfy the postulates* (a_1) *and* (a_2).

The space $M_\varkappa$ is a Kantorovitch space ([8], p. 156) corresponding to the following partial ordering: $x_1\leqslant x_2$ means that $x_1(t)\leqslant x_2(t)$ almost everywhere. The space $M_\varkappa$ is regular in the sense of Kantorovitch. It is easy to prove that

2.7.2. *The space $M_\varkappa$ satisfies the postulates* (a_2), (a_3), (b_2), (b_2'), (b_3), (b_4) *and* (b_5), *but does not satisfy the postulates* (a_1), (a_2') *and* (b_1);

2.7.3. *The space M_π satisfies the postulates* (a_1), (a_3), (b_2), (b_2') *and* (b_4), *but does not satisfy the postulates* (a_2), (a_2'), (b_1) *and* (b_5).

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11) $\operatorname{ess\,sup}|x(t)|$ denotes the greatest lower bound of the numbers k for which the set $E_t\{|x(t)|>k\}$ is of measure 0.

12) $\lim_{n\to\infty}\mathrm{as}\, x_n(t)$ denotes the asymptotic limit of the sequence $\{x_n(t)\}$.

2.8. The space M^*. This space is composed of the real functions $x=x(t)$ defined in an interval $\langle a,b\rangle$, two non-identical functions being considered as different elements of the space. This space with the norm $\|x\|=\sup_{a\leqslant t\leqslant b}|x(t)|$ is a Banach space.

Consider instead of the $\varkappa$-convergence the same modified by omission of the word „almost" in its definition (ii).

The space $M_\varkappa^*$ with so modified $\varkappa$-convergence is a Kantorovitch space corresponding to an analogous definition of partial ordering as in the case of the space $M_\varkappa$.

2.8.1. *The space $M_\varkappa^*$ satisfies the postulates* (a_2), (a_3), (b_1), (b_2), (b_2'), (b_4) *and* (b_5), *but does not satisfy the postulates* (a_1), (a_2') *and* (b_3).

Proof. We prove only that (b_3) is not satisfied. Represent the interval $I_0=\langle a,b\rangle$ as the sum $I_0=\sum_{\nu=1}^{\infty}I_\nu$ of open on the right and disjoint intervals, such that $I_{\nu+1}$ adheres to I_ν at the right. Continue the same process with every one of the intervals I_ν, and so on. Given any finite sequence $\alpha,\dots,\alpha_p$ of positive integers, we obtain thus an interval $I_{\alpha_1,\dots,\alpha_p}$, open on the right and such that

(i) $I_{\alpha_1,\dots,\alpha_p}=\sum_{\nu=1}^{\infty}I_{\alpha_1,\dots,\alpha_p,\nu}$,

(ii) if $(\alpha_1-\beta_1)^2+\dots+(\alpha_p-\beta_p)^2>0$ then $I_{\alpha_1,\dots,\alpha_p}\cdot I_{\beta_1,\dots,\beta_p}\neq 0$,

(iii) $I_{\alpha_1,\dots,\alpha_{p+1}}$ adheres to $I_{\alpha_1,\dots,\alpha_p}$ at the right.

Put

$$x_{pq}(t)=\begin{cases}1 & \text{for } t\,\epsilon\sum_{\alpha_1,\dots,\alpha_p=1}^{\infty}\bar I_{\alpha_1,\dots,\alpha_p,q}\\ 0 & \text{elsewhere.}\end{cases}$$

Obviously $|x_{pq}(t)|\leqslant 1$, and $\lim_{p\to\infty}x_{pq}(t)=0$ for each t; hence $(\varkappa)\lim_q x_{pq}=0$. Let now $q_p\to\infty$; the sequence $\{x_{pq_p}\}$ is $\varkappa$-bounded, however, it is not $\varkappa$-convergent to 0, since the sequence $\{x_{pq_p}(t)\}$ does not converge to 0 everywhere. In fact, $x_{pq_p}(t)=1$ for $t\,\epsilon\,\bar I_{q_1,\dots,q_p}$; hence $\lim_{p\to\infty}x_{pq_p}(t)=1$ for any $t\,\epsilon\prod_{p=1}^{\infty}\bar I_{q_1,\dots,q_p}=E$. It is obvious that $E\neq 0$.

2.9. The space V^* [13]. This space is composed of all the functions $x=x(t)$ which are equivalent to functions of bounded variation in $\langle a,b\rangle$; two equivalent functions of V^* are considered as one element of the space. Denote by $\operatorname{ess\,var} x(t)$ the greatest lower bound of total variation $\operatorname{var} x^*(t)$ of the functions $x^*(t)$ equivalent to $x(t)$. A function $x(t)$ belongs to V^* if and only if $\operatorname{ess\,var} x(t)<\infty$.

2.9.1. *If* $\operatorname{ess\,var} x_n(t)\leqslant K$ *for* $n=1,2,\ldots$, *and* $\lim_{n\to\infty}\operatorname{as} x_n(t)=x_0(t)$, *then* $\operatorname{ess\,var} x_0(t)\leqslant K$.

Proof. Choose a function $x_n^*(t)$ of bounded variation equivalent to $x_n(t)$ and such that $\operatorname{var} x_n^*(t)\leqslant \operatorname{ess\,var} x_n(t)+1/n$. There exists a subsequence $\{x_{n_k}(t)\}$ convergent to $x_0(t)$ almost everywhere. By the theorem of Helly [14] the sequence $\{x_{n_k}^*(t)\}$ contains a uniformly convergent subsequence $\{x_{m_k}^*(t)\}$. The function $x_0^*(t)=\lim_{k\to\infty} x_{m_k}^*(t)$ is equivalent to $x_0(t)$; moreover

$$\operatorname{var} x_0^*(t)\leqslant \varliminf_{n\to\infty} \operatorname{var} x_n^*(t)\leqslant K;$$

hence $\operatorname{ess\,var} x_0(t)\leqslant K$.

If we introduce in V^* the norm by the formula

$$\|x\|=\operatorname{ess\,sup}|x(t)|+\operatorname{ess\,var} x(t),$$

V^* becomes a Banach space.

Now we introduce the following convergence:

$(\gamma')\lim_n x_n=x_0$ means that $\operatorname{ess\,var} x_n(t)\leqslant K$ for $n=1,2,\ldots$, and $\lim_{n\to\infty}\operatorname{as} x_n(t)=x_0(t)$.

The γ'-convergence is equivalent to a strong two-norms convergence. To see this it suffices to denote by X^* the space of the integrable functions and to put $\|x\|^*=\int_a^b|x(t)|\,dt$; the conditions (n_2) and (n_3) follow from 2.9.1. Hence

2.9.2. *The γ'-convergence satisfies the postulates* (a_2), (a_3), (b_1), (b_2), (b_2'), (b_3), (b_4) *and* (b_5), *but does not satisfy* (a_1), (a_2'), as easily can be seen.

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[13]) considered first by Orlicz [13].

[14]) see, for instance, [16], p. 80.

2.10. The space L. In the well-known space L of Lebesgue integrable functions in $\langle a,b\rangle$ consider following notions of convergence:

(i) $(\varkappa)\lim_n x_n=x_0$ means that $x_n(t)\to x_0(t)$ almost everywhere, and that there exists an integrable function $x_0(t)$ such that $|x_n(t)|\leqslant x_0(t)$ for $n=1,2,\ldots$

(ii) $(\eta)\lim_n x_n=x_0$ means that $\int_a^s x_n(t)\,dt\to\int_a^s x_0(t)\,dt$ uniformly in the interval $\langle a,b\rangle$,

(iii) $(\vartheta)\lim_n x_n=x_0$ means that $\left|\int_a^s x_n(t)\,dt\right|\leqslant K$ for $n=1,2,\ldots$ and $0\leqslant s\leqslant 1$, that $\int_a^b x_n(t)\,dt\to\int_a^b x_0(t)\,dt$, and that $\lim_{n\to\infty}\operatorname{as}\int_a^s x_n(t)\,dt$ $=\int_a^s x_0(t)\,dt$.

Kantorovitch has shown ([8], p. 156) that the space $L_\varkappa$ is a regular Kantorovitch space corresponding to the same partial ordering as the space $M_\varkappa$. Hence

2.10.1. *The $\varkappa$-convergence satisfies the postulates* (a_1), (a_2), (a_2'), (a_3), (b_2), (b_2'), (b_3), (b_4) *and* (b_5), *but does not satisfy* (b_1), as easily can be seen.

