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On sequences of operations (I)

by

A. ALEXIEWICZ (Poznań).

In the theory of linear operations a group of theorems can be distinguished which are particularly often applied in various branches of Analysis.

To this class belong of course the theorems referring to the sequences of linear operations, the most important of which may be put into one of the three following forms. $\boldsymbol{K}$ denotes there generally a class of continuous operations defined in a properly chosen space, satisfying there certain conditions adapted to the following cases:

I. If $\{U_n(x)\}$ denotes an everywhere convergent sequence of operations belonging to $\boldsymbol{K}$, then their limit is an operation belonging to $\boldsymbol{K}$.

II. If a sequence $\{U_n(x)\}$ of operations belonging to $\boldsymbol{K}$ is bounded everywhere and converges in a dense set of x, then it is convergent everywhere.

III (III*). Let $\{U_{pq}(x)\}_{q=1,2,\dots}$ be a sequence of operations belonging to $\boldsymbol{K}$ for $p=1,2,\dots$ If for each p there is a point x_p such that the sequence $\{U_{pq}(x_p)\}_{q=1,2,\dots}$ is divergent (unbounded), then there is a point x_0 at which these sequences diverge (are unbounded) simultaneously for $p=1,2,\dots$

The theorems of the forms I and II formulated for a special class $\boldsymbol{K}$ may be found in the well-known paper of Lebesgue [10]. The study of operations of this class has been continued by Hobson [9], Haar [6], Hahn [7], and many other geometers. The more general class $\boldsymbol{K}$ for which the theorems of the forms I, II, III still hold is that of linear operations defined in a Banach space. Their applications to the study of the convergence of Fourier developments and orthogonal developments, to the linear interpolation, to so called singular integrals and so on, are well known.

The theorem I for the considered class $\boldsymbol{K}$ was proved for the first time by Banach in his thesis [2], the theorem II „of resonance“ and III „about condensation of singularitaties“ by him and Steinhaus [4]. Afterwards Banach [1, 3] proved I and III for a class $\boldsymbol{K}$ of linear operations defined in the spaces of type F and G [1]). Mazur and Orlicz [11] proved the theorems II and III* for the same class in a space of type F. To the same authors [12] we owe the theorems of the forms I, II, III and III* for the class $\boldsymbol{K}$ of polynomials of degree at most m in the spaces of type F.

All the preceding theorems may be demonstrated by the so called method of category which is based on the theorem of Baire and has been introduced into this domain of research by Saks (see Banach-Steinhaus [4]). Hahn [8] and Saks [18] have demonstrated theorems I and III by the same method for a class $\boldsymbol{K}$ of non-linear operations defined in a non-linear metric space.

Now arises the problem of developing a general theory involving as particular cases the preceding theorems of the forms I, II, III and III*. This is precisely the content of the first part of this paper [2]).

I study here certain classes $\boldsymbol{K}$ of continuous functions, the argument and the values of which are varying in very general metric spaces which will be specialized later on. I introduce some intrinsic properties of those classes. These properties form as weak as possible sufficient conditions for the validity of the theorems I and III.

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[1]) I use the standard terminology introduced by Banach in his treatise [1].

[2]) Presented with some insignificant alterations as Doctor Thesis, on the March 10, 1944, to the secret University in Lwów, during the terror of the German occupation.

I wish to express my deepest gratitude and my esteem to Professors A. Zierhoffer (Dean), W. Orlicz, W. Nikliborc and W. Rubinowicz (Members of the Commission of the Department), who worked in that tragical time, regardless of their personal danger. I owe my greatest thanks especially to my Professor, Mr W. Orlicz, for having suggested me the theme of this paper, and for his help, his encouragement and his stimulations.

In the three further parts of this paper I shall deal with the classes of linear and polynomial operations defined in linear spaces with the notion of limit. I consider the axioms concerning the convergence of sequences of elements in these spaces and I establish a set of axioms sufficient for the validity of theorems I, II, III and III*.

Terminology and notations.

Any subset R of a metric space whose complement is of Baire's first category will be called *residual*. The elements of a metric space will be called *points*.

Following the usage of Banach the functions from an abstract set to an arbitrary set will be called *operations*, except those whose domain is any family of sets and which will be termed *set functions*. The operations from an arbitrary set to the reals will be called *functionals*. The term *function* is reserved for the operations from the reals to the reals.

$U(x)$ being any operation defined in X, and X_0 being any subset of X, $U(x|X_0)$ will denote the same operation with the domain restricted to the set X_0.

We shall use the following notations:

1^0 $\underset{x}{E}\{w(x)\}$, the set of elements x satisfying the condition $w(x)$.

2^0 $\varrho(x,y)$, the distance between the points x and y of any metric space.

3^0 $\bar{E}$, the closure of the set E.

4^0 $K(x_0,r)$, the open sphere with centre x_0 and radius r, i. e. the set $\underset{x}{E}\{\varrho(x,x_0)<r\}$.

5^0 X^k, the Cartesian product $\underbrace{X\times\ldots\times X}_{k}$ (k times) of the space X.

6^0 $\|x\|$, the norm of the element x in Banach space.

7^0 Ξ and Υ, the spaces conjugate respectively to the Banach spaces X and Y.

8^0 $\xi(x)$ and $\eta(y)$, the elements of Ξ and Υ respectively.

9^0 $|E|$, the Lebesgue measure of the set E.

1. Unless the contrary is explicitly stated all the operations considered in this paper are supposed to be continuous and defined from a complete metric space X to a metric space Y without other restrictions depending upon the spaces considered.

The operations of a given class will be termed *equicontinuous* at the point x_0 if, given any $\varepsilon>0$, there exists a $\delta>0$ such that $\varrho(x,x_0)<\delta$ implies $\varrho(U(x),U(x_0))<\varepsilon$ for every operation $U(x)$ belonging to this class; such operations will be said to be *equicontinuous in the set* $R\subset X$, if they are equicontinuous at any point of the set R; finally, if $R=X$, they will be termed simply *equicontinuous*.

Moreover, if the operations of a family are equicontinuous in a set R, and the number δ in the above definition can be chosen independently of x_0, the equicontinuity will be said to be *uniform* in R. The equicontinuity of a sequence of operations may be understood as that of the class composed of the terms of this sequence.

The following properties of sequences of operations will play a fundamental role throughout this paper:

(r_1) Convergence in a residual set implies equicontinuity,

(r_2) Convergence in a set of the second category implies equicontinuity,

(r_3) Convergence in a set of the second category implies uniform equicontinuity,

(c) Convergence in a sphere implies convergence everywhere.

(a) and (b) being any of the above properties, (ab) will denote their logical product.

Let (x) be any one of the above properties or the logical product of some of them, and let $\boldsymbol{K}$ be any class of operations from X to Y. The class $\boldsymbol{K}$ will be said to be a *(x)-class* if every sequence of operations of the class $\boldsymbol{K}$ has the property (x). E. g. a class $\boldsymbol{K}$ is a (cr_1)-class, if each sequence of operations of this class has the properties (c) and (r_1).

In the sequel we will need the following theorems which may be considered as a generalization of a classical theorem of Arzelà:

(1.1) *Suppose the operations $U(x)$ to be equicontinuous at the point x_0, and convergent to $U(x)$ in a sphere about x_0; then $U(x)$ is continuous at x_0.*

(1.2) *Suppose the operations $U_n(x)$ to be uniformly equicontinuous and convergent to $U(x)$ in X; then $U(x)$ is uniformly continuous (in X).*

(1.3) *The space Y being complete, suppose the sequence of operations $\{U_n(x)\}$ to be equicontinuous and convergent in a set D dense in the space X; then the sequence $\{U_n(x)\}$ converges in the whole of X.*

The converse to (1.1) is not true in general: the equicontinuity of a convergent sequence at x_0 is not a necessary condition for the continuity of its limit at x_0. However, if we neglect the sets of the first category, the convergence of a sequence of operations implies its equicontinuity. This fact is established in the following

Theorem 1. *Let $\{U_n(x)\}$ be a sequence of operations convergent in a residual set A; then the sequence $\{U_n(x)\}$ is equicontinuous in a residual set B.*

Proof[3]. Consider the space Z of convergent sequences $z=\{y_n\}$ of elements of the space Y, the distance between the elements $z_1=\{y_{1n}\}$ and $z_2=\{y_{2n}\}$ being defined by the formula

$$\varrho(z_1,z_2)=\sup_{n=1,2,\dots}\frac{\varrho(y_{1n},y_{2n})}{1+\varrho(y_{1n},y_{2n})}:$$

we can easily see that the space Z is metric. The sequence $\{U_n(x)\}$ may be considered as an operation $W(x)$ from X to Z. The continuity at x_0 of the operation $W(x)$ is obviously equivalent to the equicontinuity of the sequence $\{U_n(x)\}$ at x_0. Put

$$U_{ni}(x)=\begin{cases}U_n(x) & \text{for } n\leqslant i,\\ U_i(x) & \text{for } n\geqslant i.\end{cases}$$

$$W_i(x)=\{U_{ni}(x)\}_{n=1,2,\dots}$$

Since the operations $W_i(x)$ are continuous in X, and the sequence $\{W_i(x)\}$ converges to $W(x)$ in A, we get by the well-known argument of Baire that $W(x)$ is continuous in a residual set $B\subset X$.

(1.4). Corollary. *Let $\{U_n(x)\}$ be a sequence of operations convergent in a set A of the second category; then this sequence is equicontinuous in a set B of the second category.*

[3]) This proof replaces a direct one given originally by the author, and has been suggested by S. Mazur.

