and by the inequality $s \leqslant k-s$, it is also not smaller than $k-1+E\lg_2(k-1)$. Hence and from Lemma 2 follows Lemma 4.

From the lemmas proved above follows

Theorem. The best system of tournament the purpose of which is to establish the champion and the second champion is the system S.

The number of matches sufficient for establishing the champion of the tournament by this system is $n-1$; the number sufficient for establishing the champion and the second champion is $n-1+E\lg_2(n-1)$.

ARITHMETICS OF NATURAL NUMBERS AS PART OF THE BI-VALUED PROPOSITIONAL CALCULUS

BY

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This paper[1]) contains an outline of a method of constructing — within the bi-valued propositional calculus — of an arithmetic of natural numbers, elementary and extremely narrow, but sufficient for practical purposes. Utilization of such a logical construction to the theory of relay and electronic digital machines will be discussed elsewhere.

I wish to express my thanks to J. Egerváry, Budapest, for his valuable suggestions which helped me to formulate the definition scheme of addition of natural numbers, as well as to A. Mostowski, Warsaw, for certain suggestions of general character.

1. *Falsum, verum,* negation, implication, alternation, conjunction, as well as existential and general quantifiers, are symbolised, respectively, as follows:

$$0,\ 1,\ {}',\ \rightarrow,\ \dot{+},\ .,\ \Sigma,\ \Pi\ ^{2)}.$$

The letters

$$p, p_{\mathrm{I}}, p_{\mathrm{II}}, p_{\mathrm{III}}, \ldots, q, q_{\mathrm{I}}, q_{\mathrm{II}}, q_{\mathrm{III}}, \ldots,$$

$$r, r_{\mathrm{I}}, r_{\mathrm{II}}, r_{\mathrm{III}}, \ldots, s, s_{\mathrm{I}}, s_{\mathrm{II}}, s_{\mathrm{III}}, \ldots, t, t_{\mathrm{I}}, t_{\mathrm{II}}, t_{\mathrm{III}}, \ldots,$$

are propositional variables.

[1]) Partly identical with the author's paper, *Les tautologies arithmétiques du calcul propositionnel et les circuits éléctriques*, read at the First Congress of Hungarian Mathematicians (Budapest 1950).

[2]) The method of defining Σ, Π in the language of the propositional calculus was formulated in the author's paper *Functors of the Propositional Calculus*, read at the Sixth Congress of Polish Mathematicians, Warsaw, September 20-23, 1948, see *VI Zjazd Matematyków Polskich*, Dodatek do Rocznika Polskiego Towarzystwa Matematycznego 22 (1950), p. 78-80.

In propositional functions, propositional variables that follow one another in succession are not separated by commas; thus instead of "$\varphi(p,q,r)$" it is written "$\varphi(pqr)$". Several propositional variables following one another in succession are often replaced by a capital letter; thus "$p_{II}p_I$" is replaced by "P_{II}", and "$q_{III}q_{II}q_I$" by "Q_{III}". In more general terms: 1) $p_I, q_I, r_I, \ldots$ is often replaced by $P_I, Q_I, R_I, \ldots$ respectively, 2) $p_{I\alpha}P_\alpha, q_{I\alpha}Q_\alpha, r_{I\alpha}R_\alpha, \ldots$ is written $P_{I\alpha}, Q_{I\alpha}, R_{I\alpha}, \ldots$ The abbreviations introduced in this way to the language of the propositional calculus can conveniently be called *natural variables* (instead of "natural variable" the term "variable binary development" might be used as well). Analogous abbreviations are introduced for the propositional constants following one another:

(1.01) $0_I =_{Df} 0,$

(1.02) $0_{I\alpha} =_{Df} 00_\alpha,$

(1.11) $1_I =_{Df} 1,$

(1.12) $1_{I\alpha} =_{Df} 11_\alpha.$

Thus "$\varphi(000)$" is written briefly "$\varphi(0_{III})$", and "$\psi(p_{II}p_I111)$" is abbreviated to "$\psi(P_{II}1_{III})$".