η-convergence is the convergence generated by the norm $\|x\|^*=\max_{a\leqslant s\leqslant b}\left|\int_a^s x(t)\,dt\right|$; this space, however, is a B^*-space and not a Banach space. This follows from the proposition:

2.10.2. *The space L_η satisfies the postulates* (a_1), (a_3), (b_1), (b_2), (b_2'), (b_3) *and* (b_4), *but does not satisfy the postulates* (a_2), (a_2'), (b_5).

Proof. (a_1) follows by formula $\|x\|^*\leqslant\|x\|=\int_a^b|x(t)|\,dt$. We prove now that (a_2) is not satisfied. Denote by $u(t)$ the function equal to the distance between the number t and the set of all integers. It is well-known (Van der Waerden [15]) that the function $\sum_{n=1}^{\infty}\frac{u(4^n t)}{4^n}$ is continuous and nowhere differentiable. It is easy to prove also that, given any sequence of indices $\{n_k\}$, the series $\sum_{k=1}^{\infty}\frac{u(4^{n_k}t)}{4^{n_k}}$ represents a function with the same properties. Put

$$v(t)=\begin{cases} 1 & \text{for } n\leqslant t\leqslant n+1/2,\\ -1 & \text{for } n+1/2<t<n+1,\end{cases}$$

$$x_n(t)=\frac{v(4^n t)}{2^n}.$$

In order to establish the proposition suppose for instance that $a=0$, $b=1$; then $\int_0^s x_n(t)\,dt=\frac{u(4^n s)}{2^n}$; hence $\|x_n\|^*\to 0$.

Put $\lambda_k=2^{-k}$, and let $\{n_k\}$ be any sequence of indices. We prove that the series $\sum_{k=1}^{\infty}\lambda_{n_k}x_{n_k}$ is not η-convergent. In fact, in the contrary case there would exist a function $x_0(t)\,\epsilon\,\boldsymbol{L}$ such that $\int_0^s\sum_{k=1}^{m}\lambda_{n_k}x_{n_k}(t)\,dt\to\int_0^s x_0(t)\,dt$ uniformly in s as $m\to\infty$. On the other hand $\int_0^s\sum_{k=1}^{m}\lambda_{n_k}x_{n_k}(t)\,dt\to\sum_{k=1}^{\infty}\frac{u(4^{n_k}s)}{4^{n_k}}$, and this would imply the differentiability almost everywhere of the function $\sum_{k=1}^{\infty}\frac{u(4^{n_k}t)}{4^{n_k}}$, which is impossible.

To see that (b_5) is not fulfilled, note that the sequence constructed above fulfils the condition of Cauchy and is not convergent.

The following theorems can be proved:

2.10.3. *The space $\boldsymbol{L}_\vartheta$ satisfies the postulates* (a_3), (b_1), (b_2), (b_2'), (b_3) *and* (b_4), *but does not satisfy the postulates* (a_1), (a_2), (a_2'), (b_5).

2.10.4. [15]) *The general form of the $(\boldsymbol{L}_\eta)$-linear functionals is*

$$\xi(x)=\int_a^b x(t)\,h(t)\,dt, \tag{1}$$

where $h(t)$ is an arbitrary function of bounded variation.

The general form of the $(\boldsymbol{L}_\vartheta)$-linear functionals is (1) *with $h(t)$ absolutely continuous.*

[15]) This theorem is proved in [2] in a slightly more general form.

2.11. The space $\boldsymbol{L}\{X\}$. This space is composed of the functions $x=x(t)$ from a real interval $\langle a,b\rangle$ to a Banach space X, integrable in the sense of Bochner ([5], p. 265); two equivalent functions are considered as one element of the space.

Introducing in $\boldsymbol{L}\{X\}$ the norm by the formula $\|x\|=\int_a^b\|x(t)\|\,dt$ we get a Banach space.

Consider the following notion of convergence:

$(\varkappa)\lim_n x_n=x_0$ means that $x_n(t)\to x_0(t)$ almost everywhere, and there exists a real integrable function $\gamma(t)$ such that $\|x_n(t)\|<\gamma(t)$ for $n=1,2,\ldots$

It is easy to see that $x_n\xrightarrow{\varkappa}x_0$ if and only if the sequence $\|x_n(t)-x_0(t)\|$ as elements of the space $\boldsymbol{L}$ is $\varkappa$-convergent to 0, i. e. if there exists an integrable real function $\omega(t)$ such that, given any $\varepsilon>0$, the inequality $\|x_n(t)-x_0(t)\|<\varepsilon\,\omega(t)$ holds for any n sufficiently large. It follows:

2.11.1. *The space $\boldsymbol{L}\{X\}_\varkappa$ satisfies the postulates* (a_1), (a_2), (a_2'), (a_3), (b_2), (b_2'), (b_3), (b_4) *and* (b_5), *but does not satisfy the postulate* (b_1), as easily may be seen.

2.12. The spaces $\boldsymbol{L}^p$ and $\boldsymbol{L}^p\{X\}$. In the well-known space $\boldsymbol{L}^p$ of functions integrable with the p-th power $(p>1)$ denote by σ the weak convergence, by π the convergence almost everywhere, and by $\varkappa$ the convergence defined as follows:

$(\varkappa)\lim_n x_n=x_0$ means that $x_n(t)\to x_0(t)$ almost everywhere, and that there exists a function $\gamma(t)\,\epsilon\,\boldsymbol{L}^p$ such that $|x_n(t)|<\gamma(t)$ for $n=1,2,\ldots$

The space $\boldsymbol{L}^p_\varkappa$ is a regular Kantorovitch space ([8], p. 156) corresponding to the same partial ordering as the space $\boldsymbol{M}_\varkappa$. Hence

2.12.1. *The $\varkappa$-convergence satisfies the postulates* (a_1), (a_2), (a_2'), (a_3), (b_2), (b_2'), (b_3), (b_4) *and* (b_5), *but does not satisfy the postulate* (b_1), as easily may be seen.

It is easy to prove that

2.12.2. *The π-convergence satisfies the postulates* (a_1), (a_3), (b_2), (b_2'), (b_3) *and* (b_4), *but does not satisfy the postulates* (a_2), (a_2'), (b_1), (b_5).

The σ-convergence may be characterized as follows (see Banach [3], p. 135):

$(\sigma)\lim\limits_n x_n = x_0$ if and only if $\|x_n\| < K$ for $n = 1, 2, \ldots$, and $\int\limits_a^s x_n(t)\,dt \to \int\limits_a^s x_0(t)\,dt$ for every s.

2.12.3. *The σ-convergence satisfies the postulates* (a_2), (a_3), (b_1), (b_2), (b_2'), (b_3), (b_4) *and* (b_5), *but does not satisfy the postulates* (a_1) *and* (a_2').

Proof. We prove only (b_4). Suppose that $(\sigma)\lim\limits_q x_{pq} = 0$ for $p = 1, 2, \ldots$ and that $q_p \to \infty$ implies the σ-boundedness of the sequence $\{x_{pq_p}\}$. It follows from a theorem of Banach ([3], p. 80) that $\|x_{pq}\| \leqslant M$ with M independent of p and q. Denote by Ξ the conjugate space to L^p: Ξ being separable, let $\{\xi_n\}$ be a sequence of elements of Ξ dense everywhere. Choose q_p to have $|\xi_j(x_{pq_p})| < 1/p$ for $j = 1, 2, \ldots, p$ and denote by $\zeta_n(\xi)$ the (Ξ)-linear functional of the form $\zeta_n(\xi) = \xi(x_{nq_n})$. The inequality

$$|\xi(x_{nq_n})| \leqslant \|\xi\|\,\|x_{nq_n}\|$$

implies the boundedness of the sequence $\{\zeta_n(\xi)\}$ for every ξ; moreover, $\lim\limits_{n\to\infty} \zeta_n(\xi_i) = 0$ for $i = 1, 2, \ldots$ Hence by the theorem of Banach-Steinhaus ([3], p. 79)

$$\zeta_n(\xi) \to 0 \text{ for any } \xi \in \Xi, \text{ i. e. } x_{nq_n} \overset{\sigma}{\to} 0.$$

Let X denote a Banach space. By $L^p\{X\}$ will be denoted the space of the functions $x = x(t)$ from a real interval $\langle a, b\rangle$ to the space X, integrable in Bochner sense with the p-th power. Introducing in $L^p\{X\}$ the norm by the formula $\|x\| = \left(\int\limits_a^b \|x(t)\|^p\,dt\right)^{1/p}$ we get a Banach space.

Let $\varkappa$-convergence be defined as follows:

$(\varkappa)\lim x_n = x_0$ means that $x_n(t) \to x_0(t)$ almost everywhere, and there exists a function $\gamma(t) \in L^p$ such that $\|x_n(t)\| \leqslant \gamma(t)$ for $n = 1, 2, \ldots$

As in 2.11 we can prove that

2.12.4. *The space $L^p\{X\}_\varkappa$ satisfies the postulates* (a_1), (a_2), (a_2'), (a_3), (b_2), (b_2'), (b_3), (b_4) *and* (b_5), *but does not satisfy the postulate* (b_1).

