En posant $q=0,1,2,..,h$, on obtient successivement $h+1$ progressions arithmétiques $\{q+r(h+1)\}$ à différence constante $h+1$; l'ensemble de ces progressions constitue la suite $\{n\}$ des nombres naturels. En appliquant (39) successivement à $q=0,1,\dots,h+1$, on conclut que (39) subsiste pour tous les q et tout $t\in\prod_{q=1}^{q=h+1}T_q=T^*$, donc

(40) $f\{x_n(t)<\alpha, x_{n+h}(t)<\beta\}=F(\alpha)\cdot F(\beta)$ pour tous les α,β et tout $t\in T^*$ où $|T^*|=1$.

Or, on a en vertu du théorème I ($\underset{n}{f}$ désignant la fréquence des n)

(41) $\underset{n}{f}\{x_n(t)<\alpha\}=F(\alpha)$ et $\underset{n}{f}\{x_{n+h}(t)<\beta\}=F(\beta)$ pour tous les α,β et tout $t\in T^0$ où $|T^0|=1$.

En posant $T_h=T^0\cdot T$, on a en vertu de (40) et (41)

(42) $\underset{n}{f}\{x_n(t)<\alpha, x_{n+h}(t)<\beta\}=\underset{n}{f}\{x_n(t)<\alpha\}\cdot\underset{n}{f}\{x_{n+h}(t)<\beta\}$ pour tous les α,β et pour tout $t\in T_h$ où $|T_h|=1$.

En posant $T=\prod_{h=1}^{\infty}T_h$ et en appliquant (42) à $h=1,2,\dots,$ on conclut que (42) subsiste pour tous les h entiers, pour tous les α,β réels et pour tout $t\in T$; cela équivaut à $\operatorname{ind}[x_n(t),x_{n+h}(t)]$ pour tous les h, donc à $\operatorname{ind}[x_{n+h}(t),x_{n+k}(t)]$ pour $h\neq k$, $t\in T$ étant fixe; comme $|T|=1$, le théorème IV se trouve établi.

On démontre par des moyens analogues le

Théorème V. *Si les fonctions $x_i(t)$ aux distributrices identiques sont indépendantes en bloc, la suite $\{x_i(t)\}$ est complètement aléatoire pour presque tous les t.*

Notons qu'il est facile de donner des exemples des suites aléatoires, quand on admet des *distributrices dégénérées*, c'est-à-dire qui ne prennent qu'une ou deux valeurs différentes. La condition que la distributrice d'une suite prenne au moins trois valeurs différentes exclut les suites convergentes et celles divergentes vers l'infini, qui satisfont d'une manière banale à la définition d'une suite aléatoire, tout comme les fonctions constantes satisfont à la définition de l'indépendance.

(Reçu par la Rédaction le 2. 6. 1949).

Banach spaces of functions analytic in the unit circle, I

by

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Introduction.

It is by now very well known that the concepts and methods of functional analysis play an important part in the theory of functions of a real variable. The concept of function space which has been most widely and successfully used in analysis is that of the normed linear space, called a *Banach space*, if it is complete. Most of the function spaces which have received careful attention are spaces of functions of a real variable. Very little work has been done on classes of analytic functions which form Banach spaces.

In this paper we propose to study spaces whose elements are functions analytic in the unit circle. Originally we began by studying particular spaces, with norms defined in a definite analytical manner (e. g. the bounded analytic functions, with $\|f\|$ equal to the least upper bound of $|f(z)|$ for $|z|<1$). It gradually became clear, however, that very general theorems could be proved merely by postulating a few additional properties of the space beyond the assumption that it was a normed linear space of analytic functions. We have thus been led to a theory which is quite abstract, in that it applies to a whole class of spaces, without reference to the particular analytical definition of the norm in any given space. The theory is also satisfactorily comprehensive, for it applies to the spaces which seem to be of greatest interest in the theory of functions.

This abstract theory is developed in Part I of the paper. There are four main axioms in addition to the assumption that we are dealing with a normed linear space of analytic functions. Later we consider three more axioms which have interesting consequences. One of the main goals in Part I is the determination of the representation of linear functionals. The study of this problem leads to the construction of new spaces of analytic functions, which, under certain conditions are isomorphic or equivalent to the space of linear functionals defined on a given space.

Part II of the paper [1]) deals with realizations of the abstract theory of Part I. We study the spaces H^p, $1 \leqslant p \leqslant \infty$; H^∞ is the space of bounded analytic functions. We also study the space K of analytic functions which are continuous in the closed unit circle. Representation theorems for linear functionals are obtained in case $1 \leqslant p < \infty$. In this connection we introduce „mean values" $N_p(f;r)$ in certain ways analogous to the integral means

$$\mathfrak{M}_p(f;r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\vartheta})|^p \, d\vartheta\right)^{1/p}.$$

When $1 < p < \infty$ there is a close relation between these two kinds of mean values, the link being provided by M. Riesz's theorem on conjugate harmonic functions. Indeed, we show the equivalence between M. Riesz's theorem and certain important propositions about the Banach space H^p.

For the convenience of the reader we have included a table of contents of the paper.

Table of Contents.

[1]) To appear in the next volume of Studia Mathematica.

PART I. Axiomatic Development and General Theory.

1. Notations and definitions. Throughout the paper we shall use the following notations.

Δ: the open set $|z| < 1$ in the complex plane;

$\mathfrak{A}$: the class of all complex-valued functions which are defined (single-valued) and analytic in Δ.

Clearly $\mathfrak{A}$ is a linear class, i. e. $\mathfrak{A}$ is closed under addition and multiplication by complex constants;

r, ϱ: real variables subject to the inequalities $0 \leqslant r < 1$, $0 \leqslant \varrho < 1$.

We use the symbols $f, g, \ldots, F, G, \ldots$, for elements of $\mathfrak{A}$. In general we suppress the independent variable except when we refer to the values of a function.

Definition 1.1. We define

$$u_n(z) = z^n, \qquad n = 0, 1, 2, \ldots$$

and use u_n to denote these functions as elements of $\mathfrak{A}$.

We have occasion to introduce two very simple operations which may be performed on elements of $\mathfrak{A}$. Each of these operations depends upon a parameter.

Definition 1.2. If $f \in \mathfrak{A}$ and if x is a real parameter, we write $g = U_x f$ when g is that element of $\mathfrak{A}$ defined by

$$g(z) = f(ze^{ix}), \quad z \in \Delta.$$

Definition 1.3. If $f \in \mathfrak{A}$ and if w is a complex parameter such that $|w| \leqslant 1$, we write $g = T_w f$ when g is that element of $\mathfrak{A}$ defined by $g(z) = f(zw)$, $z \in \Delta$.

It is evident that U_x and T_w are distributive operators mapping $\mathfrak{A}$ into itself. That is, $U_x f$ and $T_w f$ are elements of $\mathfrak{A}$ if $f \in \mathfrak{A}$. Also

$$U_x(af + bg) = a\,U_x f + b\,U_x g,$$

with a similar relation for T_w.

If $f \in \mathfrak{A}$, $f(z)$ can be expanded in a power series in z, convergent in Δ. The coefficients in this series are well-defined functionals of f; we need a notation for them.

Definition 1.4. If $f \epsilon \mathfrak{A}$ has the expansion

$$f(z)=\sum_{n=0}^{\infty} a_n z^n,$$

we use the notation $a_n=\gamma_n(f)$. Thus

$$\gamma_n(f)=\frac{1}{n!} f^{(n)}(0).$$

We observe the distributive property

$$\gamma_n(af+bg)=a\gamma_n(f)+b\gamma_n(g).$$

The symbol γ_n will often be used by itself when we discuss linear functionals on various spaces whose elements are members of $\mathfrak{A}$.

Definition 1.5. If f and g are elements of $\mathfrak{A}$ with developments

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \quad g(z)=\sum_{n=0}^{\infty} b_n z^n,$$

we define

$$B(f,g;z)=\sum_{n=0}^{\infty} a_n b_n z^n, \quad z \epsilon \Delta.$$

This series defines an element of $\mathfrak{A}$; to see that this is true we write $z=z_1 z_2$, with $|z_k|<1$, $k=1,2$. For example, z_1 and z_2 may be taken as the two square roots of z. Then

$$a_n b_n z_n=(a_n z_1^n)(b_n z_2^n).$$

Now $a_n z_1^n \to 0$ as $n \to \infty$, and the series

$$\sum_{n=0}^{\infty} b_n z_2^n$$

is absolutely convergent. Hence the series defining $B(f,g;z)$ is absolutely convergent.

