Gödel Kurt [1] *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, Monatshefte für Mathematik und Physik **37**, 2. H.

— [2] *Eine Interpretation des intuitionistischen Aussagenkalküls*, Ergebnisse eines mathematischen Kolloquiums, Heft **4** (1933), pp. 39-40.

Lewis C. J. and Langford, C. H. [1] *Symbolic Logic*, New York 1932.

McKinsey, J. C. C. [1] *A Solution of the Decision Problem for the Lewis Systems S. 2 and S. 4, with an Application to Topology*, The Journal of Symbolic Logic **6**, Nr 4 (1941), pp. 117-134.

— and Tarski, A. [1] *The Algebra of Topology*, Annals of Mathematics **45** (1944), pp. 141-191.

— [2] *On Closed Elements in Closure Algebras*, ibidem **47** (1946), pp. 122-162.

— [3] *Some Theorems about the Sentential Calculi of Lewis and Heyting*, The Journal of Symbolic Logic **13**, Nr 1 (1948), pp. 1-15.

MacNeille, H. M. [1] *Partially Ordered Sets*, Transactions of the American Mathematical Society **42** (1937), pp. 416-460.

Mostowski, A. [1] *Logika matematyczna*, Monografie Matematyczne, Warszawa 1948.

— [2] *Proofs of Non-deducibility in Intuitionistic Functional Calculus*, The Journal of Symbolic Logic **13**, Nr 4 (1948), pp. 204-207.

Rasiowa H. and Sikorski, R. [1] *A Proof of the Completeness Theorem of Gödel*, Fund. Math. **37** (1950), pp. 193-200.

Rieger, L. [1] *On the Lattice Theory of Brouwerian Propositional Logic*, Spisy wyd. prirodovedeckou fakultou University Karlovy, 1949, Cislo 189.

— [2] *On Free $\aleph_\xi$-Complete Boolean Algebras*, Fund. Math. **38** (1951), pp. 35-52.

Sikorski, R. [1] *Closure Algebras*, Fund. Math. **36** (1949), pp. 165-206.

Tarski, A. [1] *Pojęcie prawdy w językach nauk dedukcyjnych*, Prace Towarzystwa Naukowego Warszawskiego, Nr **34** (1933), Wydz. III, pp. 1-35.

A Characterization of L Spaces[1]).

By

R. E. Fullerton (Wisconsin, U.S.A.).

1. Introduction. Kakutani [2] has characterized a space of integrable functions as a Banach lattice satisfying the following three conditions:

(1) there exists a unit element $e>\theta$ such that $x>\theta$ implies $e\wedge x>\theta$;

(2) $x\geqslant\theta$, $y\geqslant\theta$ imply $\|x+y\|=\|x\|+\|y\|$;

(3) $x\wedge y=\theta$ implies $\|x-y\|=\|x+y\|$.

The set of points, Ω, over which the L space is defined can be assumed to have measure 1. Kakutani [3] has also given a similar type of characterization for Banach lattices of functions continuous over a bicompact Hausdorff space. More recently, Clarkson [1] has characterized a Banach space of continuous functions in terms of the shape of the unit sphere. In this characterization an order relation is introduced by means of a certain type of cone used in the construction of the unit sphere, and under this ordering the space in shown to be an M space and hence equivalent to a space of continuous functions. In this paper spaces of integrable functions will be characterized by the shape of their unit spheres, making use of methods similar to those of Clarkson. The Borel field of measurable subsets of the space Ω will be shown to correspond to the family of maximal convex subsets of the unit sphere in a manner similar to the role played by this family in the case of a space of continuous functions as investigated by Eilenberg [4]. The case in which the measure is completely atomic is of particular interest and will be treated in more detail.

[1]) Presented to the American Mathematical Society, September 9, 1948.

2. The Characterization Theorem. A cone in a linear space $\mathfrak{X}$ is a set $\mathfrak{C}$ containing a point v such that if $x\in\mathfrak{C}$ and λ is any non negative real number, then $v+\lambda x\in\mathfrak{C}$. A cone will be called a *C-cone* if for any two points x, y, of $\mathfrak{X}$ there exists a point $z\in\mathfrak{X}$ such that $(x+\mathfrak{C})\cap(y+\mathfrak{C})=z+\mathfrak{C}$. This type of cone was used by Clarkson in his paper and has received more recent attention in a paper by Krein and Rutman [5], who have called it a *minihedral cone*. As they have pointed out, Clarkson's theorem shows that a necessary and sufficient condition that a Banach space $\mathfrak{X}$ be isomorphic to a space of real-valued functions continuous over a bicompact set S is that $\mathfrak{X}$ contain a proper C-cone with an interior point.