2. Symmetrical difference and equivalences, namely those involving two propositional variables and those involving more than two propositional variables are defined:

(2.01) $(p \dot{-} q) =_{Df} (p' \cdot q \dot{+} p \cdot q'),$

(2.11) $(P_I \equiv Q_I) =_{Df} (p_I' \cdot q_I' + p_I \cdot q_I),$

(2.12) $(P_{I\alpha} \equiv Q_{I\alpha}) =_{Df} [(p_{I\alpha} \equiv q_{I\alpha}) \cdot (P_\alpha \equiv Q_\alpha)],$

(2.13) $(P_{I\alpha} \equiv Q_\alpha) =_{Df} [p'_{I\alpha} \cdot (P_\alpha \equiv Q_\alpha)],$

(2.14) $(P_\alpha \equiv Q_{I\alpha}) =_{Df} [q'_{I\alpha} \cdot (P_\alpha \equiv Q_\alpha)],$

(2.15) $(P_{I\beta\alpha} \equiv Q_\alpha) =_{Df} [p'_{I\beta\alpha} \cdot (P_{\beta\alpha} \equiv Q_\alpha)],$

(2.16) $(P_\alpha \equiv Q_{I\beta\alpha}) =_{Df} [q'_{I\beta\alpha} \cdot (P_\alpha \equiv Q_{\beta\alpha})]$ [3]).

[3]) In the definition schemata symbols consisting of strokes I, II, III, IIII, etc., may be substituted for small Greek letters. If in a definition schema such symbols are substituted for all the Greek letters, then such definition schema becomes a definition.

The proofs of the following tautologies of the propositional calculus involve no difficulties:

(2.21) $\vdash (0 \equiv 0),$ (2.31) $\vdash (1 \equiv 1),$

(2.22) $\vdash (00 \equiv 0),$ (2.32) $\vdash (01 \equiv 1),$

(2.23) $\vdash (0 \equiv 00),$ (2.33) $\vdash (1 \equiv 01),$

(2.24) $\vdash (00 \equiv 00),$ (2.34) $\vdash (01 \equiv 01),$

(2.41) $\vdash (1 \equiv 10)'.$

3. Two propositional functions, each containing three propositional variables (the "nabla" function and the "delta" function) are defined:

(3.01) $\nabla(pqr) =_{Df} [(p \dot{+} q) \cdot (q \dot{+} r) \cdot (p \dot{+} r)]$

(3.02) $\triangle(pqr) =_{Df} [(p \equiv q) \equiv r]$

The Schröder zero-one table for these functions follows:

pqr	000	001	010	011	100	101	110	111
$\nabla(pqr)$	0	0	0	1	0	1	1	1
$\triangle(pqr)$	0	1	1	0	1	0	0	1

The following propositional function, involving five propositional variables, can be defined by means of the nabla and the delta functions:

(3.11) $(t_{II}t_I = p + q + r) =_{Df} \{[t_{II} \equiv \nabla(pqr)] \cdot [t_I \equiv \triangle(pqr)]\}.$

The function defined above may be called *addition of three uni-digital numbers* [4]). The following definition

(3.12) $(t = p + q + r) =_{Df} (0t = p + q + r)$

is introduced in order to simplify the symbolism.

The proofs of the following tautologies involve no difficulties:

(3.21) $\vdash \{[(t_{II}t_I = p + q + r) \cdot (s_{II}s_I = p + q + r)] \rightarrow (t_{II}t_I \equiv s_{II}s_I)\},$

(3.22) $\vdash \{[(t = p + q + r) \cdot (s = p + q + r)] \rightarrow (t \equiv s)\},$

[4]) That is, numbers which consist of the digit in the binary system notation; *falsum* is here identified with "zero" and *verum* with "one".

(3.23) $\vdash [\nabla(pqr)\triangle(pqr)=p+q+r]$,

(3.31) $\vdash(\ 0=0+0+0)$,

(3.32) $\vdash(\ 1=0+0+1)$,

(3.33) $\vdash(\ 1=0+1+0)$,

(3.34) $\vdash(10=0+1+1)$,

(3.41) $\vdash(\ 1=1+0+0)$,

(3.42) $\vdash(10=1+0+1)$,

(3.43) $\vdash(10=1+1+0)$,

(3.44) $\vdash(11=1+1+1)$.

The tautologies (3.21) and (3.22) are the laws of the univalence of addition of uni-digital numbers, whereas the tautologies from (3.31) to (3.44), taken jointly, form the addition table for three uni-digital numbers in the binary system.