For later reference we note the following relations, which are either immediately evident, or are easily verified:

(1.1) $\gamma_n(T_w f)=w^n\gamma_n(f)$,

(1.2) $B(f,g;w)=B(g,f;w)$,

(1.3) $B(af+bg,h;z)=aB(f,h;z)+bB(g,h;z)$,

(1.4) $B(T_w f,g;z)=B(f,g;wz), \quad |w|\leqslant 1$,

(1.5) $B(f,g;z)=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(z_1 e^{i\vartheta})\, g(z_2 e^{-i\vartheta})\, d\vartheta$,

where $z=z_1 z_2$, $|z_k|<1$, $k=1,2$.

2. Spaces of type $\mathfrak{A}$. Axioms P_1-P_4. Let B be a complex normed linear space each element of which is a member of $\mathfrak{A}$. Such a space will be called a *space of type* $\mathfrak{A}$ provided it contains at least two elements.

A space of type $\mathfrak{A}$ is first of all a linear subclass of $\mathfrak{A}$. Such a linear subclass with more than one element becomes a space of type $\mathfrak{A}$ when with each element f of the subclass we associate a number $\|f\|$ having the properties of a norm, that is,

$$\|f\|\geqslant 0, \quad \|f\|=0 \text{ if and only if } f(z)\equiv 0,$$

$$\|af\|=|a|\,\|f\|,$$

$$\|f+g\|\leqslant\|f\|+\|g\|.$$

For example, the set of all the bounded functions in the class $\mathfrak{A}$ becomes a space of type $\mathfrak{A}$ if we define

$$\|f\|=\sup_{|z|<1}|f(z)|.$$

We shall be interested in spaces of B of type $\mathfrak{A}$ having certain additional properties. For the present we list four properties which B may enjoy.

P_1. There exists a constant A such that $|\gamma_n(f)|\leqslant A\|f\|$ if $f \epsilon B$ and $n=0,1,2,\ldots$ The least such constant A depends on the space B; it will be denoted hereafter by $A_1(B)$.

P_2. $u_n \epsilon B$, $n=0,1,2,\ldots$ There exists a constant A such that $\|u_n\|\leqslant A$ for all n. The least such constant A here will be denoted by $A_2(B)$.

P_3. $U_x f \epsilon B$ if $f \epsilon B$ and x is real; also, $\|U_x f\|=\|f\|$.

P_4. $T_r f \epsilon B$ if $f \epsilon B$ and $0\leqslant r<1$. There exists a constant A such that $\|T_r f\|\leqslant A\|f\|$. The least such A will be denoted by $A_4(B)$.

We shall presently proceed to develop certain of the properties of spaces of type $\mathfrak{A}$ which satisfy one or more of these four axioms.

We shall say that B is a *space of type* $\mathfrak{A}_k$ (where $k=1,2,3,4$) if it is a space of type $\mathfrak{A}$ satisfying axioms $P_1,\ldots,P_k$.

In part II of this paper we study in some detail a whole class of spaces of type $\mathfrak{A}_4$. Consequently we shall not take space at this point to discuss concrete examples of such spaces.

We denote by B^* the space of linear (continuous and distributive) functionals defined on B. Linear functionals will be represented by small Greek letters $\gamma, \lambda, \ldots$. The axiom P_1 states that $\gamma_n \epsilon B^*$ and that the sequence $\|\gamma_n\|$ is bounded. It is readily seen that

$$A_1(B)=\sup_n \|\gamma_n\|. \tag{2.1}$$

For reference we set out explicitly the definition.

$$A_2(B)=\sup_n \|u_n\|. \tag{2.2}$$

If X is any normed linear space, the space of all (bounded) linear transformations of X into itself is denoted by $[X]$. The axiom P_3 states that $U_x \epsilon [B]$ for each real x, and that U_x is an isometric operator. It is clear that the operators U_x form a one parameter commutative group, with $U_0=I$. The axiom P_4 states that $T_r \epsilon [B]$ and that the norms $\|T_r\|$ are bounded uniformly in r. Evidently we have

$$A_4(B)=\sup_r \|T_r\|. \tag{2.3}$$

We shall observe a few simple relations which the constants $A_k(B)$ must satisfy. In the first place, $\gamma_n(u_n)=1$. Therefore, if axioms P_1 and P_2 hold, we have

$$1\leqslant \|\gamma_n\|\|u_n\|, \quad 1\leqslant A_1(B)A_2(B). \tag{2.4}$$

Also, $T_r u_n = r^n u_n$. Hence, if $u_n \epsilon B$ for some particular value of n, and if P_4 holds, we have $r^n\|u_n\|\leqslant \|T_r\|\|u_n\|\leqslant A_4(B)\|u_n\|$, from which it follows that

$$1\leqslant A_4(B). \tag{2.5}$$

With $r=0$ we have $T_0 f = f(0) = \gamma_0(f)u_0$. Hence, if $u_0 \epsilon B$ and $\gamma_0 \epsilon B^*$, we have $T_0 \epsilon [B]$, and $\|T_0\|=\|\gamma_0\|\|u_0\|$. But, if P_4 holds, $\|T_0\|\leqslant A_4(B)$. Hence, in this situation,

$$\|\gamma_0\|\|u_0\|\leqslant A_4(B). \tag{2.6}$$

In many of the most interesting concrete examples, the constants $A_k(B)$ are all equal to unity. It is not difficult to construct examples in which $A_1(B)$ and $A_2(B)$ are different from unity. We have not discovered any spaces of type $\mathfrak{A}_4$ for which $A_4(B)>1$.

3. Spaces of type $\mathfrak{A}_1$. In this section we explore some of the consequences of axiom P_1.

Theorem 3.1. *If B is a space of type $\mathfrak{A}_1$ and $f \epsilon B$, $z \epsilon \Delta$, we have*

$$|f(z)|\leqslant \frac{A_1(B)\|f\|}{1-|z|}.$$

The proof follows at once from (2.1) and the expansion

$$f(z)=\sum_{n=0}^{\infty}\gamma_n(f)z^n.$$

Theorem 3.1 shows us that, for a fixed $z \epsilon \Delta$, $f(z)$ defines an element of B^*, of norm not exceeding $A_1(B)(1-|z|)^{-1}$. It also follows from Theorem 3.1 that if $\{f_n\}$ is a sequence of elements of B such that the norms $\|f_n\|$ are bounded then the sequence $\{f_n\}$ is a normal family in the sense of Montel. For, the moduli $|f_n(z)|$ are uniformly bounded on each compact subset of Δ.

Theorem 3.2. *Let B be a space of type $\mathfrak{A}_1$. Suppose f, $f_n \epsilon B$ and $\|f_n - f\|\to 0$. Then $f_n(z)\to f(z)$ uniformly on compact subsets of Δ.*

This is an immediate corollary of Theorem 3.1, as applied to the function $f_n - f$.

The theorem can be strengthened, for it suffices to assume that f_n converges weakly to f (i.e. that $\gamma(f_n)\to\gamma(f)$ for each $\gamma \epsilon B^*$).

Theorem 3.3. *Let B be a space of type $\mathfrak{A}_1$. Suppose $f, f_n \epsilon B$, and that f_n converges weakly to f. Then $f_n(z)$ converges uniformly to $f(z)$ on compact subsets of Δ.*

Proof. The weak convergence implies that $\|f_n\|$ is bounded (BANACH [1][2]), p. 133), say $\|f_n\| \leqslant A$. Hence, by Theorem 3.1, there is a uniform bound for $|f_n(z)|$ on each compact subset of Δ. Now $f_n(z) \to f(z)$ for each $z \in \Delta$ since, for fixed $z, f(z)$ is a linear functional of f. The uniformity of convergence now follows by Vitali's theorem.

The next theorem is still stronger:

Theorem 3.4. *Let B be a space of type $\mathfrak{A}_1$. Suppose that*

$$\lim_{n\to\infty} \gamma_k(f_n) = \gamma_k(f), \qquad k=0,1,2,\ldots$$

where f_n and f are members of B, and let the norms $\|f_n\|$ be bounded, say $\|f_n\| \leqslant A$. Then $f_n(z)$ converges uniformly to $f(z)$ on compact subsets of Δ.