Theorem 1. *Necessary and sufficient conditions that a Banach space $\mathfrak{X}$ be equivalent to a space $L(\Omega,m)$ of all real-valued functions defined on a set Ω, integrable with respect to a completely additive measure m with $m(\Omega)=1$, are:*

(1) *There exists on the surface of the unit sphere a maximal closed convex set F such that the cone $\mathfrak{C}=\{\lambda x|x\in F,\lambda\geqslant 0\}$ is a C-cone;*

(2) *The unit sphere consists of all points of the form $\alpha x+\beta y$; $\alpha,\beta\geqslant 0$, $\alpha+\beta=1$, $x\in F$, $y\in -F$;*

(3) *F contains an element e such that if $x\in\mathfrak{C}$, $x\neq\theta$, there exists a $y\in\mathfrak{C}$, $y\neq\theta$ such that $e\in y+\mathfrak{C}$, $x\in y+\mathfrak{C}$.*

Proof. Let $L(\Omega,m)$ be the space described above. If $x,y\in L(\Omega,m)$, $x\geqslant y$ means $x(t)\geqslant y(t)$ almost everywhere. The norm is defined in the usual manner

$$\|x\|=\int_\Omega |x(t)|\,dm(t).$$

Let F be the set of all essentially non-negative functions of unit norm. Evidently F is both closed and convex. To show that F is a maximal convex subset of the surface of the unit sphere, suppose $F\subset F'$, where F' is a convex set of elements of unit norm. Suppose $x'\in F'$ but $x'\bar{\in} F$. Then there exists a measurable subset $\sigma\subset\Omega$ such that $x'(t)<0$ for $t\in\sigma$. Let σ be the set of all points for which $x'(t)<0$, and consider $\tilde{\sigma}$, the complement of σ in Ω. Let $x\in F$, $\dfrac{x+x'}{2}\in F'$ and $\left\|\dfrac{x+x'}{2}\right\|=1$. However

$$\begin{aligned}1=\left\|\frac{x+x'}{2}\right\|&=\frac{1}{2}\int_\Omega |x'(t)+x(t)|\,dm(t)\\&=\frac{1}{2}\int_{\tilde\sigma}[x'(t)+x(t)]\,dm(t)+\frac{1}{2}\int_\sigma[x(t)-x'(t)]\,dm(t)<\\&<\frac{1}{2}\int_{\tilde\sigma}|x(t)|\,dm(t)+\frac{1}{2}\int_{\tilde\sigma}|x'(t)|\,dm(t)+\frac{1}{2}\int_\sigma|x(t)|\,dm(t)+\\&+\frac{1}{2}\int_\sigma|x'(t)|\,dm(t)=\frac{1}{2}\int_\Omega|x(t)|\,dm(t)+\frac{1}{2}\int_\Omega|x'(t)|\,dm(t)\\&=\frac{1}{2}\|x\|+\frac{1}{2}\|x'\|=1.\end{aligned}$$

This contradiction shows that $F'=F$, and F is maximal.

Let $\mathfrak{C}(\theta,F)=\{x|x=\alpha z,\|z\|=1,z\geqslant\theta,\alpha\geqslant 0\}$. This set is a cone with vertex θ generated by the set F. Let $y_1,y_2\in L(\Omega,m)$. Then $y_1+\mathfrak{C}$, $y_2+\mathfrak{C}$ are evidently cones with vertices y_1 and y_2, respectively, and consist of all elements of $L(\Omega,m)$ not less than y_1 and y_2, respectively. If $y_1\vee y_2=\max[y_1(t),y_2(t)]$, then $(y_1+\mathfrak{C})\cap(y_2+\mathfrak{C})$ is the set of all elements not less than $y_1\vee y_2$, and $(y_1+\mathfrak{C})\cap(y_2+\mathfrak{C})=(y_1\vee y_2)+\mathfrak{C}$. Thus $\mathfrak{C}$ is a C-cone.

To prove (2), assume $\|x\|\leqslant 1$, $x=x_1+x_2$, where $x_1=x\vee\theta$, $x_2=x\wedge\theta$. Then $\|x\|=\|x_1\|+\|x_2\|$. Let y be any non negative element with $\|y\|=1-\|x\|$. Let $z_1=x_1+\dfrac{y}{2}$, $z_2=x_2-\dfrac{y}{2}$. Then $\|z_1\|+\|z_2\|=$ $=\|x_1\|+\|x_2\|+\|y\|=\|x\|+1-\|x\|=1$. Also $x=z_1+z_2=\|z_1\|\dfrac{z_1}{\|z_1\|}+\|z_2\|\dfrac{z_2}{\|z_2\|}$ and since $\dfrac{z_1}{\|z_1\|}\in F$, $\dfrac{z_2}{\|z_2\|}\in -F$, (2) is true.