Proof. As the set R of the points of convergence of the sequence $\{U_n(x)\}$ satisfies the condition of Baire, there exists a sphere K such that the set $K-R$ is of the first category. It is sufficient now to apply Theorem 1 replacing the space X by the set K.

2. Unless the contrary is explicitly stated, from this section on the space Y is supposed to be complete.

Theorem 2. *Suppose $\{U_n(x)\}$ to be a sequence of operations belonging to a (r_2)-class ((r_3)-class), convergent in a set R of the second category. Then the sequence $\{U_n(x)\}$*

(2.1) *is equicontinuous (uniformly equicontinuous),*

(2.2) *converges in $\overline{R}$,*

(2.3) *the limit-operation is continuous (uniformly continuous) in R.*

Proof. The set $\overline{R}$ being closed in X, it may be considered as a complete metric space; thus (2.1) results from the definition of a (r_2)-class ((r_3)-class) and from Theorem 1; (2.2) and (2.3) result from (1.3) and (1.2).

Under the hypotheses of Theorem 2, if we suppose, moreover, that the set R is dense in X, the sequence $\{U_n(x)\}$ converges in X to a continuous (uniformly continuous) limit.

Theorem 3. *Let $\{U_n(x)\}$ be a sequence of operations belonging to a (cr_2)-class ((cr_3)-class), convergent in a set R of the second category; then the sequence $\{U_n(x)\}$ is convergent everywhere and its limit is continuous (uniformly continuous).*

Proof. This results from (2.2), since $\overline{R}$ contains a sphere.

Theorems 2 and 3 imply

(2.4) *Given any sequence $\{U_n(x)\}$ of operations belonging to a (r_2)-class ((cr_2)-class), the set of the points of convergence of this sequence is either of the first category, or it contains a sphere (is identical with the space X).*

It is easily seen that the set of the points of convergence of a sequence of operations belonging to a (r_1)-class need not be identical with the space X. As a trivial example consider the sequence of functions $U_n(x)=\max[t,(-1)^n t]$ of the real variable t, convergent for $t\geqslant 0$. The following proposition is obvious:

(2.5) *Given a (r_2)-class $\boldsymbol{K}$ of operations, every sequence of those operations converges either in a set of the first category or in the whole of X, if and only if $\boldsymbol{K}$ has the property (c).*

In Theorems 2 and 3 we can omit the hypothesis that the space Y is complete, assuming that the sequence in question is convergent everywhere. We can easily prove the following proposition:

(2.6) *The space Y being arbitrary (not necessarily complete), the limit of any convergent sequence of operations belonging to a (r_1)-class ((r_3)-class) is continuous (uniformly continuous).*

3. In this section we give an application of Theorem 1.

Let X be a complete metric space, Y a Banach space, Υ the space conjugate to Y, and let Υ_0 be the unit sphere in Υ.

An operation $U(x)$ from X to Y is called *weakly continuous*, if, given any element $\eta=\eta(y)\in\Upsilon$, the functional $\eta(U(x))$ is continuous.

Let $X_0\subset X$, and let us denote by $\boldsymbol{K}(X_0)$ the family of the functionals of the form $\eta(U(x|X_0))$ where $\eta\in\Upsilon_0$, and consider the following conditions:

(i) *the range of $U(x)$ is separable, and $\boldsymbol{K}(X)$ is a (r_1)-class,*

(ii) *given any sequence $\{x_n\}$ of elements of X, there exists a separable closed set $X_0\subset X$ such that $x_n\in X_0$ for $n=1,2,\ldots$ and $\boldsymbol{K}(X_0)$ is a (r_1)-class.*

Theorem 4. *Suppose the operation $U(x)$ to be weakly continuous and one of the conditions* (i), (ii) *satisfied; then the operation $U(x)$ is continuous.*

Proof. Suppose first the condition (i) satisfied; the linear closed span of the range of $U(x)$ being separable we can suppose without loss of generality that the space Y is so. Suppose now, the operation $U(x)$ is discontinuous at x_0; then there is an $\varepsilon>0$ and a sequence $\{x_n\}$ such that $x_n\to x_0$ and

$$\|U(x_n)-U(x_0)\|\geqslant\varepsilon. \tag{3.1}$$

By a theorem of Banach ([1], p. 55) there are elements $\eta_n\in\Upsilon_0$ such that

$$\eta_n(U(x_n)-U(x_0))=\|U(x_n)-U(x_0)\|. \tag{3.2}$$

Let $\{y_p\}$ be a sequence of elements dense in Y. Since $|\eta_n(y_p)|\leqslant\|y_p\|$ for $n=1,2,\ldots$, we can select a subsequence $\{\eta_{n_k}(y)\}$ such that $\lim\limits_{k\to\infty}\eta_{n_k}(y_p)$ exists for $p=1,2,\ldots$ Since $\|\eta_n\|\leqslant 1$, by the Banach-Steinhaus theorem ([4], p. 33) the sequence $\{\eta_{n_k}(y)\}$

converges everywhere. The condition (i) being satisfied the sequence $\{\eta_{n_k}(U(x))\}$ is equicontinuous at x_0, contrarily to (3.1) and (3.2).

Let now the condition (ii) be satisfied, and suppose there exists a sequence $x_n \to x_0$ for which the inequality (3.1) holds. We will prove that the range of the operation $U(x|X_0)$ is separable. Let $\{z_n\}$ be a sequence dense in X_0 and put $y_n = U(z_n)$; denoting by Y_1 the set of the linear combinations with rational coefficients of the elements y_n, we see that the set Y_1 is denumerable. If $x \in X_0$, there exists a sequence $z_{n_k} \to x$; since $U(x)$ is weakly continuous, we have $\eta(U(z_{n_k})) \to \eta(U(x))$ for each $\eta = \eta(y) \in \Upsilon$; hence by a theorem of Banach ([1], p. 134) there exists a sequence of linear combinations with rational coefficients of the elements $y_{n_k} = U(z_{n_k})$ convergent to $U(x)$. Thus the set Y_1 is dense in the range of $U(x|X_0)$. To prove the theorem it is sufficient to apply the first part of this proof.

Remarks. Theorem 4 is true also if X is a B_0-space[4].

Suppose Υ_1 is a subset of Υ satisfying the following conditions: (a) there is a constant M such that $\|\eta\| \leqslant M$ for each $\eta \in \Upsilon_1$, (b) there is a constant m such that given any $y \in Y$, there exists an element $\eta \in \Upsilon_1$ such that $\eta(y) \geqslant m\|y\|$. Then, we can replace in Theorem 4 the condition (i) by the following weaker one:

(i') *the range of $U(x)$ is separable and the family of the functionals of the form $\eta(U(x))$, where $\eta \in \Upsilon_1$, is a (r_1)-class.*

4. The theorems of section 2 enable us to state some theorems on condensation of singularities.

Theorem 5. *Let $\{U_{pq}(x)\}_{q=1,2,\dots}$ be a sequence of operations belonging for fixed p to a (r_2)-class K_p. If for each p there exists an element x_p at which the sequence $\{U_{pq}(x)\}_{q=1,2,\dots}$ is not equicontinuous, then there exists a residual set R such that the sequences $\{U_{pq}(x)\}_{q=1,2,\dots}$ are divergent whatever be p and $x \in R$.*

Proof. Let H_p be the set of convergence of the sequence $\{U_{pq}(x)\}_{q=1,2,\dots}$. This set is of the first category, since, in the contrary case, the sequence $\{U_{pq}(x)\}_{q=1,2,\dots}$ would be, by Theorem 2, equicontinuous. It suffices to put

$$R = X - (H_1 + H_2 + \dots).$$

[4]) Concerning the definition see, for instance, Eidelheit [5], p. 140.

From (2.4) follows immediately

Theorem 6. *Let $\{U_{pq}(x)\}_{q=1,2,\dots}$ be a sequence of operations belonging for fixed p to a (cr_2)-class K_p. If for each p there exists an element x_p such that the sequence $\{U_{pq}(x_p)\}_{q=1,2,\dots}$ diverges, then there exists a residual set R such that the sequences $\{U_{pq}(x)\}_{q=1,2,\dots}$ are divergent whatever be p and $x \in R$.*

Theorem 7. *Let $\{U_{pq}(x)\}_{q=1,2,\dots}$ be a sequence of operations belonging for fixed p to a (r_2)-class K_p. If for each p there exists an element x_p and a set D_p dense in X such that the sequence $\{U_{pq}(x)\}_{q=1,2,\dots}$ diverges for $x = x_p$ and converges for $x \in D_p$, then there exists a residual set R such that the sequences $\{U_{pq}(x)\}_{q=1,2,\dots}$ diverge whatever be p and $x \in R$.*

Proof. By Theorem 2, the set H_p of convergence of the sequence $\{U_{pq}(x)\}_{q=1,2,\dots}$ is of the first category; it is sufficient to put

$$R = X - (H_1 + H_2 + \dots).$$

5. In this section some theorems concerning the condensation of the „unboundedness" will be established. Hence we must restrict the space Y to be such that the notion of bounded sequences may be introduced.

The space Y is supposed to be a F-space (Banach [1], p. 33). The sequence $\{y_n\}$ of elements is called *bounded*[5]) if $\vartheta_n y_n \to 0$ for any sequence $\{\vartheta_n\}$ of real numbers $\vartheta_n \to 0$.

We need the following properties of sequences of operations:

(b) Boundedness in a sphere implies boundedness in the whole of X.

If any sequence of operations of a family K has the property (b), K will be said to be a *(b)-class.*

The family K of operations will be said to be an *(h)-class* if it has the following property:

(h) If $U(x) \in K$ and λ is a real number, then $\lambda U(x) \in K$.