Proof. Let S be a compact subset of Δ, and choose $r<1$ so that $|z| \leqslant r$ when $z \in S$. Now, for $z \in S$ and for any natural number m,

$$|f_n(z)-f(z)| \leqslant \sum_{k=0}^{m} |\gamma_k(f_n-f) z^k| + \sum_{k=m+1}^{\infty} \|\gamma_k\| (\|f_n\| + \|f\|) r^k$$
$$\leqslant \sum_{k=0}^{m} |\gamma_k(f_n-f)| + 2(A+\|f\|) A_1(B) \frac{r^{m+1}}{1-r}.$$

If $\varepsilon > 0$ is given, we can choose m so large that

$$2(A+\|f\|) A_1(B) \frac{r^{m+1}}{1-r} < \frac{\varepsilon}{2}.$$

Then choose $n_0(\varepsilon)$ so that $n \geqslant n_0(\varepsilon)$ implies

$$|\gamma_k(f_n-f)| < \frac{\varepsilon}{2m+2}, \qquad k=0,1,\ldots,m.$$

Then $n \geqslant n_0(\varepsilon)$ and $z \in S$ imply $|f_n(z)-f(z)| < \varepsilon$; this completes the proof.

Theorem 3.5. *Let B_1 and B_2 be complete spaces of type $\mathfrak{A}_1$, and suppose that each element of B_1 is also an element of B_2. If $f \in B_1$, let $\|f\|_1$ and $\|f\|_2$ denote the norms of f as an element of B_1 and B_2 respectively. Then there exists a constant A depending only on B_1 and B_2 such that $\|f\|_2 \leqslant A\|f\|_1$ for each $f \in B_1$.*

Proof. The assertion is that the identity mapping of B_1 onto itself is bounded as an operator mapping B_1 into B_2. Since both spaces are complete it is sufficient to show that the mapping is closed (BANACH [1], p. 41). Suppose that $f_n, f \in B_1$, $g \in B_2$, and that $\|f_n - f\|_1 \to 0$, $\|f_n - g\|_2 \to 0$. From Theorem 3.2 we conclude that $f=g$, since the limit of $f_n(z)$ is unique. This proves that the mapping is closed, and completes the proof of the theorem.

[2]) Numbers in square brackets refer to the bibliography at the end of this paper.

We now consider the expression $B(f,g;z)$ (see Definition 1.5). Fixing $z \in \Delta$, $g \in \mathfrak{A}$, and taking $f \in B$, we have

$$B(f,g;z) = \sum_{n=0}^{\infty} \gamma_n(f)\gamma_n(g) z^n,$$

$$|B(f,g;z)| \leqslant \sum_{n=0}^{\infty} \|\gamma_n\| \|f\| |\gamma_n(g)| |z^n| \leqslant \|f\| A_1(B) \sum_{n=0}^{\infty} |\gamma_n(g)| |z|^n.$$

Thus $B(f,g;z)$ defines a linear functional of f over B if B is of type $\mathfrak{A}_1$.

Definition 3.1. When B is of type $\mathfrak{A}_1$ we define, for fixed $g \in \mathfrak{A}$ and $z \in \Delta$,

$$N(g;z) = \sup_{\|f\|=1} |B(f,g;z)|.$$

It is immediately evident that

(3.1) $N(g+h;z) \leqslant N(g;z) + N(h;z)$,

(3.2) $N(ag;z) = |a| N(g;z)$.

From (1.2) and (1.4) it is clear that

(3.3) $N(T_w g;z) = N(g;wz)$, $|w| \leqslant 1$.

We shall have a good deal more to say about $N(g;z)$ later on, when we assume that B is of type $\mathfrak{A}_3$.

4. Spaces of type $\mathfrak{A}_2$. In this section we shall make use of axioms P_1 and P_2. We also assume that B is a Banach space, i.e. that it is complete.

We shall deal with the notion of analyticity for functions defined on Δ, with values in a complex Banach space. Many features of the classical theory of functions are known to extend at once to such a situation (WIENER [6]; HILLE [2], p. 52-64).

Theorem 4.1. *Let B be a Banach space of type $\mathfrak{A}_2$. Suppose $f \in \mathfrak{A}$ and $w \in \Delta$. Then $T_w f \in B$. As a function of w, $T_w f$ is analytic in Δ, with the power series expansion*

$$T_w f = \sum_{n=0}^{\infty} \gamma_n(f) w^n u_n. \tag{4.1}$$

Proof. The series appearing in (4.1) is certainly convergent in B, since it is absolutely convergent and B is complete. The absolute convergence is clear, for by (2.2)

$$\|\gamma_n(f)\, w^n u_n\| \leqslant |\gamma_n(f)|\,|w|^n A_2(B),$$

and the series

$$f(w)=\sum_{n=0}^{\infty}\gamma_n(f)\, w^n$$

is absolutely convergent.

To show that the value of the series in (4.1) is the element $T_w f$ of $\mathfrak{A}$, we fix w and write

$$g=\sum_{k=0}^{\infty}\gamma_k(f)\, w^k u_k, \qquad s_n=\sum_{k=0}^{n}\gamma_k(f)\, u_k.$$

Then

$$T_w s_n=\sum_{k=0}^{n}\gamma_k(f)\, w^k u_k,$$

and so $\|T_w s_n - g\|\to 0$. Let $T_w s_n = g_n$. Then $g_n(z)\to g(z)$, by Theorem 3.2. But we see that $g_n(z)\to f(wz)$. Therefore $g(z)=f(wz)$, or $g=T_w f$. This proves that $T_w f$ is in B and has the series expansion (4.1). The analyticity in w of $T_w f$ is thereby proved.

Theorem 4.2. *Let B be a Banach space of type $\mathfrak{A}_2$, and let $f(z)$ be analytic in some circle $|z|<R$, where $R>1$. Then $f\epsilon B$.*

Proof. Define $g(z)=f(Rz)$, $z\epsilon\Delta$. Then $g\epsilon\mathfrak{A}$ and $T_w g=f$ if $w=R^{-1}$. Now apply Theorem 4.1.

Theorem 4.3. *Let B be a Banach space of type $\mathfrak{A}_2$. Suppose f, $f_n\epsilon\mathfrak{A}$ and let $f_n(z)\to f(z)$ uniformly on compact subsets of Δ. Then $\|T_w f_n - T_w f\|\to 0$ uniformly in w on compact subsets of Δ.*

Proof. Given $\varepsilon>0$ and a compact subset S of Δ, choose $r<1$ so that S lies in the circle $|w|\leqslant r$. Choose ϱ so that $r<\varrho<1$, and let

$$\delta=\frac{(\varrho-r)\,\varepsilon}{A_2(B)\,\varrho}.$$

For any $f\epsilon\mathfrak{A}$ we have, by Cauchy's formulas,

$$\gamma_k(f)=\frac{1}{2\pi\varrho^k}\int_0^{2\pi} f(\varrho e^{i\vartheta})\, e^{-ik\vartheta}\, d\vartheta.$$

Now choose $n_0=n_0(\delta,\varrho)$ so that $n>n_0$ and $|z|\leqslant\varrho$ imply $|f_n(z)-f(z)|<\delta$. Then

$$|\gamma_k(f_n-f)|\leqslant\frac{1}{2\pi\varrho^k}\int_0^{2\pi}\delta\, d\theta=\frac{\delta}{\varrho^k}.$$

Therefore, if $|w|\leqslant r$ and $n>n_0$, we have

$$\|T_w f_n - T_w f\| = \Big\|\sum_{k=0}^{\infty} w^k\gamma_k(f_n-f)\, u_k\Big\| \leqslant \sum_{k=0}^{\infty} r^k\frac{\delta}{\varrho^k}A_2(B)=\frac{A_2(B)\,\delta\,\varrho}{\varrho-r}=\varepsilon.$$

This completes the proof.

Before stating the next theorem we observe that Abel's theorem on the continuity of power series holds in case the coefficients in the power series are elements of a Banach space. We have only to inspect the proof to see that this extension of the classical theorem is valid (Titchmarsh [5], p. 229).

Let w_0 be a point for which $|w_0|=1$. By *salient of Δ at w_0* we mean the part of Δ lying in the angular region (of angle less than 180^0) between two chords of the unit circle which meet at w_0.

Theorem 4.4. *Let B be a Banach space of type $\mathfrak{A}_2$. Let f be an element of B such that*

$$f=\sum_{n=0}^{\infty}\gamma_n(f)\, u_n, \tag{4.2}$$

the series converging in the metric of B. Then $\|T_w f - f\|\to 0$ as $w\to 1$ within a salient of Δ at $w=1$.