2.13. The space H^p. This space consists of the functions $x = x(t)$ satisfying in $\langle a, b\rangle$ the Hölder condition

$$|x(t_1) - x(t_2)| \leqslant M|t_1 - t_2|^p,$$

where $0 < p \leqslant 1$. Two functions are considered as one element of the space if and only if they are identical. If we introduce the norm

$$\|x\| = |x(0)| + \sup_{a \leqslant t_1 < t_2 \leqslant b} \left|\frac{x(t_1) - x(t_2)}{t_1 - t_2}\right|,$$

H^p becomes a Banach space.

Consider the following convergence:

$(\gamma')\lim x_n = x_0$ means that $\|x_n\| \leqslant K$ for $n = 1, 2, \ldots$, and $x_n(t) \to x_0(t)$ uniformly in $[a, b]$.

Putting $X^* = C$ [16]) and $\|x\|^* = \max|x(t)|$ we easily see that γ' is a strong two-norms convergence. This follows from the lemma:

2.13.1. *Let $x_n(t) \in H^p$ and $\|x_n\| \leqslant K$ for $n = 1, 2, \ldots$, and suppose that the sequence $\{x_n(t)\}$ converges in a set dense in $\langle a, b\rangle$. Then there exists an element $x_0(t) \in H^p$ such that $x_n(t) \to x_0(t)$ uniformly in $\langle a, b\rangle$, and that $\|x_0\| \leqslant K$.*

Proof. The hypothesis implies the uniform equicontinuity of the sequence $\{x_n(t)\}$ in $\langle a, b\rangle$. By the theorem of Arzelà $\{x_n(t)\}$ converges uniformly to a continuous function $x_0(t)$. The remaining part of the lemma follows by passing to the limit in the formula

$$|x_n(0)| + \left|\frac{x_n(t_1) - x_n(t_1)}{t_1 - t_2}\right| \leqslant K,$$

valid for $a \leqslant t_1 < t_2 \leqslant b$.

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[16]) C denotes, as usual, the space of functions continuous in the interval $\langle a, b\rangle$.

Hence

2.13.2. *The γ'-convergence satisfies the postulates* (a_2), (a_3), (b_1), (b_2), (b_2'), (b_3), (b_4) *and* (b_5), *but does not satisfy the postulates* (a_1) *and* (a_2'), as easily may be seen.

We can easily prove that if $\|x_n\| \leqslant K$ for $n=1,2,\ldots$, and $\lim_{n\to\infty} \text{as}\, x_n(t) = x_0(t)$, then $x_n(t) \to x_0(t)$ uniformly in $\langle a,b\rangle$. Hence we can replace in the definition of γ'-convergence the condition

$$x_n(t) \to x_0(t)$$

uniformly in $\langle a,b\rangle$ by $\lim_{n\to\infty} \text{as}\, x_n(t) = x_0(t)$.

2.14. The space $\mathfrak{S}$. Let $\boldsymbol{s}$ denote the space of the sequences $x=\{x_n^*\}$ of real numbers. Introducing the n-th pseudonorm by formula $|x|_n = |x_n^*|$, we easily see that $\boldsymbol{s}$ is a B_0-space. It is known that the general form of the $(\boldsymbol{s})$-linear functionals is

$$\xi(x) = \sum_{n=1}^{\infty} \xi_n^* x_n^*,$$

where $\xi_n^* = 0$ for $n > N$. Given any $\xi(x)$, we denote by $b(\xi)$ the greatest n for which $\xi_n^* \neq 0$.

Let $\mathfrak{S}$ be the space conjugate to $\boldsymbol{s}$; $\mathfrak{S}$ consists of the sequences $\xi = \{\xi_n^*\}$ such that $\xi_n^* = 0$ for almost every n. Mazur and Orlicz [12] have shown that the (strong) ν-convergence in $\mathfrak{S}$ may be characterized as follows: if $\xi_n = \{\xi_{ni}^*\}_{i=1,2,\ldots}$ and $\xi_0 = \{\xi_{0i}^*\}_{i=1,2,\ldots}$, then $(\nu)\lim_n \xi_n = \xi_0$ means that $b(\xi_n) \leqslant K$ for $n=1,2,\ldots$, and that

$$\lim_{n\to\infty} \xi_{ni}^* = \xi_{0i}^*$$

for $i=1,2,\ldots$

2.14.1. *The ν-convergence satisfies the postulates* (a_1), (a_2), (a_2'), (b_1), (b_2), (b_2'), (b_3) *and* (b_5), *but does not satisfy the postulates* (a_3) *and* (b_4).

2.14.2. *The general form of the* $(\mathfrak{S}_\nu)$*-linear functionals is*

$$\zeta(\xi) = \sum_{n=1}^{\infty} \zeta_n^* \xi_n^*,$$

with arbitrary ζ_n^*.

2.15. The space $\boldsymbol{l}$. In the space $\boldsymbol{l}$ of sequences $x=\{x_n^*\}$ such that $\|x\| = \sum_{n=1}^{\infty} |x_n^*| < +\infty$ consider the following notions of convergence: if $x_n = \{x_{ni}^*\}_{i=1,2,\ldots}$ and $x_0 = \{x_{0i}^*\}_{i=1,2,\ldots}$, then

$(\varkappa)\lim_n x_n = x_0$ means that $\lim_{n\to\infty} x_{ni}^* = x_{0i}^*$ for $i=1,2,\ldots$, and there exists an element $z=\{z_i^*\} \,\epsilon\, \boldsymbol{l}$ such that $|x_{ni}^*| \leqslant z_i^*$ for $n=1,2,\ldots$ and $i=1,2,\ldots$

$(\eta)\lim_n x_n = x_0$ means that $\lim_{n\to\infty} \sum_{i=1}^{m} x_{ni}^* = \sum_{i=1}^{m} x_{0i}^*$ uniformly with respect to m.

Kantorovitch ([8], p. 165) has shown that the space $\boldsymbol{l}_\varkappa$ is a regular Kantorovitch space corresponding to the following partial ordering: $x_1 = \{x_{1i}^*\} \leqslant x_2 = \{x_{2i}^*\}$ means that $x_{1i}^* \leqslant x_{2i}^*$ for $i=1,2,\ldots$ Hence

2.15.1. *The $\varkappa$-convergence satisfies the postulates* (a_1), (a_2), (a_2'), (a_3), (b_2), (b_2'), (b_3), (b_4) *and* (b_5), *but* — as easily seen — *does not satisfy the postulate* (b_1).

The η-convergence is generated by the norm

$$\|x\|^* = \sup_{n=1,2,\ldots} \Big|\sum_{i=1}^{n} x_i^*\Big|;$$

this norm is, however, a B^*-norm. It is easy to show that

2.15.2. *The η-convergence satisfies the postulates* (a_1), (a_3), (b_1), (b_2), (b_2'), (b_3) *and* (b_4), *but does not satisfy the postulates* (a_2), (a_2'), *and* (b_5).

2.16. The independence of postulates. The examples of the spaces $\boldsymbol{M}_\varkappa$, $\boldsymbol{M}_\pi$, and $\mathfrak{S}_\nu$ show that the postulates (a_1), (a_2) and (a_3) are independent of one another. By the properties of the spaces $\boldsymbol{M}_\varkappa$, $\boldsymbol{M}_\varkappa^*$, $\mathfrak{S}_\nu$ and $\boldsymbol{L}_\eta$ it follows that the postulates (b_1), (b_3), (b_4) and (b_5) do not follow from the remaining of the postulates (b_1)-(b_5). The problem of the independence of (b_2) remains open.

3. Independence of statements I^l, II^l and III^l. Let α and β denote the convergence generated by the norm in the F-spaces X and Y respectively. Then $\mathrm{I}^l(X_\alpha, Y_\beta)$, $\mathrm{II}^l(X_\alpha, Y_\beta)$ and $\mathrm{III}_i^l(X_\alpha, Y_\beta)$ are true for $i=1$ and 2. It is not the case in general $\varLambda$-spaces.

3.1. Theorem. *In general $\varLambda$-spaces the statements* I^l, II^l *and* III_1^l *are independent of one another.*

Proof. I^l *does not follow from* II^l *and* III^l_1 [17]). Put $X=\boldsymbol{L}$ and $Y=\boldsymbol{S}_\pi$ [18]). By a theorem of Banach ([4], p. 32) it follows that $\mathrm{II}^l(\boldsymbol{L},\boldsymbol{S}_\pi)$ is true. We prove later on (see Part IV of this paper, Theorem 3.2) that the statement $\mathrm{III}^l_1(\boldsymbol{L},\boldsymbol{S}_\pi)$ is true.