Analogously as in section 2, we define the (br_1)-, (hr_1)-, (bhr_1)-classes, etc.

[5]) In the sense of Banach; see Mazur and Orlicz [11].

Theorem 8. *The set S of the points of boundedness of a sequence $\{U_n(x)\}$ of operations belonging to a (hbr_2)-class $\boldsymbol{K}$ is either of the first category, or identical with X.*

Proof. Suppose the set S to be of the second category; then there is a sphere $K\subset\bar{S}$. We will prove that $K\subset S$. In the contrary case, there would exist an element $x_0\,\epsilon\,K-S$. Thus we may find real numbers ε, λ_n and indices k_n such that $\varepsilon>0$, $\lambda_n\to 0$ and

$$(5.1)\qquad \|\lambda_n U_{k_n}(x_0)\|\geqslant\varepsilon.$$

Since the operations $V_n(x)=\lambda_n U_{k_n}(x)$ belong to $\boldsymbol{K}$ and since $V_n(x)\to 0$ in the set S of the second category, the operations $V_n(x)$ are, by the (r_2)-property, equicontinuous. Hence there is a number $\delta>0$ such that $\varrho(x,x_0)<\delta$ implies $\|V_n(x)-V_n(x_0)\|<{}^{\varepsilon}/_3$ for $n=1,2,\dots$ Obviously $x_0\,\epsilon\,\bar{S}-S$. It follows that there exists an element $x_1\,\epsilon\,S$ for which $\varrho(x_1,x_0)<\delta$. Choose M so that $\|V_n(x_1)\|<{}^{\varepsilon}/_3$ for $n>M$. Hence, for almost all n we have

$$\|V_n(x_0)\|\leqslant\|V_n(x_1)-V_n(x_0)\|+\|V_n(x_1)\|<{}^{2}/_{3}\,\varepsilon<\varepsilon,$$

contrarily to (5.1). Thus $K\subset S$ and by the (b)-property $X\subset S$.

From the above proof we infer easily

(5.2) *The set of the points at which a sequence of operations of a (hr_2)-class is bounded is either of the first category, or contains a sphere.*

Theorem 8 implies the

Theorem 9. *Let $\{U_{pq}(x)\}$ be a sequence of operations belonging for fixed p to a (hbr_2)-class $\boldsymbol{K}_p$. If for each p there exists an element x_p such that the sequence $\{U_{pq}(x_p)\}_{q=1,2,\dots}$ is unbounded, then there exists a residual set $R\subset X$ such that the sequences $\{U_{pq}(x)\}_{q=1,2,\dots}$ are unbounded whatever be p and $x\,\epsilon\,R$.*

The theorem (5.2) implies the following one:

(5.3) *Let $\{U_{pq}(x)\}_{q=1,2,\dots}$ be a sequence of operations belonging for fixed p to a (hr_2)-class $\boldsymbol{K}_p$. If for each p there exists en element x_p and a set D_p dense in X such that the sequence $\{U_{pq}(x)\}_{q=1,2,\dots}$ is bounded for $x\,\epsilon\,D_p$ and unbounded for $x=x_p$, then there exists a residual set $R\subset X$ such that the sequences $\{U_{pq}(x)\}_{q=1,2,\dots}$ are unbounded whatever be p and $x\,\epsilon\,R$.*

6. The theorems of the foregoing section can be proved under changed hypotheses. We can namely weaken in Theorem 8 the hypothesis of $\boldsymbol{K}_p$ being a (hbr_2)-class strengthening at the same time the hypotheses about X and Y.

Theorem 10. *Let X be connected*[6]*), Y a Banach space, and let $\{U_n(x)\}$ be a sequence of operations belonging to a (hr_2)-class $\boldsymbol{K}$. The set of the points at which $\overline{\lim\limits_{n\to\infty}}\|U_n(x)\|<\infty$ is either of the first category, or identical with X.*

Proof. Write

$$S=\underset{x}{E}\{\overline{\lim_{n\to\infty}}\|U_n(x)\|<\infty\},\qquad R=X-S.$$

It is to be proved that, if S is of the second category, then $R=0$. Suppose the contrary. The space X being connected, we have $RS'+R'S\neq 0$. Hence, we have to consider two cases:

1° Suppose $RS'\neq 0$. Let $x_0\,\epsilon\,RS'$; thus

$$\overline{\lim_{n\to\infty}}\|U_n(x_0)\|=\infty,$$

and there are points $x_n\,\epsilon\,S$ such that $x_n\to x_0$. A sequence of indices $\{n_k\}$ may be chosen so that $\lim\limits_{k\to\infty}\|U_{n_k}(x_0)\|=\infty$. There are real numbers $\lambda_n\to 0$ for which $\overline{\lim\limits_{k\to\infty}}\|\lambda_k U_{n_k}(x_0)\|=\infty$. The operations $V_k(x)=\lambda_k U_{n_k}(x)$ belong to $\boldsymbol{K}$ and

$$(6.1)\qquad \overline{\lim_{k\to\infty}}\|V_k(x_0)\|=\infty.$$

If $x\,\epsilon\,S$, then $\lim\limits_{k\to\infty}\|V_k(x)\|=0$; by the (r_2)-property the sequence $\{V_n(x)\}$ is equicontinuous, thus there exists a $\delta>0$ such that $\varrho(x,x_0)<\delta$ implies $\|V_k(x)-V_k(x_0)\|\leqslant 1$ for each k.

Choose m to have $\varrho(x_m,x_0)<\delta$. Since $x_m\,\epsilon\,S$ we have $\lim\limits_{k\to\infty}\|V_k(x_m)\|=0$, and the inequality

$$\|V_k(x_0)\|\leqslant\|V_k(x_0)-V_k(x_m)\|+\|V_k(x_m)\|$$

gives contrarily to (6.1)

$$\overline{\lim_{k\to\infty}}\|V_k(x_0)\|\leqslant 1+\overline{\lim_{k\to\infty}}\|V_k(x_m)\|=1.$$

[6]) A space is said to be *connected* if it is not the sum of two closed, non vacuous and disjoint subsets.

2° Suppose $R'S \neq 0$, and let $x_0 \epsilon R'S$; thus $\varlimsup_{n\to\infty} \|U_n(x_0)\| < \infty$ and there exist elements $x_p \to x_0$ such that $\varlimsup_{n\to\infty} \|U_n(x_p)\| = \infty$ for $p = 1, 2, \ldots$

Given any p, there exists a sequence $\{n_{pk}\}_{k=1,2,\ldots}$ such that $\lim_{k\to\infty} \|U_{n_{pk}}(x_p)\| = \infty$, and we may find real numbers t_{pk} $(p, k = 1, 2, \ldots)$ such that

$$\lim_{k\to\infty} t_{pk} = 0 \quad \text{for} \quad p = 1, 2, \ldots,$$

$$\lim_{k\to\infty} \|t_{pk} U_{n_{pk}}(x_p)\| = \infty,$$

$$|t_{pk}| < 1/p \quad \text{for} \quad k = 1, 2, \ldots \tag{6.2}$$

Range the double sequences $\{n_{pk}\}$ and $\{t_{pk}\}$ in a simple $\{s_k\}$ and $\{t_k\}$ respectively in such a manner that the equalities $t_{pk} = t_i$, $n_{pk} = s_j$ imply $j = i$. Thus the sequence $\{t_k U_{s_k}(x)\}$ contains all the sequences $\{t_{pk} U_{n_{pk}}(x)\}_{k=1,2,\ldots}$. The operations $V_k(x) = t_k U_{s_k}(x)$ belong to $\boldsymbol{K}$: moreover

$$\varlimsup_{k\to\infty} \|V_k(x_p)\| = \infty \quad \text{for} \quad p = 1, 2, \ldots \tag{6.3}$$

By (6.2) we have $\lim_{k\to\infty} t_k = 0$, and since $x \epsilon S$ implies

$$\varlimsup_{k\to\infty} \|U_{s_k}(x_0)\| < \infty,$$

we get

$$\lim_{k\to\infty} \|V_k(x_0)\| = 0.$$

The class $\boldsymbol{K}$ having the property (r_2), and $V_k(x)$ being convergent to 0 in the set S of the second category, the operations $V_k(x)$ are equicontinuous. Hence there is a $\delta > 0$ such that $\varrho(x, x_0) < \delta$ implies $\|V_k(x) - V_k(x_0)\| \leqslant 1$ for $k = 1, 2, \ldots$ From the inequality

$$\|V_k(x_p)\| \leqslant \|V_k(x_p) - V_k(x_0)\| + \|V_k(x_0)\|,$$

and p being sufficiently large we get contrarily to (6.3)

$$\varlimsup_{k\to\infty} \|V_k(x_p)\| \leqslant 1 + \varlimsup_{k\to\infty} \|V_k(x_0)\| = 1.$$

It can be shown by a trivial example that without the hypothesis of X being connected, Theorem 10 is in general false.

Theorem 10 implies the

Theorem 11. *Let X be connected, Y a Banach space, and let $\{U_{pq}(x)\}_{q=1,2,\ldots}$ be a sequence of operations belonging for fixed p to a (hr_2)-class $\boldsymbol{K}_p$. If for each p there exists an element x_p such that the sequence $\{U_{pq}(x_p)\}_{q=1,2,\ldots}$ is unbounded, then there exists a residual set R such that the sequences $\{U_{pq}(x)\}_{q=1,2,\ldots}$ are unbounded whatever be p and $x \epsilon R$.*

Theorem 11 may be generalized for a wider class of spaces.