4. The following definition and the two definition schemata are adopted:

(4.01) $(T_{\mathrm{II}}=P_{\mathrm{I}}+Q_{\mathrm{I}})=_{\mathrm{Df}}\{[t_{\mathrm{II}}\equiv(p_{\mathrm{I}}\cdot q_{\mathrm{I}})]\cdot[t_{\mathrm{I}}\equiv(p_{\mathrm{I}}\mathbin{\dot{-}}q_{\mathrm{I}})]\}$,

(4.02) $(T_{\mathrm{II}\alpha}=P_{\mathrm{I}\alpha}+Q_{\mathrm{I}\alpha})=_{\mathrm{Df}}\sum_{r}[(t_{\mathrm{II}\alpha}t_{\mathrm{I}\alpha}=p_{\mathrm{I}\alpha}+q_{\mathrm{I}\alpha}+r)\cdot(rT_{\alpha}=P_{\alpha}+Q_{\alpha})]$,

(4.03) $(T_{\gamma}=P_{\alpha}+Q_{\beta})=_{\mathrm{Df}}$

$$=_{\mathrm{Df}}\sum_{T_{\mathrm{II}\delta}P_{\mathrm{I}\delta}Q_{\mathrm{I}\delta}}[(T_{\mathrm{II}\delta}=P_{\mathrm{I}\delta}+Q_{\mathrm{I}\delta})\cdot(T_{\mathrm{II}\delta}\equiv T_{\gamma})\cdot(P_{\mathrm{I}\delta}\equiv P_{\alpha})\cdot(Q_{\mathrm{I}\delta}\equiv Q_{\beta})].$$

The function defined by (4.01) is called *addition* of two uni-digital numbers. The propositional function schema defined by (4.02) is called *addition* of an $(\alpha+1)$-digital number to an $(\alpha+1)$-digital number (number of digits in the binary system). The propositional function schema defined by (4.03) is called *addition* of an α-digital number to a β-digital number.

The following tautologies of the bi-valued propositional calculus deserve attention:

(4.11) $\vdash[(p\cdot q)(p\mathbin{\dot{-}}q)=p+q]$ (see 4.01),

(4.12) $\vdash[(r_{\mathrm{II}}r_{\mathrm{I}}=p+q)\equiv(r_{\mathrm{II}}r_{\mathrm{I}}=p+q+0)]$ (see 3.11, 4.01),

(4.13) $\vdash(\ 0=0+0)$ (see 4.11, 4.03),

(4.14) $\vdash(\ 1=1+0)$ (see 4.11, 4.03),

(4.15) $\vdash(\ 1=0+1)$ (see 4.11, 4.03),

(4.16) $\vdash(10=1+1)$ (see 4.11).

The tautologies from (4.13) to (4.16), taken jointly, form the addition table of uni-digital numbers in the binary system.

Attention should also be paid to the following tautology, by means of which the addition table of bi-digital numbers can easily be constructed:

(4.21) $\vdash\{(r_{\mathrm{II}}r_{\mathrm{I}}=p_{\mathrm{I}}+q_{\mathrm{I}})\rightarrow[\nabla(p_{\mathrm{II}}q_{\mathrm{II}}r_{\mathrm{II}})\triangle(p_{\mathrm{II}}q_{\mathrm{II}}r_{\mathrm{II}})\ r_{\mathrm{I}}=p_{\mathrm{II}}p_{\mathrm{I}}+q_{\mathrm{II}}q_{\mathrm{I}}]\}$.

By substituting in the tautology (4.21)

propositional constants	1, 0, 1, 0, 0, 0
for propositional variables	$p_{\mathrm{II}}, p_{\mathrm{I}}, q_{\mathrm{II}}, q_{\mathrm{I}}, r_{\mathrm{II}}, r_{\mathrm{I}}$

the following tautology of the propositional calculus is obtained:

(4.22) $\vdash(100=10+10)$;

in the binary system it is the analogon of the classic formula:

$$4=2+2.$$

It must be added with reference to the tautology (4.21) that each substitution of the stroke symbol (I, II, III, IIII, ...) for the letter α in the following schema will give a tautology:

(4.23) $(R_{\mathrm{I}\alpha}=P_{\alpha}+Q_{\alpha})\rightarrow[\nabla(p_{\mathrm{I}\alpha}q_{\mathrm{I}\alpha}r_{\mathrm{I}\alpha})\triangle(p_{\mathrm{I}\alpha}q_{\mathrm{I}\alpha}r_{\mathrm{I}\alpha})R_{\alpha}=P_{\mathrm{I}\alpha}+Q_{\mathrm{I}\alpha}]$.