Proof. Observe that the series in (4.2) is what we obtain when we set $w=1$ in the series (4.1). The present theorem is thus seen to be an immediate corollary of Theorem 4.1 and the extended version of Abel's theorem.

There are spaces (e. g. the spaces H^p, $1<p<\infty$, which we consider in Part II) such that 4.2 holds for each f in the space.

Theorem 4.5. *If B is a Banach space of type $\mathfrak{A}_2$, $T_w\epsilon[B]$ for each $w\epsilon\Delta$. As a function on Δ to $[B]$, T_w is analytic, with the series expansion*

$$T_w=\sum_{n=0}^{\infty} w^n E_n,$$

the coefficients being elements of $[B]$ defined by

$$E_n(f)=\gamma_n(f)\, u_n.$$

Proof. From Theorem 4.1 we have

$$\|T_w f\| \leqslant A_1(B)\,A_2(B)\|f\|\sum_{n=0}^{\infty}|w|^n = \frac{A_1(B)\,A_2(B)\,\|f\|}{1-|w|}.$$

This shows that $T_w \epsilon [B]$. Since $T_w f$ is analytic in Δ for fixed f, it follows from a theorem of the author (Taylor [3], p. 576) that T_w is analytic and that its power series coefficients are the ones indicated above.

5. A convexity theorem. We now prove

Theorem 5.1. *Let $\mathfrak{M}$ be a nonvoid subset of $\mathfrak{A}$ with the property that $U_x f \epsilon \mathfrak{M}$ if x is real and $f \epsilon \mathfrak{M}$. Corresponding to each $f \epsilon \mathfrak{M}$ let $M(f;z)$ be a member of $\mathfrak{A}$ with the properties:*

(a) $M(U_x f; z) = M(f; z e^{ix})$, *$x$ real, $z \epsilon \Delta$,*

(b) *for fixed $z \epsilon \Delta$, $|M(f;z)|$ is bounded as f varies over $\mathfrak{M}$.*

Let $M(z) = \sup\limits_f |M(f;z)|$. Then we conclude:

(1) $M(z) = M(|z|)$,

(2) *$M(r)$ is a nondecreasing function of r,*

(3) *either $M(r) \equiv 0$, or $M(r) > 0$ when $0 < r < 1$; in the latter case $\log M(r)$ is convex as a function of $\log r$.*

Proof. For any real x, $U_x f$ runs over all of $\mathfrak{M}$ when f runs over all of $\mathfrak{M}$ (for $f = U_x(U_{-x} f)$). Hence, from (a) we conclude that $M(ze^{ix}) = M(z)$; assertion (1) now follows.

To prove (2), assume $0 \leqslant r_1 < r_2 < 1$, $f \epsilon \mathfrak{M}$. The function $|M(f;z)|$ assumes its maximum value on the circle $|z| = r_k$ at some point $z = z_k$ $(k=1,2)$. By the maximum modulus theorem we have $|M(f;z_1)| \leqslant |M(f;z_2)|$. But $|M(f;z_2)| \leqslant M(z_2) = M(r_2)$. Therefore

$$|M(f;r_1)| \leqslant |M(f;z_1)| \leqslant M(r_2),$$

whence $M(r_1) \leqslant M(r_2)$.

To prove (3) we follow a similar argument, depending on the three circles theorem of Hadamard. If $0 < r_1 < r_2 < r_3 < 1$, let $M_k = M(r_k)$, $k = 1, 2, 3$. Write

$$n_{ij} = \log\frac{r_i}{r_j},$$

and let $|M(f;z)|$ assume its maximum on the circle $|z| = r_k$ at $z = z_k$. By Hadamard's theorem

$$|M(f;z_2)|^{n_{31}} \leqslant |M(f;z_1)|^{n_{32}}\,|M(f;z_3)|^{n_{21}}.$$

Therefore

$$|M(f;z_2)|^{n_{31}} \leqslant M_1^{n_{32}} M_3^{n_{21}},$$

and so, since $|M(f;r_2)| \leqslant |M(f;z_2)|$, we easily conclude that

$$M_2^{n_{31}} \leqslant M_1^{n_{32}} M_3^{n_{21}}. \tag{5.1}$$

From this inequality and (2) it follows that if $M(r) = 0$ for some r such that $0 < r < 1$, then $M(r) \equiv 0$. The proof of (3) is thus complete, the convexity as asserted being a consequence of (5.1).

Remark. $M(r)$ has the additional property that if it is constant on any interval, say $r_2 \leqslant r \leqslant r_3$, where $r_2 < r_3$, then it is constant on the interval $0 < r \leqslant r_3$. This property is shared by any function enjoying the properties (2) and (3) of Theorem 5.1. In fact, if $0 < r_1 < r_2$, we have (5.1) holding $M_1 \leqslant M_2$, and $M_2 = M_3$. Now $n_{31} - n_{21} = n_{32}$. We may assume $M_3 \neq 0$; hence we conclude from (5.1) that $M_2 \leqslant M_1$. But then $M_1 = M_2$. This proves the constancy of $M(r)$ on $0 < r \leqslant r_3$.

Theorem 5.2. *Let B be a space of type $\mathfrak{A}$ satisfying axiom P_3 and the further axiom that, for each $z \epsilon \Delta$, $f(z)$ defines a linear functional of f over B (which is certainly the case if axiom P_1 holds). Let*

$$m(z) = \sup_f \frac{|f(z)|}{\|z\|}, \quad f \neq 0, \quad f \epsilon B.$$

Then $m(z)$ has the properties (1)—(3) of the function $M(z)$ of Theorem 5.1; furthermore, $m(r) > 0$ if $0 < r < 1$.

Proof. Take $\mathfrak{M}$ as the set of non-zero elements of B; define

$$M(f;z) = \frac{f(z)}{\|z\|}, \quad z \epsilon \Delta, \quad f \epsilon \mathfrak{M}.$$

The proof now follows by application of Theorem 5.1. We cannot have $m(r) \equiv 0$, since no element of $\mathfrak{M}$ vanishes identically.

6. Spaces of type $\mathfrak{A}_3$. In this section we deal with spaces in which axioms P_1, P_2 and P_3 hold.

Theorem 6.1. *Let B be a space of type $\mathfrak{A}_3$. The function $N(g;z)$ of Definition 3.1 has the following properties:*

(1) $N(g;z)=N(g;|z|)$, $g\in\mathfrak{A}$, $z\in\Delta$;

(2) *$N(g;r)$ is a nondecreasing function of r;*

(3) *$N(g;r)=0$ for all r if and only if $g=0$;*

(4) *if $g\neq 0$, $\log N(g;r)$ is a convex function of $\log r$, $0<r<1$;*

(5) *$N(g;r)$ is continuous in r.*

Proof. We take $\mathfrak{M}$ as in the proof of Theorem 5.2, and define

$$M(f;z)=\frac{B(f,g;z)}{\|f\|},$$

where g is fixed in $\mathfrak{A}$. Then $M(z)=N(g;z)$. We apply Theorem 5.1. It remains only to discuss (3) and (5). Clearly $g=0$ implies $N(g;r)=0$ identically in r. On the other hand, if $g\neq 0$, we have $\gamma_n(g)\neq 0$ for some n. Take $f=u_n$. Then $B(u_n,g;r)=\gamma_n(g)r^n\neq 0$ if $r\neq 0$. Since $u_n\neq 0$, this implies $N(g;r)\neq 0$. Thus (3) is established.

In proving (5) we may assume $g\neq 0$, by (3). Continuity in r on the range $0<r<1$ is a consequence of (4), since a convex function is continuous. To prove continuity at $r=0$ we observe that from the definition of $N(g;r)$ we have

$$N(g;r)\leqslant\sum_{n=0}^{\infty}\|\gamma_n\||\gamma_n(g)|r^n.$$

Thus

$$\lim_{r\to 0}N(g;r)\leqslant\|\gamma_0\||\gamma_0(g)|=N(g;0).$$

The conclusion follows, since $N(g;r)$ is nondecreasing.