A F-space Y will be said to be a *F^+-space*, if it satisfies the following property:

(f) there exists a $\delta > 0$ such that $t_n \to \infty$ and $\|t_n y_n\| < \delta$ implies $y_n \to 0$ [7]).

Examples of F^+-spaces are:

1° the space L^α, $0 < \alpha < 1$, composed of the functions $x(t)$ defined in an interval $\langle a, b\rangle$ for which $\int_a^b |x(t)|^\alpha dt < \infty$, the norm being defined by the formula $\|x\| = \int_a^b |x(t)|^\alpha dt$;

2° the space l^α, $0 < \alpha < 1$, composed of the sequences $x = \{\hat{x}_n\}$ such that $\sum_{n=1}^{\infty} |\hat{x}_n|^\alpha < \infty$, the norm being defined by the formula $\|x\| = \sum_{n=1}^{\infty} |\hat{x}_n|^\alpha$.

Introducing some not essential modifications in the proof of Theorem 10 we can replace in Theorem 11 the hypothesis of Y being a Banach space by that of Y being a F^+-space.

If we omit in Theorem 11 the hypothesis of Y being a Banach space, or a F^+-space, Theorem 11 may be false. Let e. g. $X = \langle -1, +1\rangle$ and let Y be the space $\boldsymbol{S}$ of measurable functions (Banach [1], p. 9). The space $\boldsymbol{S}$ is not a F^+-space. Given two numbers ϑ and β such that $0 \leqslant \vartheta \leqslant 1, -\infty < \beta < +\infty$ put for $x \epsilon X$

$$U_{\beta\vartheta}(x; t) = \begin{cases} \beta & \text{if } \vartheta < x \leqslant 1, \quad \text{or} \quad \vartheta \leqslant t \leqslant x, \\ 0 & \text{elsewhere.} \end{cases}$$

[7]) Mr. Mazur has communicated to the author that this condition is equivalent to the existence of a bounded neighbourhood of the element 0.

$U_{\beta\vartheta}(x;t)$ may be considered as an operation $U_{\beta\vartheta}(x)$ from X to $\boldsymbol{S}$; let $\boldsymbol{K}_\vartheta$ be the family of operations $U_{\beta\vartheta}(x)$ where $-\infty<\beta<+\infty$. Each family $\boldsymbol{K}_\vartheta$ is equicontinuous, for

$$\|U_{\beta\vartheta}(x_1)-U_{\beta\vartheta}(x_2)\| = \begin{cases} \int\limits_{x_1}^{x_2}\frac{|\beta|}{1+|\beta|}\,dt\leqslant|x_1-x_2| & \text{for } -1\leqslant\vartheta\leqslant x_1\leqslant x_2\leqslant 1, \\ \int\limits_{\vartheta}^{x_2}\frac{|\beta|}{1+|\beta|}\,dt\leqslant x_2-\vartheta\leqslant|x_1-x_2| & \text{for } -1\leqslant x_1\leqslant\vartheta\leqslant x_2\leqslant 1, \\ 0\leqslant|x_1-x_2| & \text{for } x_1\leqslant x_2\leqslant\vartheta\leqslant 1. \end{cases}$$

The family $\boldsymbol{K}_\vartheta$ has the property (h), hence it is a (hr_2)-class. Write $\alpha_p=1-1/p$, $V_{pq}(x)=U_{q\alpha_p}(x)$; the operations of the sequence $\{V_{pq}(x)\}_{q=1,2,\dots}$ belong to the class $\boldsymbol{K}_{\alpha_p}$, and we easily see that this sequence is bounded for $-1\leqslant x\leqslant\alpha_p$ and unbounded for $\alpha_p<x\leqslant 1$. In fact, $V_{pq}(x)=0$ for $-1\leqslant x\leqslant\alpha_p$ and

$$\|q^{-1}V_{pq}(x)\|=\int\limits_{\alpha_p}^{x}\frac{q^{-1}q}{1+q^{-1}q}\,dt=\frac{1}{2}\left(x-1+\frac{1}{p}\right)\qquad\text{for } \alpha_p<x\leqslant 1.$$

There does not exist any element $x\neq 1$ for which the sequences $\{V_{pq}(x)\}_{q=1,2,\dots}$ are simultaneously unbounded.

7. From this section on we shall present some applications of the foregoing theorems.

Let X be an abstract set. Suppose, that for certain pairs of elements x_1,x_2 there is defined their *sum* x_1+x_2 satisfying the following conditions:

(p_1) There is an element 0 such that $x+0$ and $0+x$ exist for each x,

(p_2) $0+0=0$,

(p_3) If x_1+x_2 exists, then x_2+x_1 exists and $x_1+x_2=x_2+x_1$,

(p_4) If $x_1+x=x_2+x$, then $x_1=x_2$,

(p_5) If there exists x_1+x_2 and $(x_1+x_2)+x_3$, there exist x_2+x_3 and $x_1+(x_2+x_3)$; moreover, $(x_1+x_2)+x_3=x_1+(x_2+x_3)$.

The set X will be said to be a *pseudogroup*.

It can be shown that: (a) $x+0=x$; (b) if it is possible to define for the elements $x_1,\dots,x_n$ the sum $x_1+x_2+\dots+x_n$ (in the usual manner), the commutative law holds for this sum.

Any element x for which $x_1=x+x_2$ will be denoted by x_1-x_2; this element is uniquely determined if existing. Suppose now the space X is simultaneously a pseudogroup and a complete metric space, and, moreover, the following conditions [8]) are satisfied:

(p_6) Given any $x_0\in X$ and $\varepsilon>0$, there exists a $\delta>0$ such that any element $x\in K(0,\delta)$ is of the form $x=x_1-x_2$, where $x_1,x_2\in K(x_0,\varepsilon)$,

(p_7) Given any $\delta>0$, there exists an $\omega>0$ such that for each pair x_1,x_2 of elements the inequality $\varrho(x_1,x_2)<\omega$ implies the existence of elements z_1,z_2 such that

$$x_2=(x_1-z_1)+z_2,\qquad z_1,z_2\in K(0,\delta).$$

Every pseudogroup satisfying the above conditions will be called a *pseudogroup of Saks*.

Let Y be a F-space. An operation from a pseudogroup of Saks X to Y will be called *additive* if the existence of x_1+x_2 implies $U(x_1+x_2)=U(x_1)+U(x_2)$; it follows that $U(x)$ being additive, and x_1-x_2 existing, we have $U(x_1-x_2)=U(x_1)-U(x_2)$.

We will denote by $\boldsymbol{A}$ the class of the additive and continuous operations from X to Y.

(7.1) *If the sequence $\{U_n(x)\}$ of operations belonging to $\boldsymbol{A}$ is equicontinuous at x_0, then:*

(7.2) *this sequence is uniformly equicontinuous,*

(7.3) *given any $\varepsilon>0$, there is a $\delta>0$ such that $\varrho(x,0)<\delta$ implies $\|U_n(x)\|<\varepsilon$ for $n=1,2,\dots$*

Proof. Denote by $\delta(x_0,\varepsilon)$ and $\omega(\delta)$ the least upper bounds of the numbers δ and ω, the existence of which is assured by (p_6) and (p_7) respectively. There is a number η such that $\varrho(x,x_0)<\eta$ implies $\|U_n(x)-U_n(x_0)\|<\varepsilon/2$ for $n=1,2,\dots$ Write $\delta=1/2\,\delta(x_0,\eta)$ and let $\varrho(x,0)<\delta$. By (p_6) there are elements x_1,x_2 such that $\varrho(x_1,x_0)<\eta$, $\varrho(x_2,x_0)<\eta$ and $x=x_1-x_2$; hence

$$\|U_n(x_i)-U_n(x_0)\|<\varepsilon/2\quad\text{for}\quad i=1,2\quad\text{and}\quad n=1,2,\dots$$

[8]) These conditions present a generalization of two conditions considered by Saks [17] in a particular case.

It follows $\|U_n(x)\| = \|U_n(x_1) - U_n(x_2)\| < \varepsilon$. Hence (7.3) is proved. To prove (7.2) put $\omega = \omega(\delta)$ and let $\varrho(x_1, x_2) < \omega$; by (p_7) there are elements z_1, z_2 such that $\varrho(z_1, 0) < \delta$, $\varrho(z_2, 0) < \delta$ and $x_2 = (x_1 - z_1) + z_2$; hence

$$\|U_n(z_1)\| < \varepsilon, \ \|U_n(z_2)\| < \varepsilon \quad \text{for} \quad n = 1, 2, \ldots$$

and

$$\|U_n(x_1) - U_n(x_2)\| = \|U_n(x_1) - U_n[(x_1 - z_1) + z_2]\| =$$
$$= \|U_n(x_1) - U_n(x_1) + U_n(z_1) - U_n(z_2)\| < 2\varepsilon.$$

Thus we have shown that *the family A is a (r_2)-class.* From Theorem 2 we obtain

Theorem 12. *Let $U_n(x)$ be additive and continuous operations from a pseudogroup of Saks X to a F-space Y. If the sequence $\{U_n(x)\}$ converges in a residual set $R \subset X$, it converges in the whole of X, the limit-operation is additive and continuous, and* (7.3) *holds* [9]).

Proof. It remains only to prove that the limit is additive; this results from the passage to the limit.

A pseudogroup of Saks will be called *atomless* if the following condition holds:

(p_8) Given any $x \epsilon X$ and $\varepsilon > 0$ there exist elements $x_1, x_2, \ldots, x_n$ such that $x = x_1 + x_2 + \ldots + x_n$ and $\varrho(x_i, 0) < \varepsilon$ for $i = 1, 2, \ldots, n$.