If $e(t)$ is almost everywhere positive on Ω, (3) is obvious if we let $y=e\wedge x>\theta$.

In the proof of the sufficiency it will be shown that if (1), (2), and (3), hold, it is possible to introduce an order relation in the space under which the space becomes an abstract L space with a unit and by Kakutani's theorem, is equivalent to a space $L(\Omega,m)$ as described.

An abstract L space is a Banach space with an order relation satisfying the following postulates:

K 1. $x\geqslant y$, $y\geqslant x$, imply $y=x$;

K 2. $x\geqslant y$, $y\geqslant z$, imply $x\geqslant z$;

K 3. $x \geqslant y$ and $\lambda \geqslant 0$ imply $\lambda x \geqslant \lambda y$;

K 4. $x \geqslant y$ implies $x+z \geqslant y+z$ for any $z \in \mathfrak{X}$;

K 5. $x_n \geqslant y_n$, $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, imply $x \geqslant y$;

K 6. To any pair of elements x and y there exists a supremum $z = x \vee y$ such that $z \geqslant x$, $z \geqslant y$, and if $z' \geqslant x$, $z' \geqslant y$, then $z' \geqslant z$;

K 7. There exists an infimum $x \wedge y$ with properties dual to K6;

K 8. $x \geqslant \theta$, $y \geqslant \theta$, imply $\|x+y\| = \|x\| + \|y\|$;

K 9. $x \wedge y = \theta$ implies $\|x+y\| = \|x-y\|$;

K 10. There exists an element $e > \theta$ such that if $x > \theta$, then $e \wedge x > \theta$.

The order in $\mathfrak{C}$ is defined by assuming $x \geqslant y$ if $x - y \in \mathfrak{C}(\theta, F)$. Conditions (1), (2), and (3), are assumed to hold, and postulates K 1 to K 10 will be shown to hold under the ordering as defined.

K 3 is obvious.

K 1. This statement means that if $x-y \in \mathfrak{C}$ and $y-x \in \mathfrak{C}$, then $y = x$. Assume that $w \in \mathfrak{C}$ and $-w \in \mathfrak{C}$, $w \neq \theta$. By K 3 it is possible to assume $\|w\| = 1$ and hence $w \in F$. This would imply also that $-w \in F$ and hence $(w-w)/2 = \theta \in F$ contrary to the hypothesis that all elements of F are of unit norm. Hence $x - y = \theta$.

K 2. This is equivalent to the statement that if $x-y \in \mathfrak{C}$ and $y-z \in \mathfrak{C}$, then $x-z = (x-y)+(y-z) \in \mathfrak{C}$, or if $x_1 \in \mathfrak{C}$, $x_2 \in \mathfrak{C}$, then $x_1 + x_2 \in \mathfrak{C}$. This is true if $\mathfrak{C}$ is convex, since then $(x_1+x_2)/2 \in \mathfrak{C}$ and $x_1 + x_2 \in \mathfrak{C}$.

To prove $\mathfrak{C}$ convex, let $x, y \in \mathfrak{C}$. It may be assumed that $\|x\| = 1$ and that $y = \lambda y'$, where $\|y'\| = 1$ and $\lambda \geqslant 0$. Assume $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$.

$$\alpha x + \beta y = \alpha x + \beta\lambda y' = (\alpha + \beta\lambda)\left(\frac{\alpha}{\alpha+\beta\lambda}x + \frac{\beta\lambda}{\alpha+\beta\lambda}y'\right).$$

However

$$\frac{\alpha}{\alpha+\beta\lambda} \geqslant 0, \quad \frac{\beta\lambda}{\alpha+\beta\lambda} \geqslant \theta \quad \text{and} \quad \frac{\alpha}{\alpha+\beta\lambda} + \frac{\beta\lambda}{\alpha+\beta\lambda} = 1.$$

Since F is convex and both x and y are in F, then

$$\frac{\alpha}{\alpha+\beta\lambda}x + \frac{\beta\lambda}{\alpha+\beta\lambda}y' \in F,$$

and hence

$$\alpha x + \beta y \in \mathfrak{C}.$$

K 4 is obvious.

K 5 holds if $\mathfrak{C}$ is closed, i. e. if $x_n - y_n \in \mathfrak{C}$ and $\lim_{n\to\infty}(x_n - y_n) = x - y$ imply $x - y \in \mathfrak{C}$.