It is thus known how to construct the addition tables for tridigital, quadri-digital, etc. numbers.

5. Multiplication of natural numbers, beginning with multiplication by a uni-digital number, is defined:

(5.01) $(T_{\beta}=P_{\alpha}\times Q_{\mathrm{I}})=_{\mathrm{Df}}[q_{\mathrm{I}}'\cdot(T_{\beta}\equiv 0)\mathbin{\dot{+}}q_{\mathrm{I}}\cdot(T_{\beta}\equiv P_{\alpha})]$,

(5.02) $(T_{\mathrm{I}\alpha\beta}=P_{\alpha}\times Q_{\mathrm{I}\beta})=_{\mathrm{Df}}$

$$=_{\mathrm{Df}}\sum_{R_{\alpha\beta}}\{\sum_{S_{\alpha}}[(S_{\alpha}=P_{\alpha}\times q_{\mathrm{I}\beta})\cdot(T_{\mathrm{I}\alpha\beta}=R_{\alpha\beta}+S_{\alpha}0_{\beta})]\cdot(R_{\alpha\beta}=P_{\alpha}\times Q_{\beta})\},$$

(5.03) $(T_{\gamma}=P_{\alpha}\times Q_{\beta})=_{\mathrm{Df}}$

$$=_{\mathrm{Df}}\sum_{T_{\delta\varepsilon}P_{\delta}Q_{\varepsilon}}[(T_{\delta\varepsilon}=P_{\delta}\times Q_{\varepsilon})\cdot(T_{\delta\varepsilon}\equiv T_{\gamma})\cdot(P_{\delta}\equiv P_{\alpha})\cdot(Q_{\varepsilon}\equiv Q_{\beta})].$$

Attention should be paid to the following schemata:

(5.11) $(T_{\beta}=P_{\alpha}\times 0)\equiv(T_{\beta}\equiv 0)$ (see 5.01),

(5.12) $(T_{\beta}=P_{\alpha}\times 1)\equiv(T_{\beta}\equiv P_{\alpha})$ (see 5.01).

The multiplication tables for multiplication of α-digital numbers by a uni-digital number are obtained directly from these schemata:

$$(5.21)\qquad 0=P_\alpha\times 0 \qquad \text{(see 5.11)},$$

$$(5.22)\qquad P_\alpha=P_\alpha\times 1 \qquad \text{(see 5.12)}.$$

The following schema of tautology, obtained from the definition schema (5.02):

$$(5.31)\qquad [(R_{\alpha\beta}=P_\alpha\times Q_\beta)\cdot(S_\alpha=P_\alpha\times q_{\mathrm{I}\beta})\cdot(T_{\mathrm{I}\alpha\beta}=R_{\alpha\beta}+S_\alpha 0_\beta)]\to$$
$$\to(T_{\mathrm{I}\alpha\beta}=P_\alpha\times Q_{\mathrm{I}\beta})$$

seems useful, since it allows to pass from the multiplication table for multiplication of α-digital numbers by β-digital numbers to the multiplication table for multiplication of α-digital numbers by $(\beta+1)$-digital numbers, e. g., the table for multiplication of α-digital numbers by bi-digital numbers is obtained by means of (5.21), (5.22), the following special case of the schema (5.31) being applied:

$$(5.32)\qquad [(R_{\alpha\mathrm{I}}=P_\alpha\times Q_{\mathrm{I}})\cdot(S_\alpha=P_\alpha\times q_{\mathrm{II}})\cdot(T_{\mathrm{I}\alpha\mathrm{I}}=R_{\mathrm{I}\alpha\mathrm{I}}+S_\alpha 0)]\to$$
$$\to(T_{\mathrm{I}\alpha\mathrm{I}}=P_\alpha\times Q_{\mathrm{II}}) \qquad \text{(see 5.31, 1.01)}.$$