(7.4) *If the pseudogroup of Saks X is atomless, the family A has the properties* (*c*), (*h*) *and* (*b*).

Proof. Let $\{U_n(x)\}$ be a sequence of operations belonging to A.

Ad (*c*). Suppose the sequence converges for every $x \epsilon K(x_0, \varepsilon)$. By (p_6) there is a $\delta > 0$ such that $x \epsilon K(0, \delta)$ implies $x = x_1 - x_2$, where $x_1 \epsilon K(x_0, \varepsilon)$ and $x_2 \epsilon K(x_0, \varepsilon)$; since the sequences $\{U_n(x_1)\}$ and $\{U_n(x_2)\}$ are convergent, the same holds for the sequence $\{U_n(x)\}$ if $x \epsilon K(0, \delta)$; the convergence in the whole of X follows now by (p_8).

Ad (*h*). Trivial.

Ad (*b*). Follows from (*h*) and (*c*).

Theorems 2, 6, and 9 imply

[9]) This theorem is true also under the hypothesis of Y being a pseudogroup of Saks.

Theorem 13. *Let $\{U_n(x)\}$ be a sequence of additive and continuous operations from an atomless pseudogroup of Saks to a F-space, convergent in a set of the second category; then the sequence is convergent everywhere.*

Theorem 14. *Let $\{U_{pq}(x)\}_{q=1,2,\ldots}$ be a sequence of additive and continuous operations from an atomless pseudogroup of Saks to a F-space. If, given any p, there exists an element x_p such that the sequence $\{U_{pq}(x_p)\}_{q=1,2,\ldots}$ is divergent (is unbounded), then there exists a residual set such that the sequence $\{U_{pq}(x)\}_{q=1,2,\ldots}$ is divergent (is unbounded) whatever be p and $x \epsilon R$.*

Suppose now X and Y are F-spaces. Any F-space is an atomless pseudogroup of Saks in which the addition is defined for every pair of elements; in this case, the notion of additive and continuous operation is identical with that of linear operation.

Thus Theorems 13 and 14 imply the following theorem due to Banach ([3], p. 108), Mazur and Orlicz ([11], p. 156):

(7.5) *Let $\{U_n(x)\}$ be a sequence of linear operations convergent in a set R of the second category, then $\{U_n(x)\}$ converges everywhere to a linear operation.*

(7.6) *Let $\{U_{pq}(x)\}_{q=1,2,\ldots}$ be a sequence of linear operations. If for any p there exists an element x_p such that the sequence $\{U_{pq}(x_q^p)\}_{=1,2,\ldots}$ is divergent (is unbounded), then there exists a residual set $R \subset X$ such that the sequences $\{U_{pq}(x)\}_{q=1,2,\ldots}$ are divergent (are unbounded) whatever be p and $x \epsilon R$.*

8. Let S be an abstract set, $\mathfrak{S}$ a *σ-field* [10]) composed of subsets of S, $\mu(e)$ a totally additive measure in $\mathfrak{S}$, such that $\mu(S) < \infty$.

All the sets considered in this section are supposed to belong to $\mathfrak{S}$.

The subclass $\mathfrak{S}_0$ composed of the elements $e \epsilon \mathfrak{S}$ such that $\mu(e) = 0$ is an ideal in $\mathfrak{S}$; we denote the quotient-field $\mathfrak{S}/\mathfrak{S}_0$ by $\mathfrak{S}^\mu$. Thus the elements of $\mathfrak{S}^\mu$ are classes e^* of elements of $\mathfrak{S}$ such that if $e_1, e_2 \epsilon e^*$ then $\mu(e_1 - e_2) + \mu(e_2 - e_1) = 0$.

[10]) $\mathfrak{S}$ is a *σ-field* if: 1° $S \epsilon \mathfrak{S}$, 2° the difference of two sets of $\mathfrak{S}$ belongs to $\mathfrak{S}$, 3° the sum of every sequence of disjoints sets of $\mathfrak{S}$ belongs to $\mathfrak{S}$.

In $\mathfrak{E}^\mu$ we introduce a metric in the following manner: if $e_1^*, e_2^* \epsilon \mathfrak{E}^\mu$, choose $e_1 \epsilon e_1^*$, $e_2 \epsilon e_2^*$ and put

$$\varrho(e_1^*, e_2^*) = \mu(e_1 - e_2) + \mu(e_2 - e_1):$$

this definition does not depend upon a particular choice of e_1 and e_2. It is easy to show that $\mathfrak{E}^\mu$ is a complete metric space [11]).

Let $\mathfrak{E}_1$ be any subset of $\mathfrak{E}$. This set will be said to be *of the first category*, or *of the second category*, or *residual*, or *closed*, or *separable* respectively if the set $\mathfrak{E}_1/\mathfrak{E}_0$ is so in $\mathfrak{E}^\mu$.

Introduce now in $\mathfrak{E}^\mu$ in the usual manner the addition but only for those pairs e_1^*, e_2^* of the elements, for which $e_1 \epsilon e_1^*$ and $e_2 \epsilon e_2^*$ implies $\mu(e_1 e_2) = 0$.

It is easy to see that $\mathfrak{E}^\mu$ *is a pseudogroup of Saks.*

In fact the conditions (p_1)-(p_5) being evidently satisfied it remains to prove (p_6) and (p_7).

Ad (p_6). Let $\varepsilon > 0$, $e_0^* \epsilon \mathfrak{E}^\mu$, and put $\delta = \varepsilon$; if $e^* \epsilon K(0, \delta)$, choose $e \epsilon e^*$ and $e_0 \epsilon e_0^*$; then $e = (e_0 + e) - (e_0 - e)$; and if $e_0 + e \epsilon h_1^*$ and $e_0 - e \epsilon h_2^*$, we have $e^* = h_1^* - h_2^*$ and

$$\varrho(e_0^*, h_1^*) = \mu(e - e_0) \leqslant \mu(e) < \varepsilon,$$

$$\varrho(e_0^*, h_2^*) = \mu(e_0 - (e_0 - e)) \leqslant \mu(e) < \varepsilon.$$

Ad (p_7). Given an $\delta > 0$, put $\omega = \delta$, and given $e_1^*, e_2^* \epsilon \mathfrak{E}^\mu$ such that $\varrho(e_1^*, e_2^*) < \omega$, choose e_1, e_2 and then h_1^*, h_2^* so as to have

$$e \epsilon e_1^*, \qquad e_2 \epsilon e_2^*, \qquad e_1 - e_2 \epsilon h_1^*, \qquad e_2 - e_1 \epsilon h_2^*.$$

Then $e_2^* = (e_1^* - h_1^*) + h_2^*$ and $\varrho(h_1^*, 0) < \delta$, $\varrho(h_2^*, 0) < \delta$.

The condition (p_8) is equivalent to the following:

(a) given $e \epsilon \mathfrak{E}$ and $\varepsilon > 0$ there exist elements $e_1, e_2, \ldots, e_n \epsilon \mathfrak{E}$ such that $\mu(e_i e_k) = 0$ for $i \neq k$, and $e = e_1 + e_2 + \ldots + e_n$, $\mu(e_i) < \varepsilon$ for $i = 1, 2, \ldots, n$.

This condition being satisfied, the measure will be called *non singular* (such is e. g. the Lebesgue-measure).

Let Y be a F-space; a set function $\varphi(e)$ defined from $\mathfrak{E}$ to Y will be said *μ-additive* if $\mu(e_1 e_2) = 0$ implies $\varphi(e_1 + e_2) = \varphi(e_1) + \varphi(e_2)$.

11) considered first by Nikodym [13].

This condition implies $\varphi(e) = 0$ if $\mu(e) = 0$.

The function $\varphi(e)$ is called *totally μ-additive* if

(t_1) $\qquad \mu(e) = 0$ implies $\varphi(e) = 0$,

(t_2) $\qquad \varphi(\sum_{n=1}^{\infty} e_n) = \sum_{n=1}^{\infty} \varphi(e_n)$ if $e_i e_k = 0$ for $i \neq k$.

It is called *μ-absolutely continuous* if $\mu(e) \to 0$ implies $\varphi(e) \to 0$.

Every μ-absolutely continuous function satisfies the condition (t_1); moreover, if it is μ-additive, it satisfies also (t_2). For real-valued functions the converse is true: if a real-valued function satisfies (t_1) and (t_2), it is μ-absolutely additive.

A family $\boldsymbol{K}$ of set functions is called *equi-μ-absolutely continuous* if $\mu(e_n) \to 0$, $\varphi_n \epsilon \boldsymbol{K}$ implies $\varphi_n(e_n) \to 0$.

$\varphi(e)$ being any additive and μ-absolutely continuous set function, let $e^* \epsilon \mathfrak{E}^\mu$; if $e_1, e_2 \epsilon e^*$, then $\varphi(e_1) = \varphi(e_2)$. Hence $\varphi(e)$ may be considered as an additive operation $U(e^*)$ from $\mathfrak{E}^\mu$ to Y; moreover, we will show that $U(e^*)$ is continuous. Let $\varrho(e_n^*, e_0^*) \to 0$; choose $e_n \epsilon e_n^*$ and $e_0 \epsilon e_0^*$. Then we have $\mu(e_n - e_0) + \mu(e_0 - e_n) \to 0$, hence $U(e_n^*) \to U(e_0^*)$, for

$$\|\varphi(e_n) - \varphi(e_0)\| \leqslant \|\varphi(e_n - e_0) - \varphi(e_0 - e_0 e_n)\| \leqslant$$

$$\leqslant \|\varphi(e_n - e_0)\| + \|\varphi(e_0 - e_n)\| \to 0.$$

The converse is obviously true: if $U(e^*)$ is an additive and continuous operation from $\mathfrak{E}^\mu$ to Y, and if we put $\varphi(e) = U(e^*)$ for $e \epsilon e^*$, we obtain a μ-additive and μ-continuous set function.