Let $w_n = x_n - y_n$, $w = x - y$. If $w = \theta$, K 5 is obvious. If $w \neq \theta$, we may assume without loss of generality that $\|w\| = 1$. It must be shown that $w \in F$. Since $\lim_{n\to\infty}\|w_n\| = 1$, it may be assumed that $\{\|w_n\|\}$ is bounded away from zero. Let $w_n' = \frac{w_n}{\|w_n\|}$. Then $w_n' \in F$.

$$\|w_n' - w\| \leqslant \|w_n' - w_n\| + \|w_n - w\| = \left\|\frac{w_n}{\|w_n\|} - w_n\right\| + \|w_n - w\|$$

$$= \left|\frac{1}{\|w_n\|} - 1\right| \cdot \|w_n\| + \|w_n - w\| = |1 - \|w_n\|| + \|w_n - w\| \to 0.$$

Hence $\lim_{n\to\infty} w_n' = w$. Since $w_n' \in F$ and F is closed, $w \in F$.

K 6. Let $(x+\mathfrak{C}) \cap (y+\mathfrak{C}) = z + \mathfrak{C}$. Then $z \geqslant x$, $z \geqslant y$, and if $w \geqslant x$, $w \geqslant y$, then $w \in (x+\mathfrak{C}) \cap (y+\mathfrak{C}) = z+\mathfrak{C}$ and $w \geqslant z$. Thus $x \vee y = z$.

K 7 is dual to K 6 where the infimum is defined by $x \wedge y = -[(-x) \vee (-y)]$.

K 8. Assume $x \geqslant \theta$, $y \geqslant \theta$. Then $x = \|x\| \cdot x_1$, $y = \|y\| \cdot y_1$, where $x_1 \in F$, $y_1 \in F$.

$$\|x+y\| = \|\|x\| \cdot x_1 + \|y\| \cdot y_1\|$$

$$= (\|x\| + \|y\|)\left\|\frac{\|x\|}{\|x\|+\|y\|}x_1 + \frac{\|y\|}{\|x\|+\|y\|}y_1\right\| = \|x\| + \|y\|$$

for, since F is convex, the second factor of the third quantity is equal to 1.

K 9. $x \wedge y = 0$ means that $x \in \mathfrak{C}$, $y \in \mathfrak{C}$, but $x \bar{\in} z + \mathfrak{C}$, $y \bar{\in} z + \mathfrak{C}$ for any $z \in \mathfrak{C}$, $z \neq \theta$. Assume first that $\|x\| = \|y\| = 1$. Let

$$\mathfrak{L} = \{\alpha x + \beta y \mid \alpha + \beta = 1\}$$

and let $\mathfrak{L}' = \mathfrak{L} - x$. $\mathfrak{L}'$ is a line in $\mathfrak{X}$ passing through the origin and a translate of $\mathfrak{L}$. Let u and $-u$ be the points of $\mathfrak{L}'$ for which $\|u\| = 1$. By hypothesis (2) there exists a line $\mathfrak{M}$ through u, which intersects F and $-F$. It is evident that $u \bar{\in} F$. Since $\mathfrak{M}$ contains three points of unit norm, the entire segment S containing these three points lies on the surface of the unit sphere. Since the sphere is symmetric, $-S$ has the same properties. Let v_1 and v_2 be the points in which M and $-M$ intersect F. Let $\mathfrak{L}_1'' = \mathfrak{L}' + v_1$, and $\mathfrak{L}_2'' = \mathfrak{L}' + v_2$. Since F is convex, at least one of these lines will contain no points of norm less than one. Assume that $\mathfrak{L}_1''$ is this line. Let v_1 and w be the points of intersection of $\mathfrak{L}_1''$ with $\mathfrak{M}$ and $-\mathfrak{M}$. Evidently $\|v_1 - w\| = \|u + u\| = 2$. Since $\|x\| = \|y\| = 1$, $\|x - y\| \leqslant 2$.