Theorem 6.2. *Let B be a Banach space of type $\mathfrak{A}_3$. Suppose $f\in\mathfrak{A}$. Then*

(1) *$\|T_w f\|=\|T_r f\|$ if $|w|=r<1$;*

(2) *$\|T_r f\|$ is a nondecreasing continuous function of r;*

(3) *if $f\neq 0$ and $0<r<1$, then $\|T_r f\|>0$, and $\log\|T_r f\|$ is a convex function of $\log r$;*

(4) *$\|T_r f\|$ is strictly increasing on an open subinterval of $(0,1)$ if and only if $|\gamma_0(f)|\|u_0\|<\|T_r f\|$ for each r of the subinterval.*

Proof. We appeal to Theorem 4.1. Let $w=re^{ix}$. Then

$$T_w f=U_x T_r f.$$

Assertion (1) now follows from axiom P_3. In view of (1) and the fact that $T_w f$ is analytic in w, (2) follows from the maximum modulus theorem for functions with values in a Banach space (Hille [2], p. 59; Taylor [4], Theorem 2.1). The first part of (3) is evident. The second part is merely the Hadamard three circles theorem for the function $T_w f$. The classical proof of this theorem (Titchmarsh [5], p. 172) extends at once to functions with values in a complex Banach space. To prove (4) we observe that, by the maximum modulus theorem, the equality

$$\|T_{r_1}f\|=\|T_{r_2}f\|,$$

where $0\leqslant r_1<r_2<1$, can occur only if $\|T_r f\|$ is constant on the interval $0\leqslant r\leqslant r_2$. Assertion (4) follows from this observation, because $\|T_0 f\|=|\gamma_0(f)|\|u_0\|$.

Theorem 6.3. *Let B be a Banach space of type $\mathfrak{A}_3$. Then*

(1) *$\|T_w\|=\|T_r\|$ if $|w|=r<1$,*

(2) *$\|T_r\|$ is a nondecreasing continuous function of r,*

(3) *$\|T_r\|>0$ if $r>0$; $\log\|T_r\|$ is a convex function of $\log r$.*

The proof depends on Theorem 4.5. We omit details, for the argument is almost identical with the proof of Theorem 6.2.

We conclude this section with the proof of an inequality complementary to (2.4.)

Theorem 6.4. *Suppose that B is a Banach space of type $\mathfrak{A}_3$. Then*

$$r^n\|\gamma_n\|\|u_n\|\leqslant\|T_r\|,\qquad n=0,1,2,\ldots$$

If B satisfies the axiom P_4, we have $\|\gamma_n\|\|u_n\|\leqslant A_4(B)$. In particular, if $A_4(B)=1$, we have $\|\gamma_n\|\|u_n\|=1$.

Proof. From Theorem 4.1 we have, by Cauchy's formulas for the coefficients in a power series,

$$\gamma_n(f)u_n=\frac{1}{2\pi i}\int_{|w|=r}\frac{T_w f}{w^{n+1}}\,dw.$$

Thus, by Theorem 6.3(1),

$$|\gamma_n(f)|\|u_n\|\leqslant\frac{\|T_r\|\|f\|}{r^n},$$

The conclusions of the theorem now follow; for the last assertion we use (2.4).

7. The spaces B' and B^0. In this section we deal with a space B of type $\mathfrak{A}_3$, and we show how to construct certain related spaces, which are of type $\mathfrak{A}_4$. These spaces are closely related to the study of linear functionals on B.

Definition 7.1. Let B be a space of type $\mathfrak{A}_3$, not necessarily complete. We define B' as the class of all elements $F \epsilon \mathfrak{A}$ such that $N(F;r)$ is uniformly bounded as a function of r. If $F \epsilon B'$ we write

$$N(F) = \sup_r N(F;r), \qquad 0 \leqslant r < 1.$$

We know (Theorem 6.1) that $N(F;r)$ is a nondecreasing function of r. Hence, if $F \epsilon B'$,

$$N(F) = \lim_{r \to 1} N(F;r).$$

Later we shall find it convenient to write $\|F\|'$ instead of $N(F)$

Theorem 7.1 *If B is a space of type $\mathfrak{A}_3$ the class B' forms a complete normed linear space with $N(F)$ as the norm of F.*

Proof. The fact that B' is a normed linear space is evident from the relations (3.1), (3.2), and Theorem 6.1, part (3). To prove that B' is complete, assume that $\{F_n\}$ is a Cauchy sequence in B'. Then $N(F_n)$ is bounded, say $N(F_n) \leqslant A$. Now

$$|\gamma_k(F_n - F_m) r^k| = |B(u_k, F_n - F_m; r)| \leqslant \|u_k\| N(F_n - F_m) \leqslant A_2(B) N(F_n - F_m),$$

whence it follows that $\{\gamma_k(F_n)\}$ is a Cauchy sequence for each k.

Let us define

$$a_k = \lim_{n \to \infty} \gamma_k(F_n).$$

We observe that the convergence here is uniform in k. The sequence $\{a_k\}$ is bounded, for

$$|\gamma_k(F_n) r^n| \leqslant A_2(B) N(F_n) \leqslant A_2(B) A,$$

by an argument like that given above, and therefore $|a_k| \leqslant A_2(B) A$. We now define

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

This function is a member of $\mathfrak{A}$. We shall show that $F \epsilon B'$ and that $N(F_n - F) \to 0$ as $n \to \infty$.

First of all we observe that, for fixed r, and $f \epsilon \mathfrak{A}$,

$$\lim_{n \to \infty} B(f, F_n; r) = B(f, F; r).$$

For

$$|B(f, F_n; r) - B(f, F; r)| = \left|\sum_{k=0}^{\infty} \gamma_k(f)[\gamma_k(F_n) - a_k] r^k\right|$$

$$\leqslant \left(\sum_{k=0}^{\infty} |\gamma_k(f)| r^k\right) \sup_k |\gamma_k(F_n) - a_k|;$$

the assertion follows, since $\gamma_k(F_n) - a_k \to 0$ uniformly in k. Now, if $f \epsilon B$,

$$|B(f, F_n; r)| \leqslant N(F_n)\|f\| \leqslant A\|f\|.$$

Letting $n \to \infty$, we conclude $|B(f, F; r)| \leqslant A\|f\|$, whence $F \epsilon B'$, for $N(F;r) \leqslant A$. Now suppose $\varepsilon > 0$, and let us choose $n_0(\varepsilon)$ so that $m, n \geqslant n_0(\varepsilon)$ imply $N(F_n - F_m) < \varepsilon$. Then, if $f \epsilon B$,

$$|B(f, F_n - F_m; r)| \leqslant N(F_n - F_m)\|f\| < \varepsilon\|f\|.$$

Letting $m \to \infty$, we conclude that $|B(f, F_n - F; r)| \leqslant \varepsilon\|f\|$ if $n \geqslant n_0(\varepsilon)$. It now follows that $N(F_n - F) \leqslant \varepsilon$. This completes the proof that B' is complete.

Theorem 7.2. *Let B be a space of type $\mathfrak{A}_3$. Then the space B' is a space of type $\mathfrak{A}_4$. The constants $A_k(B')$ satisfy the following relations:*

(a) $A_1(B') \leqslant A_2(B)$,

(b) $A_2(B') = A_1(B)$,

(c) $A_4(B') = 1$.

Proof. We have $\gamma_n(F) r^n = B(u_n, F; r)$. Hence, if $F \epsilon B'$, we have $|\gamma_n(F) r^n| \leqslant \|u_n\| N(F)$, and so $|\gamma_n(F)| \leqslant \|u_n\| N(F)$. This means that $\gamma_n \epsilon (B')^*$. If we denote the norm of γ_n in this sense by $N(\gamma_n)$, we see that $N(\gamma_n) \leqslant \|u_n\| \leqslant A_2(B)$. Thus Axiom P_1 holds for B', and (a) is true.

For any $f \epsilon B$ we have $B(f, u_n; r) = \gamma_n(f) r^n$. Thus $N(u_n; r) = \|\gamma_n\| r^n$. It follows that $u_n \epsilon B'$ and $N(u_n) = \|\gamma_n\|$. This settles (b). Taking $w = e^{ix}$ in (3.3), we have $N(U_x F; \varrho) = N(F; \varrho e^{ix}) = N(F; \varrho)$; the last equality comes from Theorem 6.1(1). Thus $F \epsilon B'$ implies $U_x F \epsilon B'$ and $N(U_x F) = N(F)$; Axiom P_3 is satisfied. Again, from (3.3) and

Theorem 6.1(2), $N(T_rF;\varrho)=N(F;r\varrho)\leqslant N(F;\varrho)$. Therefore, if $F\epsilon B'$, we have $T_rF\epsilon B'$ and $N(T_rF)\leqslant N(F)$. We conclude that Axiom P_4 holds for B', with $A_4(B')\leqslant 1$. The reverse inequality also holds, by (2.5). The proof is now complete.