From Theorems 13 and 14 we obtain the following theorems proved by Saks ([18], p. 967) in the case when Y is the space of real numbers:

Theorem 15. *Let $\{\varphi_n(e)\}$ be a sequence of additive and μ-absolutely continuous set functions, convergent in a residual set $R \epsilon \mathfrak{E}$; then this sequence converges everywhere to an additive and μ-absolutely continuous set function, and this sequence is equi-μ-absolutely continuous.*

Moreover, if we suppose the measure to be non-singular, we may replace the hypothesis of R being residual by that of R being of the second category.

Theorem 16. *Suppose the measure $\mu(e)$ is non-singular, and let $\{\varphi_{pq}(e)\}_{q=1,2,\dots}$ be a sequence of additive and μ-absolutely continuous set functions. Suppose that, given any p, there exists a set e_p such that the sequence $\{\varphi_{pq}(e_p)\}_{q=1,2,\dots}$ is divergent (is unbounded); then there exists a residual set $R\epsilon\mathfrak{E}$ such that the sequences $\{\varphi_{pq}(e)\}_{q=1,2,\dots}$ are divergent (unbounded) whatever be p and $e\epsilon R$.*

Let now Y be a Banach-space. A set-function $\varphi(e)$ from $\mathfrak{E}$ to Y will be termed *weakly μ-absolutely continuous (weakly totally μ-additive)* if, given any linear functional $\eta(y)$ in Y, the real-valued set function $\eta(\varphi(e))$ is also μ-absolutely continuous (totally μ-additive).

We deduce from Theorem 4 the following theorem of Pettis [16]:

Theorem 17. *Any additive and weakly μ-absolutely continuous set function from $\mathfrak{E}$ to a Banach space Y is μ-absolutely continuous.*

Proof. Consider the function $\varphi(e)$ as an additive operation $U(e^*)$ from $\mathfrak{E}^\mu$ to Y. This operation is weakly continuous. In order to apply Theorem 4 it suffices to prove the following proposition:

(8.1) *Given any sequence $\{w_n\}$ of sets of $\mathfrak{E}$, there exists a σ-field $\mathfrak{E}_1\subset\mathfrak{E}$ composed of subsets of the set $S_0=\sum_{n=1}^{\infty}w_n$, separable and closed in $\mathfrak{E}$.*

Let $\mathfrak{E}_2=\{S_0,w_1,w_2,\dots\}$ and let $\mathfrak{E}_3$ be the smallest field[12]) containing $\mathfrak{E}_2$. Sierpiński has shown ([19], p. 14) that $\mathfrak{E}_3=(\mathfrak{E}_2)_{\varrho s\varrho s}$; this class is obviously denumerable. Let $\mathfrak{E}_1$ be the class of the sets for which there exist sets $h_n\epsilon\mathfrak{E}_3$ such that $\mu(e-h_n)+\mu(h_n-e)\to 0$. The set $\mathfrak{E}_1$ is separable and closed.

We shall prove that $\mathfrak{E}_1$ is a σ-field. Let $e_1,e_2,\epsilon\mathfrak{E}_1$ and choose $e_{1n},e_{2n}\epsilon\mathfrak{E}_3$ to have

$$\mu(e_i-e_{in})+\mu(e_{in}-e_i)\to 0 \quad \text{for } n\to\infty,\ i=1,2.$$

The formulae

$$(e_1-e_2)-(e_{1n}-e_{2n})\subset(e_1-e_{1n})+(e_{2n}-e_2),$$

$$(e_{1n}-e_{2n})-(e_1-e_2)\subset(e_{1n}-e_1)+(e_2-e_{2n})$$

[12]) A class $\mathfrak{E}$ of sets is a *field* if it contains the sum and the union of any two sets belonging to it. $\mathfrak{E}_\varrho$ denotes the class of the sets of the form e_1-e_2 where $e_1,e_2\epsilon\mathfrak{E}$. $\mathfrak{E}_s$ is the class of the sets of the form $e_1+e_2+\dots+e_n$ where $e_1,e_2,\dots,e_n\epsilon\mathfrak{E}$.

give $e_1-e_2\epsilon\mathfrak{E}_1$, for

$$\mu((e_1-e_2)-(e_{1n}-e_{2n}))+\mu((e_{1n}-e_{2n})-(e_1-e_2))\leqslant$$
$$\leqslant\mu(e_1-e_{1n})+\mu(e_{2n}-e_2)+\mu(e_2-e_{2n})+\mu(e_{1n}-e_1)\to 0.$$

Let now $\{e_n\}$ be a sequence of disjoint sets of $\mathfrak{E}_1$ and put $e=\sum_{n=1}^{\infty}e_n$; given an $\varepsilon>0$, let $\sum_{n=p+1}^{\infty}\mu(e_n)<\varepsilon$. Choose the sets $h_n\epsilon\mathfrak{E}_3$ to have $\mu(h_n-e_n)+\mu(e_n-h_n)<\varepsilon/p$. The set $h=h_1+h_2+\dots+h_p$ belongs to $\mathfrak{E}_3$ and we have

$$\mu(e-h)+\mu(h-e)=$$
$$=\mu(e+h-eh)\leqslant\mu(h+e_1\dots+e_p-eh)+\sum_{n=p+1}^{\infty}\mu(e_n),$$

and since

$$h+e_1+\dots+e_p-eh\subset h+e_1+\dots+e_p-h(e_1+\dots+e_p)\subset$$
$$\subset(e_1+h_1-e_1h_1)+\dots+(e_p+h_p-e_ph_p),$$

we get $\mu(e-h)+\mu(h-e)\leqslant p\varepsilon p^{-1}+\varepsilon=2\varepsilon$, hence $e\epsilon\mathfrak{E}_1$.

Theorem 18. *A μ-additive[13]) set function from $\mathfrak{E}$ to a Banach space Y is μ-absolutely continuous, if and only if it is weakly totally μ-additive.*

Proof. This follows from the validness of Theorem 18 for real-valued functions.

From Theorems 15 and 18 we obtain easily

Theorem 19. *Let $\{\varphi_n(e)\}$ be a weakly convergent[14]) sequence of μ-additive and μ-absolutely continuous set functions from $\mathfrak{E}$ to a Banach space, convergent everywhere. Then the limit-function is also μ-additive and μ-absolutely continuous.*

9. Let N be the set of the sequences $x=\{\varepsilon_n\}$ composed of zeros and ones, the distance between the points $x_1=\{\varepsilon_{1n}\}$ and $x_2=\{\varepsilon_{2n}\}$ being defined by the formula

[13]) Remark that, according to our definition, the μ-additivness of $\varphi(e)$ implies $\varphi(e)=0$ for $\mu(e)=0$.

[14]) A sequence $\{x_n\}$ of elements of a Banach space is said to be *weakly convergent* if there exists an element x_0 such that $\xi(x_n)\to\xi(x_0)$ for any linear functional $\xi(x)$; see Banach [1], p. 133.

$$\varrho(x_1, x_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} |e_{1n} - e_{2n}|.$$

N is a complete metric space[15]). In this space we define the addition in the usual manner, but only for those elements $x_1 = \{\varepsilon_{1n}\}$ and $x_2 = \{\varepsilon_{2n}\}$ for which $\varepsilon_{1n} + \varepsilon_{2n} \leq 1$.

(9.1). *The space N is a pseudogroup of Saks.*

Proof. Ad (p_1)-(p_5). Trivial.

Ad (p_6). Let $x_0 = \{\varepsilon_{0n}\} \epsilon N$: choose p such that $2^{-p} < 2\varepsilon$ and put $\delta = 2^{-(p+1)}$. If $x = \{\varepsilon_n\} \epsilon K(0, \delta)$, we have $\varepsilon_n = 0$ for $n = 1, 2, \ldots, p$; put

$$\varepsilon_{1n} = \begin{cases} \varepsilon_{0n} & \text{for} \quad n = 1, 2, \ldots, p, \\ \varepsilon_n & \text{for} \quad n > p. \end{cases}$$

$$\varepsilon_{2n} = \begin{cases} \varepsilon_{0n} & \text{for} \quad n = 1, 2, \ldots, p, \\ 0 & \text{for} \quad n > p, \end{cases}$$

$$x_1 = \{\varepsilon_{1n}\}, \qquad x_2 = \{\varepsilon_{2n}\}.$$

Thus $x = x_1 - x_2$, $\varrho(x_1, x_0) < \varepsilon$ and $\varrho(x_2, x_0) < \varepsilon_0$.

Ad (p_7). Given a $\delta > 0$ choose p so that $2^{-p} < 2\varepsilon$, and put $\omega = 2^{-(p+1)}$. Let $x_1 = \{\varepsilon_{1n}\}$, $x_2 = \{\varepsilon_{2n}\}$ and $\varrho(x_1, x^2) < \omega$, then $\varepsilon_{1n} = \varepsilon_{2n}$ for $n = 1, 2, \ldots, p$. Write

$$\zeta_{1n} = \begin{cases} 0 & \text{if} \quad n = 1, 2, \ldots, p, \\ 0 & \text{if} \quad n > p \quad \text{and} \quad \varepsilon_{1n} = \varepsilon_{2n}, \\ \varepsilon_{1n} & \text{if} \quad n > p \quad \text{and} \quad \varepsilon_{1n} \neq \varepsilon_{2n}, \end{cases}$$

$$\zeta_{2n} = \begin{cases} 0 & \text{if} \quad n = 1, 2, \ldots, p, \\ 0 & \text{if} \quad n > p \quad \text{and} \quad \varepsilon_{1n} = \varepsilon_{2n}. \\ \varepsilon_{2n} & \text{if} \quad \varepsilon_{1n} \neq \varepsilon_{2n}. \end{cases}$$

$$z_1 = \{\zeta_{1n}\}, \qquad z_2 = \{\zeta_{2n}\}.$$

We have then $x_2 = (x_1 - z_1) + z_2$ and $z_1, z_2 \epsilon K(0, \delta)$.