The statement $\mathrm{I}^l(\boldsymbol{L},\boldsymbol{S}_\pi)$ is however false. In fact, denote by $s_n(x)$ the n-th partial sum of the Fourier development of the function $x=x(t)$, and by $U_n(x)$ the n-th polynomial of Fejér corresponding to this function. It is obvious that $s_n(x)$, and hence $U_n(x)$, are $(\boldsymbol{L},\boldsymbol{S}_\pi)$-linear operations. By the classical Fejér-Lebesgue theorem $\lim\limits_{n\to\infty} U_n(x)=x$ for each $x\in\boldsymbol{L}$. The limit operation $U(x)=x$ is however not $(\boldsymbol{L},\boldsymbol{S}_\pi)$-linear since the convergence in mean does not imply the convergence almost everywhere.

II^l *does not follow from* I^l *and* III^l_1. Put $X_\alpha=\boldsymbol{M}_{\gamma'}$, and let Y_β be the space $\boldsymbol{R}$ of the reals. The truthfulness of $\mathrm{I}^l(\boldsymbol{M}_{\gamma'},\boldsymbol{R})$ has been proved by Orlicz [19]), and the truthfulness of $\mathrm{III}^l_1(\boldsymbol{M}_{\gamma'},\boldsymbol{R})$ follows from the results of section 8. The statement $\mathrm{II}^l(\boldsymbol{M}_{\gamma'},\boldsymbol{R})$ however is not true. Fichtenholz ([6], p. 199) has shown that the general form of the $(\boldsymbol{M}_{\gamma'})$-linear functionals is

$$\xi(x)=\int_0^1 x(t)\,h(t)\,dt,$$

where $h(t)$ belongs to $\boldsymbol{L}$. Denote by D the class of the step-functions; this set is dense in $\boldsymbol{M}_{\gamma'}$. Define the function $h_n(t)$ as follows:

$$h_n(0)=h_n\left(\frac{2}{4n}\right)=h_n\left(\frac{4}{4n}\right)=h_n(1)=0,\qquad h_n\left(\frac{1}{4n}\right)=n,\qquad h_n\left(\frac{3}{4n}\right)=-n,$$

and $h_n(t)$ is linear in the intervals $\langle 0,1/4n\rangle$, $\langle 1/4n,2/4n\rangle$, $\langle 2/4n,3/4n\rangle$ and $\langle 3/4n,1\rangle$. Put

$$\xi_n(x)=\int_0^1 x(t)\,h_n(t)\,dt;$$

it is a $(\boldsymbol{M}_{\gamma'},\boldsymbol{R})$-linear operation. Since

$$\int_0^1|h_n(t)|\,dt=1/2\quad\text{and}\quad\lim_{n\to\infty}\int_0^s h_n(t)\,dt=0\qquad\text{for } 0\leqslant s\leqslant 1,$$

[17]) stated without proof by Kantorovitch ([9], p. 258).

[18]) In this case $\boldsymbol{L}$ and $\boldsymbol{S}$ denote respectively the space of the functions integrable and measurable in the interval $\langle 0,2\pi\rangle$.

[19]) See [19]; in section 7.1 another proof of this theorem is given.

it follows that

$$|\xi_n(x)|\leqslant\int_0^1|x(t)|\,|h_n(t)|\,dt\leqslant\frac{1}{2}\|x\|\quad\text{for every } x\in\boldsymbol{M}_{\gamma'}.$$

Moreover,

$$\lim_{n\to\infty}\int_0^1 x(t)\,h_n(t)\,dt=0\qquad\text{for each } x\in D.$$

The sequence $\{\xi_n(x)\}$ is however not convergent in the whole of $\boldsymbol{M}_{\gamma'}$, since the set-functions $\varphi_n(e)=\int_e h_n(t)\,dt$ are not equi-absolutely continuous.

III^l_1 *does not follow from* I^l *and* II^l. Put $X_\alpha=\mathfrak{S}_\nu$ and $Y_\beta=\boldsymbol{R}$. The truthfulness of the statements $\mathrm{I}^l(\mathfrak{S}_\nu,\boldsymbol{R})$ and $\mathrm{II}^l(\mathfrak{S}_\nu,\boldsymbol{R})$ follows from a theorem of Mazur and Orlicz [20]). The general form of $(\mathfrak{S}_\nu)$-linear functionals being $\zeta(\xi)=\sum\limits_{\nu=1}^{\infty}\zeta^*_\nu\xi^*_\nu$ with $\xi=\{\xi^*_\nu\}$ and arbitrary ζ^*_n, write

$$\zeta_{pq}(\xi)=q\,\xi^*_p.$$

Putting

$$\xi^*_{ik}=\begin{cases}1 & \text{for } i=k\\ 0 & \text{for } i\neq k\end{cases}\qquad\text{and } \xi_p=\{\xi^*_{pk}\}_{k=1,2,\dots},$$

we see that the sequence $\{\zeta_{pq}(\xi)\}_{q=1,2,\dots}$ of $(\mathfrak{S}_\nu,\boldsymbol{R})$-linear operations is unbounded (hence divergent) for $\xi=\xi_p$. No element $\xi=\{\xi^*_n\}$, however, exists for which the sequences $\{\zeta_{pq}(\xi)\}_{q=1,2,\dots}$ would be divergent simultaneously, since $\zeta_{pq}(\xi)=0$ for $p>b(\xi)$.

4. Some sufficient conditions for I^l and II^l. In the following lines X_α and Y_β are Λ-spaces, and $U(x)$, $U_n(x)$ and $U_{pq}(x)$ operations from X_α to Y_β. $\{U_n(x)\}$ being any sequence of (X_α,Y_β)-linear operations, following conditions will be useful:

(Q_1) If the sequence $\{U_n(x)\}$ is β-convergent everywhere, then $x_n\xrightarrow{\alpha}0$ implies $U_n(x_n)\xrightarrow{\beta}0$.

(Q_2) If the sequence $\{U_n(x)\}$ is β-bounded everywhere, then α-boundedness of the sequence $\{x_n\}$ implies β-boundedness of the sequence $\{U_n(x_n)\}$.

(Q_3) If the sequence $\{U_n(x)\}$ is β-bounded everywhere, then $x_n\xrightarrow{\alpha}0$ implies $\{U_n(x_n)\}\xrightarrow{\beta}0$.

[20]) See [14]; this follows also from the results of section 6 of the present paper.

It is obvious that (Q_3) implies (Q_1), and that (Q_1) implies (Q_2). If the space X_α satisfies the postulate (a_1), and Y_β satisfies the postulate (b_1), then (Q_1) implies (Q_3).

We use in the following considerations the property of additive operations which are (X_α, Y_β)-continuous at one point of being (X_α, Y_β)-linear. Hence, to prove that an additive operation is (X_α, Y_β)-linear it is sufficient to prove its (X_α, Y_β)-continuity at $x=0$.

4.1. *If the space Y_β satisfies the postulate (b_2') and the condition (Q_1) is satisfied, then theorem $I^1(X_\alpha, Y_\beta)$ holds.*

Proof. Let $\{U_n(x)\}$ be a sequence of (X_α, Y_β)-linear operations, β-convergent to $U(x)$ everywhere, and let $x_n \xrightarrow{\alpha} 0$. Since $(\beta)\lim\limits_q U_q(x_p) = U(x_p)$ for $p=1,2,\ldots$, and (Q_1) is satisfied by hypothesis, $q_p \to \infty$ implies $(\beta)\lim\limits_p U_{q_p}(x_p)=0$, and (b_2') implies

$$(\beta)\lim_p U(x_p)=0.$$

Thus $U(x)$ is (X_α, Y_β)-continuous at $x=0$.

4.2. *If the space X_α satisfies the postulate (a_1), the space Y_β satisfies the postulates (b_1) and (b_2), and the condition (Q_2) is satisfied, then $I^1(X_\alpha, Y_\beta)$ holds.*

Proof. Suppose it is not the case. Then there exists a sequence $\{U_n(x)\}$ of (X_α, Y_β)-linear operations and a sequence $\{x_n\}$ such that $x_n \xrightarrow{\beta} 0$, $U_n(x) \to U(x)$ everywhere, and $U_n(x_n)$ is not α-convergent to 0. By (b_1) we may suppose that every subsequence of $\{U(x_n)\}$ is not β-convergent to 0; (a_1) implies the existence of sequences $\{\lambda_k\}$ and $\{n_k\}$ such that $\lambda_k \to \infty$ and $y_k = \lambda_k x_{n_k} \xrightarrow{\alpha} 0$. By (Q_2) $q_k \to \infty$ implies the β-boundedness of the sequence $\{U_{q_k}(y_k)\}$. Since $(\beta)\lim\limits_q U_q(y_k) = U(y_k)$ for $k=1,2,\ldots$, the postulate (b_2) implies the β-boundedness of the sequence $\{U(y_k)\}$, in particular $\lambda_k^{-1}U(y_k) = U(x_{n_k}) \xrightarrow{\beta} 0$, which is impossible.