Theorem 7.3. *Let B be a space of type $\mathfrak{A}_3$. If $g\epsilon\mathfrak{A}$ and $w\epsilon\Delta$, then $T_w g\epsilon B'$ and*

$$N(T_w g)=N(g;|w|). \tag{7.1}$$

An element $g\epsilon\mathfrak{A}$ is in B' if and only if $N(T_r g)$ is bounded as a function of r: in this event

$$N(g)=\lim_{r\to 1} N(T_r g).$$

Proof. From (3.3) and Theorem 6.1(1) we have $N(T_w g;r)=N(g;rw)=N(g;r|w|)$. As $r\to 1$ we see by Theorem 6.1(5) that $T_w g\epsilon B'$ and that (7.1) is true. The rest of the theorem is now obvious.

The property of B' expressed in this last sentence is used as an axiom later (Axiom P_7, § 9). It has some interesting consequences (e. g., Theorem 9.2).

The next theorem is related to Theorem 3.5.

Theorem 7.4. *Let B_1 and B_2 be two spaces of type $\mathfrak{A}_3$. Denote the norm in B_k by $\|\cdot\|_k$, and that in B'_k by $N_k(\cdot)$. Suppose that each element of B_1 is contained in B_2 and that there exists a constant A such that $\|f\|_2\leqslant A\|f\|_1$ if $f\epsilon B_1$. Then the class B'_2 is contained in the class B'_1, and if $F\epsilon B'_2$, $N_1(F)\leqslant AN_2(F)$. Furthermore, if $g\epsilon\mathfrak{A}$,*

$$N_1(g;r)\leqslant AN_2(g;r). \tag{7.2}$$

Proof. If $f\epsilon B_2$ and $g\epsilon\mathfrak{A}$, we have

$$|B(f,g;r)|\leqslant\|f\|_2 N_2(g;r).$$

Thus, if $f\epsilon B_1$,

$$|B(f,g;r)|\leqslant A\|f\|_1 N_2(g;r),$$

whence (7.2) follows. If now $F\epsilon B'_2$ we have $N_1(F;r)\leqslant AN_2(F)$, and the remaining conclusion of the theorem follows.

Definition 7.2. Let B be a space of type $\mathfrak{A}_3$. We define B^0 as the class of all $F\epsilon\mathfrak{A}$ such that $\lim\limits_{r\to 1} B(f,F;r)$ exists for each $f\epsilon B$.

The set B^0 is not of much interest unless we put some additional restriction on B. If, for example, B consists of all polynomials in z, with any norm, then it is easily seen that B^0 is the entire class $\mathfrak{A}$.

A situation of real interest is obtained by assuming that B is complete.

Theorem 7.5. *Let B be a Banach space of type $\mathfrak{A}_3$. Then B^0 is a linear subset of B'. If we adopt the norm $N(F)$ for elements of B^0, then B^0 is a complete space of type $\mathfrak{A}_4$. Furthermore,*

(a) $A_1(B^0)\leqslant A_1(B')$,

(b) $A_2(B^0)=A_2(B')$,

(c) $A_4(B^0)=1$.

Proof. That B^0 is a linear subset of $\mathfrak{A}$ follows from (1.2) and (1.3). Now the norm of $B(f,F;r)$ as a functional of f over B is $N(F;r)$. Hence the hypothesis $F\epsilon B^0$ implies that $N(F;r)$ is bounded as function of r, by the Banach-Steinhaus theorem (now often called the *principle of uniform boundedness*) (Banach [1], Théorème 5, p. 80; Hille [2], Theorem 2.12.2, p. 26). Hence B^0 is a linear subset of B'.

To show that B^0 is complete, or what is equivalent, that B_0 is closed in B', we suppose $F_n\epsilon B^0$, $F\epsilon B'$, $N(F_n-F)\to 0$. We shall show that, for each $f\epsilon B$,

$$\lim_{r,\varrho\to 1}|B(f,F;r)-B(f,F;\varrho)|=0;$$

this will imply $F\epsilon B^0$. Now

$$\begin{aligned}&|B(f,F;r)-B(f,F;\varrho)|\\ &\leqslant|B(f,F-F_n;r)|+|B(f,F_n;r)-B(f,F_n;\varrho)|+|B(f,F_n-F;\varrho)|\\ &\leqslant 2N(F_n-F)\|f\|+|B(f,F_n;r)-B(f,F_n;\varrho)|.\end{aligned}$$

With f fixed in B and $\varepsilon>0$ given, chose n so large that $2N(F_n-F)\|f\|<\varepsilon/2$. With n thus fixed, we may choose r and ϱ so near 1 that

$$|B(f,F_n;r)-B(f,F_n;\varrho)|<\varepsilon/2,$$

since $F_n\epsilon B^0$. This concludes the proof that B^0 is closed in B'.

Of the remainder of the proof we consider only (c) and the axioms P_3, P_4, the other considerations being extremely obvious.

By (1.2) and (1.4) we have $B(f,U_xF;r)=B(U_xf,F;r)$. From this it follows that P_3 holds for B^0. Next, $B(f,T_rF;\varrho)=B(f,F;r\varrho)\to B(f,F;r)$ as $\varrho\to 1$. Hence $T_rF\epsilon B^0$ if $F\epsilon\mathfrak{A}$. It is now clear that P_4 holds for B^0. The relation (c) is true by virtue of (2.5), Theorem 7.2 (c), and the fact that $A_4(B^0)\leqslant A_4(B')$.

For a case where B^0 is a proper subset of B', see § 13.

We write B'' for $(B')'$. In dealing with B'' it is convenient to introduce some new notation. The functions $N(g;r)$ and $N(F)$ depend upon the underlying space B. If we wish to show this dependence explicitly we shall write $N(g;r:B)$ for $N(g;r)$ and $N(F;B)$ for $N(F)$. The norm in B'' is then introduced in accordance with definitions 3.1 and 7.1, with B' replacing B. It is also convenient to use the notations

$$\|F\|'=N(F:B),\quad F\epsilon B',$$
$$\|f\|''=N(f:B'),\quad f\epsilon B'',$$

for the norms in B' and B'' respectively.

Theorem 7.6. *Let B be a space of type $\mathfrak{A}_3$. Then B is a subset of B''. If $f\epsilon B$ we have $\|f\|''\leqslant\|f\|$.*

Proof. Suppose $f\epsilon B$. Then, if $F\epsilon B'$,

$$|B(F,f;r)|=|B(f,F;r)|\leqslant\|F\|'\|f\|,$$

so that $N(f;r:B')\leqslant\|f\|$. It follows that $f\epsilon B''$ and $\|f\|''\leqslant\|f\|$.

Theorem 7.7. *Let B be a space of type $\mathfrak{A}_3$. Let f be an element of $\mathfrak{A}$ such that $T_rf\epsilon B$ and*

$$\sup_{0\leqslant r<1}\|T_rf\|=A<\infty.$$

Then $f\epsilon B''$ and $\|f\|''\leqslant A$.

Proof. With an f as assumed in the theorem, and any $F\epsilon B'$, we have

$$|B(F,f;r\varrho)|=|B(T_rf,F;\varrho)|\leqslant\|T_rf\|\|F\|'.$$

Hence, letting $\varrho\to 1$, we have

$$|B(F,f;r)|\leqslant A\|F\|',\ N(f;r:B')\leqslant A,$$

whence the truth of the theorem follows.

8. Banach spaces of type $\mathfrak{A}_4$. In this section we assume that B is complete and that axioms P_1-P_4 hold. Our first concern is with the relation between the spaces B^0, B', and the space B^* of all linear functionals defined on B. We recall that if X and Y are normed linear spaces, the set of linear operators which map X into Y is also a normed linear space. We denote it by $[X,Y]$.

Theorem 8.1. *Let B be a Banach space of type $\mathfrak{A}_4$. If γ is an element of B^*, consider the function defined by*

$$G(z)=\sum_{n=0}^{\infty}\gamma(u_n)z^n,\quad z\epsilon\Delta. \tag{8.1}$$

Then $G\epsilon B'$ and $\|G\|'\leqslant A_4(B)\|\gamma\|$.

Proof. Certainly $G\epsilon\mathfrak{A}$, for $|\gamma(u_n)|\leqslant\|\gamma\|\|u_n\|\leqslant\|\gamma\|A_2(B)$. Observe the relation $\gamma(u_n)=\gamma_n(G)$. From it and Theorem 4.1 we derive the useful relation

$$\gamma(T_wf)=B(f,G;w),\quad w\epsilon\Delta,\quad f\epsilon\mathfrak{A}. \tag{8.2}$$

From (8.2) we see that if $f\epsilon B$,

$$|B(f,G;r)|\leqslant\|\gamma\|\|T_rf\|\leqslant\|\gamma\|A_4(B)\|f\|.$$

It now follows that $G\epsilon B'$ and that $\|G\|'\leqslant A_4(B)\|\gamma\|$, as asserted.