Hence follow the schemata of the tautologies:

$$(5.31)\qquad 0=P_\alpha\times 00 \qquad \text{(see 5.32, 5.21)},$$

$$(5.42)\qquad P_\alpha=P_\alpha\times 01 \qquad \text{(see 5.32, 5.21, 5.22)},$$

$$(5.43)\qquad P_\alpha 0=P_\alpha\times 10 \qquad \text{(see 5.32, 5.22, 5.21)},$$

$$(5.44)\qquad (R_\beta=P_\alpha+P_\alpha 0)\equiv(R_\beta=P_\alpha\times 11) \qquad \text{(see 5.02, 5.32, 5.22)}.$$

6. Before the raising of natural numbers to a power is defined, the squaring and iterated squaring are introduced:

$$(6.01)\qquad (T_\delta=P_\alpha^{2^{\mathrm{I}}})=_{\mathrm{Df}}(T_\delta=P_\alpha\times P_\alpha),$$

$$(6.02)\qquad (T_\delta=P_\alpha^{2^{\mathrm{I}\beta}})=_{\mathrm{Df}}\sum_{R_\gamma}[(T_\delta=R_\gamma^{2^{\mathrm{I}}})\cdot(R_\gamma=P_\alpha^{2^\beta})].$$

In the propositional function schema, defined by means of (6.01) or (6.02), the figure "2" has no independent meaning of its own, and is used merely for traditional reasons.

The following tautologies:

$$(6.11)\qquad \vdash(0=0^{2^{\mathrm{I}}}) \qquad \text{(see 5.21, 6.01)},$$

$$(6.12)\qquad \vdash(1=1^{2^{\mathrm{I}}}) \qquad \text{(see 5.22, 6.01)},$$

$$(6.13)\qquad \vdash(100=10^{2^{\mathrm{I}}}) \qquad \text{(see 5.43, 6.01)},$$

as well as the following schema of tautology:

$$(6.21)\qquad 1=1^{2^\alpha}$$

are given by way of example.

7. The raising to a power in the case of uni-digital exponents is defined first, followed by an auxiliary definition schema, and by the definition schema for those cases in which the exponent has $(\beta+1)$-digits.

$$(7.01)\qquad (T_\varepsilon=P_\alpha^{Q_{\mathrm{I}}})=_{\mathrm{Df}}[q'_{\mathrm{I}}\cdot(T_\varepsilon\equiv 1)\dot{+}q_{\mathrm{I}}\cdot(T_\varepsilon\equiv P_\alpha)],$$

$$(7.02)\qquad (T_\varepsilon=P_\alpha^{2^\beta\cdot Q_{\mathrm{I}}})=_{\mathrm{Df}}\sum_{R_\gamma}[(T_\varepsilon=R_\gamma^{2^\beta})\cdot(R_\gamma=P_\alpha^{Q_{\mathrm{I}}})],$$

$$(7.03)\qquad (T_\varepsilon=P_\alpha^{Q_{\mathrm{I}\beta}})=_{\mathrm{Df}}\sum_{R_\gamma S_\delta}[(T_\varepsilon=R_\gamma\times S_\delta)\cdot(R_\gamma=P_\alpha^{2^\beta\cdot q_{\mathrm{I}\beta}})\cdot(S_\delta=P_\alpha^{Q_\beta})].$$

Attention is drawn first to the principles of raising to a power in the case of uni-digital exponents:

$$(7.11)\qquad (T_\varepsilon=P_\alpha^0)\equiv(T_\varepsilon\equiv 1) \qquad \text{(see 7.01)},$$

$$(7.12)\qquad (T_\varepsilon=P_\alpha^1)\equiv(T_\varepsilon\equiv P_\alpha) \qquad \text{(see 7.01)},$$

$$7.21)\qquad (1\equiv P_\alpha^0) \qquad \text{(see 7.11)},$$

$$(7.22)\qquad (P_\alpha\equiv P_\alpha^1) \qquad \text{(see 7.12)}.$$

The table for raising α-digital numbers to a $(\beta+1)$-digital power is obtained from the table for raising α-digital numbers to a β-digital power by means of the following schema of tautology:

$$(7.31)\qquad [(T_\varepsilon=R_\gamma\times S_\delta)\cdot(R_\gamma=P_\alpha^{2^\beta\cdot q_{\mathrm{I}\beta}})\cdot(S_\delta=P_\alpha^{Q_{\mathrm{I}\beta}})]\to(T_\varepsilon=P_\alpha^{Q_{\mathrm{I}\beta}}) \qquad \text{(see 7.03)}.$$

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