4.3. *If the space Y_β satisfies the postulates (b_1), (b_3), (b_5), and if the condition (Q_3) is satisfied, then $II^1(X_\alpha, Y_\beta)$ holds.*

Proof. Let D be a set dense in X_α, and $\{U_n(x)\}$ a sequence of (X_α, Y_β)-linear operations β-bounded everywhere and β-convergent in D.

Suppose there exists an element x_0 such that the sequence $\{U_n(x_0)\}$ is β-divergent.

By (b_5) there exist two sequences $p_n \to \infty$ and $q_n \to \infty$ such that the sequence $\{U_{p_n}(x_0) - U_{q_n}(x_0)\}$ does not β-converge to 0. By (b_1) we can suppose that every subsequence of it has the same property. There exist elements $x_n \in D$ such that $x_n \xrightarrow{\alpha} x_0$. We then have $(\beta)\lim\limits_n [U_{p_n}(x_m) - U_{q_n}(x_m)] = 0$ for $m=1,2,\ldots$, and, by (Q_3), $n_m \to \infty$ implies the β-boundedness of the sequence

$$\{U_{p_{n_m}}(x_m) - U_{q_{n_m}}(x_m)\}.$$

By (b_3) there exists a sequence $n_m \to \infty$ such that

$$[U_{p_{n_m}}(x_m) - U_{q_{n_m}}(x_m)] \xrightarrow{\beta} 0.$$

By the additivity of $U_n(x)$ we have

$$U_{p_{n_m}}(x_0) - U_{q_{n_m}}(x_0) = U_{p_{n_m}}(x_0 - x_m) + [U_{p_{n_m}}(x_m) - U_{q_{n_m}}(x_m)] + U_{q_{n_m}}(x_m - x_0).$$

By (Q_2) the first and the last term of the right-hand side β-converges to 0. The second term β-converges also to 0, as we have already seen. Hence $[U_{p_{n_m}}(x_0) - U_{q_{n_m}}(x_0)] \xrightarrow{\beta} 0$, which is impossible.

5. The condition (Q_2). We now analyse more precisely the condition (Q_2). Note that the postulate (b_4) implies the following consequence:

5.1. *If $(\beta)\lim\limits_q y_{pq} = y_p$ for $p=1,2,\ldots$, and $\lambda_q \to 0$, then there exist sequences $\{q_i\}$ and $\{t_p\}$ such that $i_p \geqslant t_p$ implies*

$$(\beta)\lim_p \lambda_{q_{i_p}} y_{pq_{i_p}} = 0.$$

5.2. Theorem. *If the space X_α satisfies the postulate (a_2) and the space Y_β satisfies the postulates (b_1), (b_2) and (b_4), then the condition (Q_2) is satisfied.*

Proof. Suppose the contrary. Then there exists an everywhere β-bounded sequence $\{U_n(x)\}$ of (X_α, Y_β)-linear operations, and an α-bounded sequence $\{x_n\}$, for which the sequence $\{U_n(x_n)\}$ is not β-bounded. Hence there exists a sequence $\vartheta_n \to 0$ such that $\vartheta_n U_n(x_n)$ is not β-convergent to 0, and, by (b_1), such that every partial sequence of it has the same property. Thus we can assume that $\vartheta_n > 0$. Put

$$\tau_n = \sqrt{\vartheta_n}, \qquad y_{pq} = U_p\left(\sqrt{\tau_q}\, x_q\right).$$

Since $\sqrt{\tau_q}\, x_q \overset{\alpha}{\to} 0$, we have

$$(\beta) \lim_q y_{pq} = (\beta) \lim_q U_p\left(\sqrt{\tau_q}\, x_q\right) = 0,$$

and by (b_4) there exists a sequence of indices $\{s_n\}$ such that

$$(2) \qquad (\beta) \lim_p \sum_{n=p}^{\omega_p} \varepsilon_n \sqrt{\tau_{s_n}}\, U_p\left(\sqrt{\tau_{s_n}}\, x_{s_n}\right) = (\beta) \lim_p \sum_{n=p}^{\omega_p} \varepsilon_n U_p\left(\tau_{s_n} x_{s_n}\right) = 0,$$

ε_n being zeros or ones and $\omega_p \geqslant p$. Arrange the elements of the form

$$\varepsilon_1 \tau_{s_1} x_{s_1} + \varepsilon_2 \tau_{s_2} x_{s_2} + \ldots + \varepsilon_n \tau_{s_n} x_{s_n},$$

where $\varepsilon_n = 0$ or 1 and $n = 1, 2, \ldots$, in a sequence $\{z_n\}$, and put $y^*_{pq} = \sqrt{\tau_q}\, U_{s_q}(z_p)$. Since $(\beta) \lim_q y^*_{pq} = 0$ for $p = 1, 2, \ldots$, Theorem 5.1 implies the existence of two sequences of indices $\{r_p\}$ and $\{t^*_p\}$ such that $n_p = s_{q_{r_p}}$ and $n_p \geqslant t^*_p$ implies

$$(3) \qquad (\beta) \lim_p \tau_{n_p} U_{n_p}(z_p) = 0$$

(it is sufficient to put $t^*_p = s_{q_{t_p}}$). We now construct a sequence $\{\nu_p\}$ extracted from $\{n_p\}$ as follows: put $\nu_1 = n_1$ and suppose $\nu_1, \ldots, \nu_{k-1}$ determined. Choose M so that all the elements of the form

$$\varepsilon_1 \tau_{\nu_1} x_{\nu_1} + \varepsilon_2 \tau_{\nu_2} x_{\nu_2} + \ldots + \varepsilon_{k-1} \tau_{\nu_{k-1}} x_{\nu_{k-1}}$$

with $\varepsilon_i = 0$ or 1, and $i = 1, 2, \ldots, k-1$, appear in the sequence $z_1, z_2, \ldots, z_M$, and put

$$(4) \qquad l_k \geqslant \max(t^*_1, t^*_2, \ldots, t^*_M), \qquad \nu_k = n_{l_k};$$

ε_i being zeros or ones, we have by (2)

$$(5) \qquad (\beta) \lim_k \tau_{\nu_k} U_{\nu_k}(\varepsilon_1 \tau_{\nu_1} x_{\nu_1} + \ldots + \varepsilon_{k-1} \tau_{\nu_{k-1}} x_{\nu_{k-1}}) = 0,$$

and formula (2) implies

$$(6) \qquad (\beta) \lim_k \sum_{n=k+1}^{\omega_k} \varepsilon_n U_{\nu_k}(\tau_{\nu_n} x_{\nu_n}) = 0.$$

In (5) and (6) the sequence $\{\varepsilon_n\}$ may be chosen arbitrarily; hence they remain true if we replace the sequence $\{\nu_k\}$ by any subsequence $\{\nu^*_k\}$. From (a_2) follows the existence of such a sequence for which the series $\sum_{k=1}^{\infty} \tau_{\nu^*_k} x_{\nu^*_k}$ is α-convergent; let x_0 be its limit. The operation $U_{\nu^*_k}(x)$ is (X_α, Y_β)-linear; hence

$$U_{\nu^*_k}(x) = \sum_{r=1}^{\infty} U_{\nu^*_k}(\tau_{\nu^*} x_{\nu^*_r}),$$

the series being β-convergent. By (5) d an(b_2) the sequence

$$\left\{ \sum_{r=k+1}^{\infty} U_{\nu^*_k}(\tau_{\sigma^*_r} x_{\sigma^*_r}) \right\}$$

is β-bounded. We have

$$\tau_{\nu^*_k} U_{\nu^*_k}(x_0) = \vartheta_{\nu^*_k} U_{\nu^*_k}(x_{\nu^*_k}) + \tau_{\nu^*_k} U_{\nu^*_k}(\tau_{\nu^*_1} x_{\nu^*_1} + \ldots + \tau_{\nu^*_{k-1}} x_{\nu^*_{k-1}})$$
$$+ \tau_{\nu^*_k} \sum_{r=k+1}^{\infty} U_{\nu^*_k}(\tau_{\sigma^*_r} x_{\sigma^*_r}).$$

In this formula the two last terms of the right-hand side are β-convergent to 0. The first term however is not β-convergent to 0. It follows that the sequence $\{U_{\nu^*_k}(x_0)\}$ is β-unbounded, contrarily to hypothesis.