Definition 8.1. The passage from γ to G as indicated in (8.1) defines an operator which we denote by Γ: $G=\Gamma(\gamma)$.

Theorem 8.2. *Let B be a Banach space of type $\mathfrak{A}_4$. Then $\Gamma\epsilon[B^*,B']$ and $\|\Gamma\|=A_4(B)$. The operator Γ has an inverse (i. e., $\Gamma(\gamma)=0$ implies $\gamma=0$) if and only if the linear subspace of B spanned by $\{u_n\}$ is dense in B (or, equivalently, if and only if $\{u_n\}$ is total in B).*

Proof. Clearly $\Gamma\epsilon[B^*,B']$ and $\|\Gamma\|\leqslant A_4(B)$, by Theorem 8.1. From (8.2) we see that

$$|\gamma(T_rf)|\leqslant\|f\|\|G\|'\leqslant\|f\|\|\Gamma\|\|\gamma\|.$$

We can choose γ so that $\|\gamma\|=1$ and $|\gamma(T_rf)|=\|T_rf\|$. Therefore $\|T_rf\|\leqslant\|f\|\|\Gamma\|$. It follows that $\|T_r\|\leqslant\|\Gamma\|$ and hence that $A_4(B)\leqslant\|\Gamma\|$. Thus $A_4(B)=\|\Gamma\|$.

The supposition $\Gamma(\gamma)=0$ is equivalent to $\gamma(u_n)=0$, $n=0,1,\ldots$ Hence Γ^{-1} exists if and only if the latter sequence of conditions is equivalent to $\gamma=0$. In other words, Γ^{-1} exists if and only if $\{u_n\}$ is total in B (Banach [1], p. 58). The fact that $\{u_n\}$ is total if and only if the linear subspace spanned by $\{u_n\}$ is dense in B is a well known theorem (Banach [1], Théorème 7, p. 58).

We recall that, as f varies over B, $B(f,F;r)$ defines a linear functional of norm $N(F;r)$. Hence it is clear that if $F\epsilon B^0$,

$$\gamma(f)=\lim_{r\to 1} B(f,F;r) \tag{8.3}$$

defines an element γ of B^*.

Definition 8.2. The passage from F to γ as indicated in (8.3) defines an operator which we denote by Λ: $\Lambda(F)=\gamma$.

Theorem 8.3. *Let B be a Banach space of type $\mathfrak{A}_4$. Then $\Lambda\epsilon[B^0,B^*]$. Furthermore*

$$\Gamma\Lambda(F)=F \quad \textit{if} \quad F\epsilon B^0, \tag{8.4}$$

$$\|\Lambda(F)\|\leqslant\|F\|'\leqslant A_4(B)\|\Lambda(F)\| \quad \textit{if} \quad F\epsilon B^0. \tag{8.5}$$

Thus Λ defines a one to one mapping, with bounded inverse, of B^0 onto a subspace of B^*.

Proof. From $|B(f,F;r)|\leqslant\|f\|\|F\|'$ we conclude $|\gamma(f)|\leqslant\|f\|\|F\|'$, where $F\epsilon B^0$, and $\gamma=\Lambda(F)$. Thus $\|\Lambda(F)\|\leqslant\|F\|'$. To establish (8.4) it is enough to prove $\gamma_n(G)=\gamma_n(F)$ for each n, where $G=\Gamma(\gamma)$. Now $\gamma_n(G)=\gamma(u_n)=\lim\limits_{r\to1}B(u_n,F;r)=\lim\limits_{r\to1}\gamma_n(F)r^n=\gamma_n(F)$, as required. The second inequality in (8.5) follows at once from (8.4) and the first assertion in Theorem 8.2. This completes the proof.

Theorem 8.4. *Let B be a Banach space of type $\mathfrak{A}_4$. Then the image $\Lambda(B^0)$ of B^0 in B^* is all of B^* if and only if T_rf converges weakly to f as $r\to 1$ for each $f\epsilon B$. When this is the case, the operator Γ defines a 1-1 mapping of B^* onto B^0, with $\Gamma^{-1}=\Lambda$.*

Proof. If $F\epsilon B^0$ and $\gamma=\Lambda(F)$ we have

$$\gamma(T_rf)=B(f,F;r), \quad f\epsilon\mathfrak{A}, \tag{8.6}$$

for

$$\gamma(T_rf)=\lim_{\varrho\to1}B(T_rf,F;\varrho)=\lim_{\varrho\to1}B(f,F;r\varrho).$$

From (8.6) and the definition of Λ we see that

$$\gamma(f)=\lim_{r\to1}\gamma(T_rf)$$

for each $f\epsilon B$ and each $\gamma\epsilon\Lambda(B^0)$. Hence T_rf converges weakly to f as $r\to1$ if $\Lambda(B^0)=B^*$.

Suppose now that, for each $f\epsilon B$, T_rf converges weakly to f as $r\to1$. By (8.2) we see that

$$\lim_{r\to1}B(f,\Gamma(\gamma);r)=\lim_{r\to1}\gamma(T_rf)=\gamma(f)$$

for each $f\epsilon B$ and each $\gamma\epsilon B^*$. Thus $\Gamma(\gamma)\epsilon B^0$ and $\Lambda\Gamma(\gamma)=\gamma$. This shows that $\gamma\epsilon\Lambda(B^0)$, and hence that $B^*=\Lambda(B^0)$. The last assertion of the theorem follows from (8.4) and what we have just shown.

As a corollary of Theorems 8.2 and 8.4 we have

Theorem 8.5. *Let B be a Banach space of type $\mathfrak{A}_4$. Suppose that, for each $f\epsilon B$, T_rf converges weakly to f as $r\to1$. Then the set of all finite linear combinations of $u_0, u_1, u_2,\ldots$ is everywhere dense in B.*

We can now sharpen the statement of Theorem 7.7:

Theorem 8.6. *Let B be a Banach space of type $\mathfrak{A}_4$. The space B'' consists of all elements $f\epsilon\mathfrak{A}$ such that $\|T_rf\|$ is bounded as a function of r. The following inequalities are satisfied by each element of B'':*

(a) $\|f\|''\leqslant\sup\limits_r\|T_rf\|$;

(b) $\sup\limits_r\|T_rf\|\leqslant A_4(B)\|f\|''$.

Proof. In view of Theorem 7.7 we have only to prove the inequality (b). Given $f\epsilon B''$, we know that $T_rf\epsilon B$ by Theorem 4.1. With r and f fixed we choose $\gamma\epsilon B^*$ so that $\|\gamma\|=1$ and that $\gamma(T_rf)=\|T_rf\|$. Let $G=\Gamma(\gamma)$, so that, by (8.2), $\gamma(T_rf)=B(f,G;r)$. Now $G\epsilon B'$ and $\|G\|'\leqslant A_4(B)\|\gamma\|$, by Theorem 8.1. Hence

$$\|T_rf\|=|B(f,G;r)|\leqslant\|f\|''\|G\|'\leqslant A_4(B)\|f\|''.$$

Thus (b) is established. It is to be noted that if $A_4(B)=1$, the inequalities (a) and (b) become equalities. Now we know (Theorem 7.2 (c)) that $A_4(B')=1$. We also know (Theorem 7.3) that an element $f\epsilon\mathfrak{A}$ belongs to B' if and only if $\|T_rf\|'$ is bounded as a function of r, in which case $\|f\|'=\lim\limits_{r\to1}\|T_rf\|'$. In view of Theorems 7.1, 7.2 and 8.6 we therefore have, writing

$$B'''=(B')''=(B'')',$$

Theorem 8.7. *Let B be any space of type $\mathfrak{A}_4$ (not necessarily complete). Then B' and B''' are identical Banach spaces.*

9. Axioms P_5, P_6, P_7. We now introduce three more axioms which a space of type $\mathfrak{A}$ may satisfy, and consider some consequences of subjecting a space to one of these axioms in addition to P_1-P_4.

P_5. If $f \epsilon B$ then $T_r f \epsilon B$ and $\|f\| = \sup_r \|T_r f\|$.

P_6. If $f \epsilon B$ then $T_r f \epsilon B$ and $\lim_{r\to 1} \|T_r f - f\| = 0$.