The space N does not satisfy the condition (p_8).

Let Y be a F-space. A series $\sum_{n=1}^{\infty} y_n$ is said to be *unconditionally convergent* if it converges independently of the arrangement

[15]) This space with an homeomorphical distance has been introduced by Orlicz [15].

of its terms. Orlicz has shown ([1], p. 33) that a series $\sum_{n=1}^{\infty} y_n$ is unconditionally convergent if and only if, given any element $x = \{\varepsilon_n\} \epsilon N$, the series $\sum_{n=1}^{\infty} \varepsilon_n y_n$ converges.

It is easily seen that the general form of additive and continuous operations from N to Y is

$$U(x) = \sum_{n=1}^{\infty} \varepsilon_n y_n, \tag{9.2}$$

where $\sum_{n=1}^{\infty} y_n$ is an unconditionally convergent series the terms of which belong to Y.

Let $U_n(x) = \sum_{\nu=1}^{\infty} \varepsilon_\nu y_{n\nu}$ be a sequence of additive and continuous operations in N. The condition (7.3) introduced in section 7 may be formulated in the following equivalent fashion:

(9.3) *Given any $\varepsilon > 0$, there exists a p such that $x = \{\varepsilon_n\} \epsilon N$ implies $\|\sum_{\nu=p}^{\infty} \varepsilon_\nu y_{n\nu}\| < \varepsilon$.*

From Theorem 12 follows

Theorem 20. *Let $\sum_{\nu=1}^{\infty} y_{n\nu}$ be an unconditionally convergent series. If the sequence of operations $\{\sum_{\nu=1}^{\infty} \varepsilon_\nu y_{n\nu}\}_{n=1,2,\ldots}$ converges for every $x = \{\varepsilon_n\}$ belonging to a residual set $R \subset N$, then it converges everywhere to an operation of the form* (9.2), *and the condition of* (9.3) *is satisfied.*

Theorem 16 is not true for the additive and continuous operations in N.

Example. Let $y_0 \epsilon Y$, $y_0 \neq 0$ and put:

$$\left.\begin{aligned} U_{1q}(x) &= \varepsilon_1 (-1)^q y_0 + \varepsilon_2 (-1)^q y_0 \\ U_{2q}(x) &= \varepsilon_1 (-1)^q y_0 - \varepsilon_2 (-1)^q y_0 \end{aligned}\right\} \quad \text{for } q = 1, 2, \ldots,$$

$$\left.\begin{aligned} U_{2q}(x) &= \varepsilon_2 (-1)^q y_0 \\ U_{4q}(x) &= \varepsilon_2 (-1)^q y_0 \end{aligned}\right\} \quad \text{for } q = 1, 2, \ldots,$$

$$U_{nq}(x) = U_{mq}(x) \qquad \text{if } n \equiv m \pmod 4.$$

Then the sequences $\{U_{pq}(x)\}_{q=1,2,\dots}$ are divergent for an $x=x_p$ and there does not exist any x_0 for which these sequences should be simultaneously divergent. However from (5.4) follows

Theorem 21. *Let the series* $\sum_{\nu=1}^{\infty} y_{pq\nu}$ *be unconditionally convergent for* $p,q=1,2,\dots$, *and put* $U_{pq}(x)=\sum_{\nu=1}^{\infty}\varepsilon_\nu y_{pq\nu}$. *Suppose that, given any p, there exists a set* D_p *dense in N, and an element* x_p *such that the sequence* $\{U_{pq}(x)\}_{q=1,2,\dots}$ *is divergent (is unbounded) for* $x=x_p$, *and convergent (bounded) for* $x \in D_p$. *Then there exists a set R, residual in N, such that the sequences* $\{U_{pq}(x)\}_{q=1,2,\dots}$ *are divergent (are unbounded) whatever be p and* $x \in R$.

Remark. The convergene of $\{U_{pq}(x)\}_{q=1,2,\dots}$ in a set D_p, dense in N, is equivalent to the existence of the limit $\lim_{q\to\infty} y_{pq\nu}$ for $\nu=1,2,\dots$

Theorem 22. *Every additive and weakly continuous operation U(x) from N to a Banach space Y is continuous.*

Proof. Note first that the range of $U(x)$ is separable. This follows from the separability of N and from the fact that the weak convergence of a sequence in Y implies the strong convergence of linear combinations with rational coefficients of the elements of this sequence to the same limit. Thus, the class of operations of the form $\eta(U(x))$ where $\eta \in \bar{Y}$ being by (7.1) a (r_2)-class, it is sufficient to apply Theorem 4.

The above theorem enables us to obtain immediately a theorem due to Orlicz (see Banach [1], p. 240):

(9.4) *Let Y be a Banach space. A necessary and sufficient condition for the series* $\sum_{n=1}^{\infty} y_n$ *to be unconditionally convergent is that any partial series be weakly convergent to an element of Y.*

Proof. The necessity being trivial, we prove only the condition to be sufficient. By hypothesis, given any element $x \in N$, there exists an element $y_x \in Y$ such that the series $\sum_{n=1}^{\infty}\varepsilon_n y_n$ converges weakly to y_x. Put $U(x)=y_x$; this operation is additive in N, moreover, as $U(x)$ is the weak limit of the continuous and linear operations $U_n(x)=\sum_{\nu=1}^{n}\varepsilon_\nu y_\nu$, it is also weakly continuous. Thus, by Theorem 22, $U(x)$ is continuous. Since $U(x)$ is additive, there is an unconditionally convergent series $\sum_{n=1}^{\infty} z_n$ such that $U(x)=\sum_{\nu=1}^{\infty}\varepsilon_\nu z_\nu$. To finish, we remark that $y_\nu=z_\nu$ for $\nu=1,2,\dots$

10. As an application of the results of the preceding section, we will prove two theorems concerning the linear operations defined in spaces of more general character then the F-spaces.

Let X be a linear space, and at the same time a $\mathscr{L}$-space in the sense of Fréchet [16]). Suppose that addition and multiplication by real numbers are continuous; moreover, suppose the following axiom satisfied:

(u) If $x_n\to 0$ then there exists a partial sequence $\{x_{n_k}\}$ such that the series $\sum_{n=1}^{\infty} x_{n_k}$ is *unconditionally convergent* (i. e. such that any subseries converges).

An additive and continuous operation $U(x)$ from X to a F-space Y will be called *linear*.

We will prove the following theorems due to Orlicz [17]) which generalize two theorems of Banach ([2], p. 157):

Theorem 23. *Let* $\{U_n(x)\}$ *be an everywhere convergent sequence of linear operations. Then* $x_n\to 0$ *implies* $U_n(x_n)\to 0$.

Proof. Suppose, if possible, that for a sequence $\{x_n\}$ and for $\varepsilon>0$ we have $x_n\to 0$ and $\|U_{k_n}(x_{k_n})\|\geqslant\varepsilon$. By ($u$) we can suppose the series $\sum_{\nu=1}^{\infty} x_{k_\nu}$ to be unconditionally convergent. Given any $z=\{\varepsilon_n\}\in N$, put

$$V_n(z)=U_{k_n}\Big(\sum_{\nu=1}^{\infty}\varepsilon_\nu x_{k_\nu}\Big)=\sum_{\nu=1}^{\infty}\varepsilon_\nu U_{k_n}(x_{k_\nu}).$$

The operations $V_n(z)$ being obviously additive and continuous from N to Y, we see by Theorem 20 that (9.1) holds, and this

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[16]) i. e. a space with the notion of limit.

[17]) Not yet published. The author is indebted to Mr. Orlicz for having permitted to enclose here these results.

implies $\|U_{k_n}(x_{k_n})\| < \varepsilon$ for n sufficiently large. This is, however, impossible.

Theorem 24. *Let $\{U_n(x)\}$ be a sequence of linear operations convergent everywhere to $U(x)$. Then $U(x)$ is also linear.*

Proof. It suffices to prove that $x_n \to 0$ implies $U(x_n) \to 0$. Let $\varepsilon > 0$ be arbitrary; given any n there exists a m_n such that $\|U(x_n) - U_{m_n}(x_n)\| < \varepsilon$. Write $V_n(x) = U_{m_n}(x)$; these are linear operations which converge everywhere to $U(x)$. By Theorem 24 we have $\|V_n(x_n)\| < \varepsilon$ for n sufficiently large, hence

$$\|U(x_n)\| \leqslant \|U(x_n) - U_{m_n}(x_n)\| + \|U_{m_n}(x_n)\| < 2\varepsilon.$$

11. In this section we show how the theorems of Mazur and Orlicz concerning the sequences of polynomial operations may be deduced from the ours. We state first the fundamental definitions.