In particular, the condition (Q_2) is satisfied in all the cases, if the α-convergence is strong two-norms convergence, or weak convergence in a Banach space, or strong convergence in a space conjugate to a B_0-space, or $\varkappa$-convergence in a Kantorovitch space, and if the β-convergence is convergence generated by norm in a F^*-space, or strong two-norms convergence, or $\varkappa$-convergence in a Kantorovitch space.

It follows from 5.2 that

5.3. *If the space X_α satisfies the postulates* (a$_1$) *and* (a$_2$), *and the Space Y_β satisfies the postulates* (b$_1$), (b$_2$) *and* (b$_4$), *then the condition* (Q$_3$) *is satisfied.*

6. General sufficient conditions for I^l and II^l. From sections 4 and 5 we get the following

6.1. Theorem. *If the space X_α satisfies the postulates* (a$_1$) *and* (a$_2$), *and the space Y_β satisfies the postulates* (b$_1$), (b$_2$) *and* (b$_4$), *then $I^l(X_\alpha, Y_\beta)$ is true* [21]).

In particular, $I^l(X_\alpha, Y_\beta)$ is true in all the cases of the α-convergence being

(1) convergence generated by norm in a F-space,
(2) $\varkappa$-convergence in a Kantorovitch space,
(3) $\varkappa$-convergence in the space $\boldsymbol{L}\{\boldsymbol{X}\}$ or $\boldsymbol{L}^p\{\boldsymbol{X}\}$,
(4) strong convergence in a space conjugate to a B_0-space,

and of the β-convergence being

(I) convergence generated by norm in a F^*-space,
(II) weak convergence in a Banach space,
(III) strong two-norms convergence,
(IV) $\varkappa^*$-convergence in a Kantorovitch space [22]).

6.2. Theorem. *If the space X_α satisfies the postulates* (a$_1$) *and* (a$_2$), *and the space Y_β satisfies the postulates* (b$_1$), (b$_2$), (b$_3$), (b$_4$) *and* (b$_5$), *then $II^l(X_\alpha, Y_\beta)$ is true* [23]).

In particular, $\mathrm{II}^l(X_\alpha, Y_\beta)$ holds in the case of α-convergence being one of the convergences (1)-(4) and of β-convergence being the convergence (III), or

(I*) the convergence generated by norm in a F-space,

or

(IV*) the $\varkappa^*$-convergence in a Kantorovitch space, under the supplementary hypothesis of (b$_5$) being satisfied.

[21]) This has been proved by Mazur and Orlicz [12] for the case of β-convergence being the convergence generated by norm.

[22]) The cases (2) and (IV) have been proved by Kantorovitch ([9], p. 257).

[23]) This has been proved by Mazur and Orlicz [12] for the case of β-convergence being the convergence generated by norm.

7. Special sufficient conditions for I^l and II^l. In some more specialized cases we can give other sufficient conditions for I^l and II^l to hold.

7.1. The case of β-convergence being two-norms convergence. Suppose the space X_α to satisfy the postulate (a$_2$), and β-convergence to be a strong two-norms convergence. Let $\{U_n(x)\}$ be a sequence of (X_α, Y_β)-linear operations β-convergent to $U(x)$ everywhere. By Theorem 5.2, $x_p \xrightarrow{\alpha} 0$ and $q_p \to \infty$ imply β-boundedness of the sequence $\{U_{q_p}(x_p)\}$. Since Y_β satisfies the postulate (b$_2$), and $(\beta)\lim\limits_q U_q(x_p) = U(x_p)$ for $p = 1, 2, \ldots$, the sequence $U(x_p)$ is β-bounded. Hence:

7.1.1. *Suppose the space X_α to satisfy the postulate* (a$_2$), *and the β-convergence to be strong two-norms convergence in Y. Denote by β' the convergence generated by the norm $\|y\|^*$ in Y. If $\mathrm{I}^l(X_\alpha, Y_{\beta'})$ holds, then $\mathrm{I}^l(X_\alpha, Y_\beta)$ holds also.*

Orlicz ([14], p. 78) has shown the truthfulness of $\mathrm{I}^l(\boldsymbol{M}_{\gamma'}, Y_\beta)$, the β-convergence being convergence generated by the norm in a F^*-space. Hence:

7.1.2. $\mathrm{I}^l(\boldsymbol{M}_{\gamma'}, Y_\beta)$ *holds in the case of the β-convergence being strong two-norms convergence* [24]).

For the sake of completeness we give here the proof of this theorem of Orlicz.

We may suppose that the space Y is a F-space. It suffices to prove that (Q$_1$) is satisfied. Denote by X_0 the set of all the elements of $\boldsymbol{M}_{\gamma'}$ for which $\|x\| \leqslant 1$. Introduce the distance in X_0 by the formula $\varrho(x_1, x_2) = \|x_1 - x_2\|^*$. We easily verify that X_0 is a complete metric space. We define the addition in X_0 in the usual manner, but only for such elements x_1, x_2 for which $\|x_1 + x_2\| \leqslant 1$; it is easy to see that X_0 is a pseudogroup of Saks ([1], p. 15).

Let $\{U_n(x)\}$ be a sequence of $(\boldsymbol{M}_{\gamma'}, Y_\beta)$-linear operations convergent everywhere to $U(x)$. The operations $V_n(x) = U_n(x|X_0)$, are additive and continuous in the pseudogroup of Saks X_0; it follows easily from [1], p. 16, that the condition (Q$_1$) is satisfied.

[24]) Fichtenholz ([7], p. 222) has shown that $\mathrm{I}^l(\boldsymbol{M}_{\gamma'}, \boldsymbol{M}_{\gamma'})$ is true.

It is easy to show that

7.1.3. *If the space X_α satisfies the postulates* (a_1) *and* (a_2), *and β-convergence is the weak convergence in a weakly complete*[25]) *Banach space or in a B_0-space, then* $\mathrm{II}^1(X_\alpha, Y_\beta)$ *holds.*

7.2. The case of functionals. Let $\boldsymbol{R}$ be the space of the reals with the usual definition of convergence.

7.2.1. *If the space X_α satisfies the postulate* (a_2'), *then* $\mathrm{I}^1(X_\alpha, \boldsymbol{R})$ *is true*[26]).

Proof. Let $\xi(x)$ be the limit of a convergent sequence $\xi_n(x)$ of (X_α)-linear functionals. Suppose $\xi(x)$ is not (X_α)-linear. Then there exists a sequence $x_n \overset{\alpha}{\to} 0$ and an $\varepsilon > 0$ such that $|\xi(x_n)| \geqslant \varepsilon$. We can simply suppose that $\xi(x_n) \geqslant \varepsilon$. By (a_2') there exists a sequence of indices $\{n_k\}$ such that the series $\sum_{k=1}^{\infty} x_{n_k}$ is α-convergent.

Put

$$x_k^* = \sum_{\nu=1}^{k} x_{n_\nu}.$$

Since $\lim_{n\to\infty} \xi_n(x_k^*) = \xi(x_k^*)$ for $k = 1, 2, \ldots$, there exists for each k a m_k such that $|\xi_{m_k}(x_k^*) - \xi(x_k^*)| < 1$. Hence

$$|\xi_{m_k}(x_k^*)| \geqslant |\xi(x_k^*)| - |\xi_{m_k}(x_k^*) - \xi(x_k^*)| \geqslant \xi(x_{n_1}) + \ldots + \xi(x_{n_k}) - 1 > (k-1)\varepsilon.$$

This is however impossible, since the α-boundedness of the sequence $\{x_k^*\}$ implies by 5.2 the boundedness of the sequence $\{\xi_{m_k}(x_k^*)\}$.

7.2.2. *If the space X_α satisfies the postulate* (a_2'), *and Y_β is the space of reals, then the condition* (Q_1) *is satisfied.*

Proof. Suppose the contrary; then there exists a sequence $\{\xi_n(x)\}$ of (X_α)-linear functionals, convergent to $\xi(x)$, and a sequence $\{x_n\}$ such that $x_n \overset{\alpha}{\to} 0$, and that $\xi_{m_n}(x_n) \geqslant \varepsilon$ for an $\varepsilon > 0$ and for a sequence $\{m_n\}$ of indices.

We now construct a subsequence $\{n_k\}$ extracted from $\{m_k\}$ as follows: put $n_1 = m_1$ and suppose $n_1, \ldots, n_{k-1}$ defined; choose then n_k so that

[25]) The space Y is *weakly complete* if the weak convergence in it satisfies the postulate (b_5).