P_7. If $f \epsilon \mathfrak{A}$ is such that $T_r f \epsilon B$ for each r, and $\sup_r \|T_r f\| < \infty$, then $f \epsilon B$ and $\|f\| = \sup_r \|T_r f\|$.

It is clear that when Axiom P_5 holds, P_4 does also, and that $A_4(B) = 1$. Axioms P_4 and P_7 together imply Axiom P_5. If B is a complete space of type $\mathfrak{A}_3$ in which P_6 holds, then P_5 holds also (by Theorem 6.2(2)).

Theorem 9.1. *If B is a complete space of type $\mathfrak{A}_5$ (i.e., satisfying P_1-P_5) then B is a subspace of B''.*

Proof. We know (Theorem 7.6) that $B \subset B''$ and $\|f\|'' \leqslant \|f\|$ if $f \epsilon B$. From Theorem 8.6 (b) and Axiom P_5 we have $\|f\| \leqslant \|f\|''$, since $A_4(B) = 1$. Thus $\|f\| = \|f\|''$, and the proof is complete.

Theorem 9.2. *Let B be a complete space of type $\mathfrak{A}_4$ satisfying the additional axiom P_7 (and hence P_5 also). Then the spaces B and B'' coincide.*

Proof. By Theorem 9.1 we have only to prove that B is the whole of B''. This follows from P_7 and Theorem 8.6.

We use P_6 to sharpen the results of Theorems 8.1-8.4:

Theorem 9.3. *Let B be a complete space of type $\mathfrak{A}_4$ satisfying the additional axiom P_6. Then B^0 and B' coincide, and B' is equivalent to B^* (i.e., there is a 1-1 linear isometric mapping of B' onto all of B^*).*

Proof. We first prove that $B' \subset B^0$. We already know that $B^0 \subset B'$. From (1.2) and (1.4) we see that

$$B(T_r f, F; \varrho) = B(T_\varrho f, F; r). \tag{9.1}$$

With the aid of (9.1) we find that

$$B(f,F;r) - B(f,F;\varrho)$$
$$= B(f - T_\varrho f, F; r) + B(T_r f - T_\varrho f, F; \varrho) + B(T_\varrho f - f, F; \varrho).$$

Therefore, if $f \epsilon B$ and $F \epsilon B'$,

$$|B(f,F;r) - B(f,F;\varrho)| \leqslant \{2\|f - T_\varrho f\| + \|T_r f - T_\varrho f\|\} \|F\|'.$$

From P_6 it now follows that

$$\lim_{r,\varrho\to 1} |B(f,F;r) - B(f,F;\varrho)| = 0;$$

this implies that $F \epsilon B^0$. Thus B^0 and B' coincide.

Next we observe that $A_4(B) = 1$. For $\|T_r f\| \to \|f\|$ as $r \to 1$, by P_6, and $\|T_r f\|$ is nondecreasing in r. Finally, if P_6 holds it is certainly true that $T_r f$ converges weakly to f as $r \to 1$. Theorems 8.3 and 8.4 now show that the operator Λ defines an isometric mapping of B' onto B^*; the inverse mapping of B^* onto B' is defined by the operator Γ. This completes the proof.

Theorem 9.4. *Let B be a Banach space of type $\mathfrak{A}_4$ satisfying axiom P_7. Let B_1 be a closed subspace of B which is in itself a space of type $\mathfrak{A}_4$. Then B_1'' and B are identical, and B_1' and B' are identical.*

Proof. From the remarks at the beginning of this section we see that $A_4(B) = 1$. Hence $A_4(B_1) = 1$ also. The identity of B_1'' and B now follows from Theorem 8.6, applied to B_1, and from axiom P_7. By Theorem 8.7 and the result just proved we have $B_1' = B_1''' = B'$.

It is clear from Theorem 9.2 that B_1 cannot satisfy axiom P_7 if it is a proper subspace of B.

10. Representation of linear functionals. Theorems 8.4 and 9.3 give representation theorems for linear functionals on B. The function $B(f,F;r)$ figures in the representation. This function was defined by an infinite series (Definition 1.5), but there is also an integral representation, as given in formula (1.5). In the latter formula let us put $z_1 = \varrho$, $z_2 = r\varrho^{-1}$, where $r < \varrho < 1$. Then

$$B(f,F;r) = \frac{1}{2\pi}\int_0^{2\pi} f(\varrho e^{i\vartheta}) F\left(\frac{r}{\varrho} e^{-i\vartheta}\right) d\vartheta. \tag{10.1}$$

The number ϱ is arbitrary, subject to the limitation indicated.

Theorem 10.1. *Let B be a Banach space of type $\mathfrak{A}_4$ such that $T_r f$ converges weakly to f as $r \to 1$, for each $f \in B$. Then every linear functional $\gamma \in B^*$ is representable in the form*

$$\gamma(f) = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} f(\varrho e^{i\vartheta}) F(\frac{r}{\varrho} e^{-i\vartheta})\, d\vartheta, \qquad f \in B, \tag{10.2}$$

where $r < \varrho < 1$ and $F \in B^0$. The element F of B^0 uniquely determines and is uniquely determined by γ. Furthermore,

$$\|\gamma\| \leqslant \|F\|' \leqslant A_4(B)\|\gamma\|. \tag{10.3}$$

Under the stronger hypothesis that $\lim\limits_{r\to 1}\|T_r f - f\| = 0$ for each $f \in B$ we have the same representation (10.2). *In this case, however, F may be any element of B', and $\|\gamma\| = \|F\|'$.*

The theorem is merely a restatement of Theorems 8.4 and 9.3.

There will be circumstances under which it is legitimate to make $\varrho \to 1$ under the integral sign in (10.2). Sometimes we may even take the limit with respect to r under the integral sign. The possibility of carrying out these processes depends upon the character of the functions f and F at the boundary of the unit circle.

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(Reçu par la Rédaction le 21. 9. 1949)

Independent fields and cartesian products

by

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For every $\tau \in T(\overline{\overline{T}} \geqslant 2)$ let $\boldsymbol{X}_\tau$ be a σ-field [1]) of subsets of a fixed set $\mathscr{X}$. We shall say that the fields $\boldsymbol{X}_\tau$ $(\tau \in T)$ are *σ-independent* [2]) if

$$\prod_n X_n \neq 0 \tag{*}$$

for every σ-sequence [3]) of non-empty sets $X_n \in \boldsymbol{X}_{\tau_n}$, where $\tau_k \neq \tau_l$ for $k \neq l$.

Suppose μ_τ is a σ-measure [4]) on $\boldsymbol{X}_\tau$. We shall say that the σ-fields X_τ $(\tau \in T)$ are *stochastically σ-independent* (with respect to the σ-measures μ_τ), if there is a σ-measure μ (called the *stochastic σ-extension* of all μ_τ) on the least σ-field $\boldsymbol{X}$ containing all the σ-fields $\boldsymbol{X}_\tau$ such that μ is a common extension of all μ_τ $(\tau \in T)$ and

$$\mu(\prod_n X_n) = \prod_n \mu(X_n) \tag{$\ast\ast$}$$

for every sequence [5]) of sets $X_n \in \boldsymbol{X}_{\tau_n}$, where $\tau_k \neq \tau_l$ for $k \neq l$.

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[1]) A non-void class $\boldsymbol{P}$ of subsets of a set $\mathscr{P}$ is called a *field*, if $P, Q \in \boldsymbol{P}$ implies $\mathscr{P} - P \in \boldsymbol{P}$ and $P + Q \in \boldsymbol{P}$. A field $\boldsymbol{P}$ is called a *σ-field*, if $P_n \in \boldsymbol{P}$ $(n = 1, 2, 3, \ldots)$ implies $P_1 + P_2 + P_3 + \ldots \in \boldsymbol{P}$.

[2]) The concept of the independence of fields has been introduced by Marczewski in paper [7].

[3]) In this paper we shall write, for convenience, *a sequence* instead of *a finite sequence*, and *a σ-sequence* instead of *a finite or enumerable sequence*.

[4]) A *σ-measure* μ on a σ-field $\boldsymbol{P}$ of subsets of a set $\mathscr{P}$ is a non-negative function such that $\mu(\mathscr{P}) = 1$ and $\mu(\sum\limits_n P_n) = \sum\limits_n \mu(P_n)$ for every σ-sequence of disjoint sets $P_n \in \boldsymbol{P}$. By omitting the letter σ in the above definition we obtain the analogous definition of a *measure* on a field.

[5]) Consequently for every σ-sequence also.