Let X and Y be two linear spaces. An operation $U(x_1, x_2, \dots, x_k)$ from X^k to Y is called *k-additive and homogeneous* if it is additive and homogeneous relatively to each variable separately. $U(x_1, x_2, \dots, x_k)$ being k-additive and homogeneous, the operation $U_k(x) = U(x, x, \dots, x)$ is called *homogeneous of degree k*; any constant operation is called to be *homogeneous of degree* 0. For any operation of degree k there exists exactly one k-additive operation $U_k^*(x_1, x_2, \dots, x_k)$, homogeneous and symmetrical in all the variables, and such that $U_k(x) = U_k^*(x, x, \dots, x)$. This operation is called *the primitive* for $U_k(x)$. Any operation of the form

$$U(x) = U_0(x) + U_1(x) + \dots + U_m(x), \tag{11.1}$$

where $U_i(x)$ is homogeneous of degree i, is termed an *operation of degree m*. The representation given by formula (11.1) is called the *canonical* for $U(x)$.

Mazur and Orlicz proved ([12], p. 50-53) the following theorems:

(11.2) *Given any m, there exist real numbers a_{ik} $(i, k = 0, 1, \dots, m)$ such that* (11.1) *implies*

$$U_k(x) = a_{k_0} U(0x) + a_{k_1} U(1x) + \dots + a_{km} U(mx).$$

(11.3) *$U_k(x)$ being homogeneous of degree k, its primitive operation may be represented by the formula*

$$k!\, U_k^*(x_1, x_2, \dots, x_k) = \sum_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k = 0}^{1} (-1)^{k - (\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_k)} U_k(\varepsilon_1 x_1 + \varepsilon_2 x_2 + \dots + \varepsilon_k x_k).$$

(11.4) *$U_k(x)$ being homogeneous of degree k, we have for any rational t*

$$U_k(x + th) = \sum_{i=1}^{k} t^i \binom{k}{i} U_k^* (\underbrace{x, \dots, x}_{k-i}, \underbrace{h, \dots, h}_{i}).$$

Let now X and Y be two F-spaces. Any continuous operation of degree m is called the *polynomial* of degree m.

From (11.2) and (11.3) it follows that

$$U(x) = U_0(x) + U_1(x) + \dots + U_m(x)$$

being a polynomial of degree m, the operations $U_k(x)$ and their primitives $U_k^*(x_1, x_2, \dots, x_k)$ are continuous for $k = 0, 1, \dots, m$.

Denote by $\boldsymbol{K}_m$ the family of the polynomials of degree at most m. We will prove that *$\boldsymbol{K}_m$ is a $(cbhr_2)$-class*. This results from the following propositions:

(11.5) *Every sequence $\{U_n(x)\}$ of polynomials of degree at most m, equicontinuous at one point x_0, is equicontinuous.*

(11.6) *Every sequence $\{U_n(x)\}$ of polynomials of degree at most m, and convergent (bounded) in a sphere, is convergent (bounded) everywhere.*

Proof. Ad (11.5). Note first that it is sufficient to suppose that $x_0 = 0$, for in the contrary case we may consider the sequence of polynomials $V_n(x) = U_n(x_0 + x)$ which are also of degree at most m. Let

$$U_n(x) = U_{n0}(x) + U_{n1}(x) + \dots + U_{nm}(x)$$

be the canonical representation of $U_n(x)$. We will prove that the sequence $\{U_{nk}(x)\}_{n=1,2,\dots}$ is equicontinuous everywhere for $k = 1, 2, \dots, m$. Let $\varepsilon > 0$ be arbitrary; from (11.2) it follows that the operations $\{U_{nk}(x)\}_{n=1,2,\dots}$ are equicontinuous at 0, hence we see by (11.3) that there exists a $\delta > 0$ such that $\|x_i\| < \delta$ for $i = 1, 2, \dots, k$ imply

$$\|U_{nk}^*(x_1, x_2, \dots, x_k)\| < \varepsilon,$$

U_{nk}^* being the primitive operation for U_{nk}.

Let $z_1 \in X$ be arbitrary; choose p so as to have $\|p^{-1}z_1\| < \delta$, and then let δ_1 be such that $0 < \delta_1 < \delta$ and $\|p^{k-i}(z-z_1)\| < \delta$ for $\|z-z_1\| < \delta_1$, $i = 1, 2, \dots, k$. Then

$$\|U^*_{nk}(\underbrace{z_1,\dots,z_1}_{k-i},\underbrace{z-z_1,\dots,z-z_1}_{i})\| = \|p^{k-i}U^*_{nk}(\underbrace{p^{-1}z_1,\dots,p^{-1}z_1}_{k-i},\underbrace{z-z_1,\dots,z-z_1}_{i})\|$$

$$= \|U^*_{nk}(\underbrace{p^{-1}z_1,\dots,p^{-1}z_1}_{k-i}, p^{k-i}(z-z_1), \underbrace{z-z_1,\dots,z-z_1}_{i-1})\|$$

for $\|z-z_1\| < \delta_1$ and $i \geqslant 1$, and by (11.4)

$$\|U_{nk}(z) - U_{nk}(z_1)\| \leqslant \sum_{i=1}^{k} \binom{k}{i} \varepsilon < 2^k \varepsilon.$$

Ad (11.6). We may suppose that the sequence $\{U_n(x)\}$ is convergent (bounded) for $\|x\| \leqslant r$. By (11.2) the sequences $\{U_{nk}(x)\}_{n=1,2,\dots}$ are convergent (bounded) if $\|x\| < r/m$, $k = 0, 1, \dots, m$; by (11.3) the same holds for the sequences $\{U^*_{nk}(x_1, x_2, \dots, x_k)\}_{n=1,2,\dots}$ if $\|x_i\| \leqslant r/m^2$, $i = 1, 2, \dots, k$ and $k = 0, 1, \dots, m$. The operations $U^*_{nk}(x_1, x_2, \dots, x_k)$ being homogeneous in each variable separately, we see that the sequences $\{U^*_{nk}(x_1, x_2, \dots, x_k)\}_{n=1,2,\dots}$ are convergent for $(x_1, x_2, \dots, x_k) \in X^k$.

From Theorems 3, 6, 9 we obtain

Theorem 25. *Let $\{U_n(x)\}$ be a sequence of polynomials of degree at most m, convergent in a residual set $R \subset X$. Then the sequence converges everywhere, and the limit is a polynomial of degree at most m.*

Theorem 26. *Let $\{U_{pq}(x)\}_{q=1,2,\dots}$ be a sequence of polynomials of degree at most m_p. If, given any p, there exists an element x_p such that the sequence $\{U_{pq}(x_p)\}_{q=1,2,\dots}$ is divergent (unbounded), then there exists a residual set $R \subset X$ such that the sequences $\{U_{pq}(x)\}_{q=1,2,\dots}$ are divergent (unbounded) whatever be p and $x \in R$.*

(11.7) *Any sequence $\{U_n(x)\}$ of polynomials of degree at most m, bounded in a set $R \subset X$ of the second category, is equicontinuous.*

Proof. We prove first that the sequence is bounded everywhere. Let $\vartheta_n \to 0$, and write $V_n(x) = \vartheta_n U_n(x)$; this sequence converges to 0 in R, hence by Theorem 25 everywhere, and the limit is a polynomial $V(x)$. Since $V(x) = 0$ in $\bar{R}$ and since this set contains a sphere, $V(x) \equiv 0$.

Let $U_n(x) = U_{n1}(x) + U_{n2}(x) + \dots + U_{nm}(x)$ be the canonical representation of $U_n(x)$. It is sufficient to prove that the operations $\{U_{nk}(x)\}_{n=1,2,\dots}$ are equicontinuous for $k = 1, 2, \dots, m$, i.e. that $x_n \to 0$ implies $U_{nk}(x_n) \to 0$ as $n \to \infty$. Let k be fixed. Choose $\lambda_n \to \infty$ such that $\lambda_n x_n \to 0$ and put $W_n(x) = \lambda_n^{-k} U_{nk}(x)$; the sequence $\{W_n(x)\}$ converges to 0 everywhere; as by Theorem 2 it is equicontinuous, it follows $W_n(\lambda_n x_n) = U_n(x_n) \to 0$.

From (11.6) and (1.3) we get the following theorem due to Mazur and Orlicz ([12], p. 186):

(11.8) *Let $\{U_n(x)\}$ be a sequence of polynomials of degree at most m, bounded in a set of the second category and convergent in a set dense in X: then this sequence is convergent everywhere.*

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Sur les moyennes

par

C. RYLL-NARDZEWSKI (Lublin)

1. M. Aczél a démontré [1] que toute fonction $M(x,y)$ qui est

(i) croissante par rapport à chacune des variables x,y,

(ii) continue,

(iii) bisymétrique: $M[M(x,y), M(z,u)] = M[M(x,z), M(y,u)]$,

(iv) refléxive: $M(x,x)=x$,

(v) symétrique: $M(x,y)=M(y,x)$

est de la forme

$$M(x,y)=f^{-1}\left[\frac{f(x)+f(y)}{2}\right]. \tag{1}$$

Je me propose de montrer que l'hypothèse (iii) peut être remplacée dans le théorème de M. Aczél par la suivante:

$$M[x, M(y,z)] = M[M(x,y), M(z,x)] \tag{2}$$

et que les hypothèses (iv) et (v) deviennent alors superflues. De plus, il suffit de supposer la continuité de $M(x,y)$ seulement *sur la droite* $x=0$, ce qui entraîne déjà la continuité partout.

Plus précisément, je vais démontrer le théorème suivant:

Si une fonction $M(x,y)$, croissante par rapport à chacune des variables x,y et continue sur la droite $x=0$, satisfait à l'équation (2), *pour tous x et y réels, elle est de la forme* (1), *où $f(x)$ est une fonction continue croissante.*

Pour déduire de ce théorème celui de M. Aczél, il suffit évidemment de montrer que toute fonction $M(x,y)$ satisfaisant aux conditions (i)-(v) satisfait à la condition (2).

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