[26]) proved by Orlicz (not yet published).

$$|\xi_p(x_{n_\nu}) - \xi(x_{n_\nu})| < \varepsilon 2^{-\nu-2} \quad \text{for } \nu = 1, 2, \ldots, k-1 \text{ and } p \geqslant n_k,$$

$$\left.\begin{aligned} |\xi_{n_\nu}(x_{n_k})| &< \varepsilon 2^{-k} \\ |\xi(x_{n_k})| &< \varepsilon 2^{-k} \end{aligned}\right\} \quad \text{for } \nu = 1, 2, \ldots, k-1.$$

This is possible, since $\xi_p(x) \to \xi(x)$ everywhere, and since $\xi_p(x)$ and $\xi(x)$ are (X_α)-linear. Thus the sequence $\{n_k\}$ is defined by induction.

By (a_2') we can suppose that the series $\sum_{\nu=1}^{\infty} x_{n_\nu}$ is α-convergent. Hence, for k sufficiently large,

$$|\xi_{n_k}(x_0) - \xi(x_0)| = \Big|\sum_{\nu=1}^{\infty} [\xi_{n_k}(x_{n_\nu}) - \xi(x_{n_\nu})]\Big|$$

$$\geqslant |\xi_{n_k}(x_{n_k}) - \xi(x_{n_k})| - \sum_{\nu=1}^{k-1} |\xi_{n_k}(x_{n_\nu}) - \xi(x_{n_\nu})| - \sum_{\nu=k+1}^{\infty} |\xi_{n_k}(x_{n_\nu})| - \sum_{\nu=k+1}^{\infty} |\xi(x_{n_\nu})|$$

$$\geqslant \varepsilon - 4\frac{\varepsilon}{2^k},$$

and on the other hand $\xi_{n_k}(x_0) \to \xi(x_0)$, which is impossible.

7.3. The condition of Fichtenholz. We shall say the β-convergence in Y satisfies the *condition of Fichtenholz*[27]) if $y_n \overset{\beta}{\to} y_0$ is equivalent to $\eta(y_n) \to \eta(y_0)$ for every (Y_β)-linear functional $\eta(y)$.

7.3.1. *If the β-convergence in Y satisfies the condition of Fichtenholz and* $\mathrm{I}^1(X_\alpha, \boldsymbol{R})$ *holds, then* $\mathrm{I}^1(X_\alpha, Y_\beta)$ *holds also.*

Proof. Let $\{U_n(x)\}$ be a sequence of (X_α, Y_β)-linear operations β-convergent to $U(x)$, and let $\eta(y)$ be any (Y_β)-linear functional. Put $\eta_n(x) = \eta(U_n(x))$. The functionals $\eta_n(x)$ are (X_α)-linear, and $\eta_n(x) \to \xi(x) = \eta(U(x))$. By hypothesis, $x_n \overset{\alpha}{\to} x_0$ implies

$$\eta(U(x_n)) \to \eta(U(x));$$

hence by the condition of Fichtenholz $U(x_n) \overset{\beta}{\to} U(x_0)$.

We can prove similarly that

7.3.2. *If the condition* (Q_1) *is satisfied for any sequence of (X_α)-linear functionals, and the β-convergence satisfies the condition of Fichtenholz, then the condition* (Q_1) *is satisfied for any sequence of (X_α, Y_β)-linear operations.*

[27]) Fichtenholz ([6], p. 197) has called *regular* a convergence satisfying this condition; we prefer to call it by the name of its author.

7.3.3. *If the β-convergence in Y satisfies* (b_5) *and the condition of Fichtenholz, and* $\mathrm{II}^I(X_\alpha, \boldsymbol{R})$ *holds, then* $\mathrm{II}^I(X_\alpha, Y_\beta)$ *holds also.*

Fichtenholz has shown ([6], p. 198) that the $\varkappa$-convergence in the space $\boldsymbol{M}^*$ satisfies the condition of Fichtenholz. Hence, if $\mathrm{I}^I(X_\alpha, \boldsymbol{R})$ or $\mathrm{II}^I(X_\alpha, \boldsymbol{R})$ is true, then $\mathrm{I}^I(X_\alpha, \boldsymbol{M}^*_\varkappa)$ or $\mathrm{II}^I(X_\alpha, \boldsymbol{M}^*_\varkappa)$ is also true respectively.

It follows e. g. that $\mathrm{I}^I(\boldsymbol{M}_{\gamma'}, \boldsymbol{M}^*_\varkappa)$ holds.

7.4. Theorem II^I in Kantorovitch spaces. Kantorovitch has shown ([9], p. 539) that $\mathrm{II}^I(X_\alpha, Y_\beta)$ is true if both X_α and Y_β are regular Kantorovitch spaces. We give a slight generalization of this result.

Suppose the space X_α satisfies the postulates (a_1) and (a_2), and $Y_\varkappa$ is a regular Kantorovitch space. The $\varkappa^*$-convergence satisfies then the postulates (a_1), (a_2), (b_1), (b_2'), (b_3) and (b_4).

An operation $U(x)$ from X_α to $Y_\varkappa$ will be said to be $(X_\alpha, Y_{\varkappa^*})$-*quasilinear* [28]) if it is $(X_\alpha, Y_{\varkappa^*})$-continuous and satisfies the conditions

$$|U(x+y)| \leqslant |U(x)| + |U(y)|, \qquad |U(\lambda x)| = |\lambda|\,|U(x)|.$$

It is easy to prove that $U(x)$ being any $(X_\alpha, Y_{\varkappa^*})$-quasilinear operation, and the series $\sum_{\nu=1}^{\infty} x_\nu$ being α-convergent with the sum x_0, we have

$$(7) \qquad |U(x-y)| \geqslant \big||U(x)| - |U(y)|\big|, \qquad |U(x_0)| \leqslant \sum_{\nu=1}^{\infty} |U(x_\nu)|.$$

7.4.1. *If the sequence* $\{U_n(x)\}$ *of* $(X_\alpha, Y_{\varkappa^*})$-*quasilinear operations is* $\varkappa^*$-*bounded everywhere, and the sequence* $\{x_n\}$ *is an* α-*bounded, then the sequence* $\{U_n(x_n)\}$ *is* $\varkappa^*$-*bounded.*

Proof. This theorem may be proved very much like the theorem 5.2. We must first only extract from the sequence $\{x_n\}$ a sequence $\{x_{n_k}\}$ such that the series $\sum_{k=1}^{\infty} |U_p(x_{n_k})|$ be convergent for $p = 1, 2, \dots$ This can be done by the diagonal method. Then the proof goes on like that of the theorem 5.2. In the final evaluations inequalities (7) will be used.

[28]) This definition resembles a notion introduced by Mazur and Orlicz ([10], p. 157).

Since the space X_α satisfies the postulate (a_1), we deduce from 7.4.1 as in section 5 that

7.4.2. *If the sequence* $\{U_n(x)\}$ *of* $(X_\alpha, Y_{\varkappa^*})$-*quasilinear operations is* $\varkappa^*$-*convergent to* $U(x)$ *everywhere, then the operation* $|U(x)|$ *is* $(X_\alpha, Y_{\varkappa^*})$-*continuous.*

7.4.3. Theorem. *Suppose the space* X_α *satisfies the postulates* (a_1) *and* (a_2), *and the space* $Y_\varkappa$ *is a regular Kantorovitch space. Let* $\{U_n(x)\}$ *be a sequence of* $(X_\alpha, Y_{\varkappa^*})$-*linear operations,* $\varkappa$-*bounded everywhere and* $\varkappa$-*convergent in a set* D, *dense in* X_α. *Then this sequence is everywhere* $\varkappa$-*convergent.*

Proof. Write

$$V_n(x) = \sup_{i=1,2,\dots,n} |U_i(x)|, \qquad W(x) = \overline{\lim_n}\, U_n(x) - \underline{\lim_n}\, U_n(x).$$

The operations $V_n(x)$ are obviously $(X_\alpha, Y_{\varkappa^*})$-quasilinear, and $V_n(x) = |V_n(x)|$. The sequence $\{V_n(x)\}$ is $\varkappa$-bounded everywhere and non-decreasing. Hence ([8], p. 132) it is $\varkappa$-convergent everywhere. Let $V(x)$ be the limit of this sequence. By 7.4.2 $V(x)$ is $(X_\alpha, Y_{\varkappa^*})$-continuous. The inequalities

$$|W(x)| \leqslant 2V(x), \qquad \big||W(x)| - |W(y)|\big| \leqslant |W(x-y)|$$

imply further the $(X_\alpha, Y_{\varkappa^*})$-continuity of $W(x)$. By hypothesis $W(x) = 0$ in D; hence $W(x) = 0$ everywhere.

Theorem 7.4.3 remains true if we replace the space $Y_\varkappa$ by $\boldsymbol{L}\{\boldsymbol{X}\}_\varkappa$ or $\boldsymbol{L}^p\{\boldsymbol{X}\}_\varkappa$.

8. Theorems III^I_1 and III^I_2. We now give conditions for X_α and Y_β which are sufficient for the truthfulness of $\mathrm{III}^I_1(X_\alpha, Y_\beta)$ and $\mathrm{III}^I_2(X_\alpha, Y_\beta)$. We restrict our considerations to the case of the β-convergence being convergence generated by the norm in a F-space or strong two-norms convergence.

8.1. Theorem. *If the space* X_α *satisfies the postulate* (a_3), *and if* β *is the convergence generated by norm in the space* Y, *then* $\mathrm{III}^I_1(X_\alpha, Y_\beta)$ *and* $\mathrm{III}^I_2(X_\alpha, Y_\beta)$ *are true.*

Proof. Let $\{U_{pq}(x)\}_{q=1,2,\dots}$ be a sequence of (X_α, Y_β) linear operations. Suppose that, given any p, there exists an element x_p such that the sequence $\{U_{pq}(x_p)\}_{q=1,2,\dots}$ is β-divergent. By (a_3) then there exists a sequence $\{\vartheta_n\}$ of numbers different from 0, such that $\sum_{n=1}^{\infty}|\lambda_n|<\infty$ implies the α-convergence of the series $\sum_{n=1}^{\infty}\lambda_n\vartheta_n x_n$. The sequence $\{U_{pq}(\vartheta_p x_p)\}_{q=1,2,\ldots}$ is obviously β-divergent for $p=1,2,\ldots$ Write, $z=\{\lambda_n\}$ being any element of the space $\boldsymbol{l}$,

$$(8)\qquad U_{pq}^*(z)=U_{pq}\Big(\sum_{\nu=1}^{\infty}\lambda_\nu\vartheta_\nu x_\nu\Big),\qquad U_{pqn}^*(z)=U_{pq}\Big(\sum_{\nu=1}^{n}\lambda_\nu\vartheta_\nu x_\nu\Big).$$

The last operations are $(\boldsymbol{l},Y_\beta)$-linear, and $(\beta)\lim_n U_{pqn}^*(z)=U_{pq}^*(z)$ everywhere. Hence $U_{pq}^*(z)$ also are $(\boldsymbol{l},Y_\beta)$-linear. Put $z_p=\{\delta_{pq}\}_{q=1,2,\ldots}$, δ_{pq} denoting the delta of Kronecker. The sequence $\{U_{pq}^*(z_p)\}_{q=1,2,\ldots}$ is β-divergent for $p=1,2,\ldots$ Since $\mathrm{III}_1^l(\boldsymbol{l},Y_\beta)$ holds ([10], p. 156), there exists an element $z_0=\{\lambda_n^0\}\epsilon\,\boldsymbol{l}$ such that the sequences $\{V_{pq}(z_0)\}_{q=1,2,\ldots}$ are β-divergent for $p=1,2,\ldots$ It follows that the sequences $\{U_{pq}(x_0)\}_{q=1,2,\ldots}$ are β-divergent for $p=1,2,\ldots$, x_0 being the element $\sum_{\nu=1}^{\infty}\lambda_\nu^0\vartheta_\nu x_\nu$.

The proof of $\mathrm{III}_2^l(X_\alpha,Y_\beta)$ to hold is similar.

8.2. Theorem. *If the space X_α satisfies the postulate* (a_3), *and if the β-convergence is strong two-norms convergence, then* $\mathrm{III}_1^l(X_\alpha,Y_\beta)$ *and* $\mathrm{III}_2^l(X_\alpha,Y_\beta)$ *are true.*

Proof. We first prove $\mathrm{III}_2^l(X_\alpha,Y_\beta)$. Let $\{U_{pq}(x)\}_{q=1,2,\ldots}$ be a sequence of (X_α,Y_β)-linear operations. Suppose, given any p, there exists an element x_p such that the sequence $\{U_{pq}(x_p)\}_{q=1,2,\ldots}$ is β-unbounded. Choosing the sequence $\{\vartheta_n\}$ like in the proof of Theorem 8.1, define the operations $U_{pq}^*(z)$ and $U_{pqn}^*(z)$ by (8). These operations are $(\boldsymbol{l},Y_\beta)$-linear. This is obvious for $U_{pqn}^*(z)$, and for $U_{pq}^*(z)$ it follows from $U_{pqn}^*(z)\xrightarrow{\beta}U_{pq}^*(z)$ and from Theorem 6.1. Now, we can finish the proof as that of Theorem 8.1.

To prove $\mathrm{III}_1^l(X_\alpha,Y_\beta)$ let $\{U_{pq}(x)\}_{q=1,2,\ldots}$ be a sequence of (X_α,Y_β)-linear operations. Suppose, given any p, there exists an element x_p such that the sequence $\{U_{pq}(x_p)\}_{q=1,2,\ldots}$ is β-divergent. Thus the sequence of sequences $\{U_{pq}(x)\}_{q=1,2,\ldots}$ can be decomposed in two, $\{V_{pq}(x)\}_{q=1,2,\ldots}$ and $\{W_{pq}(x)\}_{q=1,2,\ldots}$, the first of which is divergent in Y^* for $x=x_p^1$, and the other is unbounded in Y for $x=x_p^2$. Putting, as in the proof of Theorem 8.1,

$$V_{pq}^*(z)=V_{pq}\Big(\sum_{\nu=1}^{\infty}\lambda_\nu\vartheta_\nu x_\nu\Big),\qquad W_{pq}^*(z)=W_{pq}\Big(\sum_{\nu=1}^{\infty}\lambda_\nu\vartheta_\nu x_\nu\Big),$$

we easily see ([10], p. 156) that there are in the space $\boldsymbol{l}$ two residual sets, R_1 and R_2, such that the sequences $\{V_{pq}^*(z)\}_{q=1,2,\ldots}$ are divergent in Y^* for every $x\,\epsilon\,R_1$ and $p=1,2,\ldots$, and that the sequences $\{W_{pq}^*(z)\}_{q=1,2,\ldots}$ are unbounded in Y for every $x\,\epsilon\,R_2$ and $p=1,2,\ldots$ We finish the proof choosing an element $z_0=\{\lambda_\nu^0\}\,\epsilon\,R_1R_2$ and putting $x_0=\sum_{\nu=1}^{\infty}\lambda_\nu^0\vartheta_\nu x_\nu$; the sequence $\{U_{pq}(x_0)\}_{q=1,2,\ldots}$ is obviously β-divergent for $p=1,2,\ldots$

From the above theorems it follows in particular that $\mathrm{III}_1^l(X_\alpha,Y_\beta)$ and $\mathrm{III}_2^l(X_\alpha,Y_\beta)$ are true in the case of β-convergence being convergence generated by norm in a F-space, or strong two-norms convergence, and of X_α being any one of the spaces considered in sections 2.1-2.3, 2.7-2.13 and 2.15.

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Linear operations in Saks spaces (I)

by

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This paper deals with metric spaces composed of elements of the unit sphere of a linear normed space, the metric of which is defined (see 1.3) by means of another norm, not necessarily homogeneous. The spaces of this kind may be considered as pseudolinear in a certain sense, and some investigations of Banach spaces can be adapted to the spaces of this kind [1]).

1.1. Let X be a linear space. A functional $\|x\|$ defined in X will be called a *B-norm* if it satisfies the following conditions:

(a) $\|x\|=0$ *if and only if* $x=0$,

(b) $\|x+y\|\leqslant\|x\|+\|y\|$,

(c) $\|\vartheta x\|=|\vartheta|\,\|x\|$, ϑ *being any real number.*

Each functional $\|x\|$ satisfying the above conditions (a), (b) and the following one:

(c′) *if the sequence* $\{\vartheta_n\}$ *of real numbers tends to* ϑ *and* $\|x_n-x\|\to 0$, *then* $\|\vartheta_n x_n-\vartheta x\|\to 0$

will be said to be a *F-norm.*

Any functional $\|x\|$ satisfying the conditions (b) and (c), or (b) and (c′), will be termed a *B-* or *F-pseudonorm* respectively.

A *Banach space* or a *Fréchet space* is a linear space X provided with a *B-* or *F*-norm (i.e. Banach norm or Fréchet norm) respectively and such that the distance

$$d(x,y)=\|x-y\|$$

makes X a complete metric space.

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[1]) The results of this paper were presented September 26th 1948 at the VI Polish Mathematical Congress in Warsaw. The second part of the present paper (to appear) will deal with investigation of sequences of operations and with applications of the results of part I.