Let T be the segment $\{\alpha v_1+\beta w|, \alpha,\beta\geqslant 0, \alpha+\beta=1\}$. Let

$$T_1=T+(x-v_1),\quad T_2=T+(x-w).$$

T_1 and T_2 are included in $\mathfrak{L}$ and of length 2 with v_1 and w translated to x respectively. Hence either T_1 or T_2 contains the segment joining x and y. Assume that T_1 is this set. Then $x\in(x-v_1)+\mathfrak{C}$ and $y\in(x-v_1)+\mathfrak{C}$. Hence x and y are in $[(x-v_1)+\mathfrak{C}]\cap\mathfrak{C}=z+\mathfrak{C}$, where $z\geqslant\theta$ and $z=\theta$, only if $x-v_1\leqslant\theta$. If $x\neq v_1$, $v_1-x>\theta$, and $1=\|v_1-x+x\|=\|v_1-x\|+\|x\|>1$ by K 8. Hence $x=v_1$ since $z\leqslant\theta$ by assumption. Similarly it can be shown that $x=w=v_2$. Hence $\|x-y\|=2$. Since evidently $\|x+y\|=2$, $\|x-y\|=\|x+y\|$ if $\|x\|=\|y\|=1$.

Since $\|x\|=\|y\|=\|-y\|=1$ and $\left\|\frac{x+y}{2}\right\|=\left\|\frac{x-y}{2}\right\|=1$, the entire segments $\{\alpha x+\beta y\}$ and $\{\gamma x-\delta y\}$, $\alpha,\beta,\gamma,\delta\geqslant 0$, $\alpha+\beta=1$, $\gamma+\delta=1$, consist of points of unit norm.

Assume that x and y are not necessarily of unit norm and $x\wedge y=\theta$. Then

$$1=\left\|\frac{\|x\|}{\|x\|+\|y\|}\cdot\frac{x}{\|x\|}+\frac{\|y\|}{\|x\|+\|y\|}\cdot\frac{y}{\|y\|}\right\|$$
$$=\left\|\frac{\|x\|}{\|x\|+\|y\|}\cdot\frac{x}{\|x\|}-\frac{\|y\|}{\|x\|+\|y\|}\cdot\frac{y}{\|y\|}\right\|=\left\|\frac{x+y}{\|x+y\|}\right\|=\left\|\frac{x-y}{\|x+y\|}\right\|$$

and

$$\|x+y\|=\|x-y\|.$$

K 10 is hypothesis (3).

This shows that all the postulates for a Kakutani L space with a unit are satisfied and that the space is equivalent to a space of integrable real functions over a totally disconnected bicompact Hausdorff space Ω, where the field of measurable subsets is the field of Borel sets and $m(\Omega)=1$.

Remark. It is to be noted that, although $\mathfrak{C}$ is convex and closed and generates the entire space, it need not have any interior points.

3. Further properties of the unit sphere.

Theorem 2. *Let $\mathfrak{X}$ be an L space with a unit. If $x\in\mathfrak{X}$, $\|x\|=1$, there exists a maximal closed convex subset H of the surface of the unit sphere containing x and such that the unit sphere consists of the closure of the convex set determined by H and $-H$.*

Proof. Let $x\in\mathfrak{X}$, $\|x\|=1$. x is representable in $L(\Omega,m)$ as an integrable real function $x(t)$ with

$$\int_\Omega |x(t)|\,dm(t)=1.$$

If $x(t)\geqslant 0$ or $x(t)\leqslant 0$ almost everywhere, the theorem follows from theorem 1. If $x(t)\geqslant 0$ for $t\in\sigma$, $x(t)<0$ for $t\in\tilde{\sigma}$, where $m(\sigma)>0$, $m(\tilde{\sigma})>0$, let H be the set of all functions essentially non negative on σ and essentially non positive on $\tilde{\sigma}$ and of unit norm. By methods similar to those used in the proof of the necessity in theorem 1, H can be shown to be both convex and closed. To prove that H and $-H$ generate the entire unit sphere let $y(t)$ be any positive function of $L(\Omega,m)$, $\|y\|=1$. Let $y_1(t)=y(t)$ for $t\in\sigma$ and zero elsewhere and let $y_2(t)=y(t)$ for $t\in\tilde{\sigma}$ and zero elsewhere. Let

$$z_1=\frac{y_1}{\|y_1\|},\qquad z_2=\frac{y_2}{\|y_2\|}.$$

Then

$$z_1\in H,\quad z_2\in -H,$$

and

$$\|y\|=\|y_1\|+\|y_2\|=1,\qquad y=\|y_1\|\cdot z_1+\|y_2\|\cdot z_2,$$

which is in the closed convex set determined by H and $-H$.

Similarly all negative functions of unit norm are in the set and, by theorem 1, the closed convex set determined by H and $-H$ is the unit sphere of X.

Corollary. *Every maximal convex set on the surface of the unit sphere in an L space with a unit generates a C-cone with vertex θ which satisfies the conditions of theorem 1.*

4. Construction of the function space. Maximal convex subsets of the surface of the unit sphere will be called *faces* of the unit sphere. To each face of the unit sphere in an L space with a unit corresponds the set of integrable functions of $L(\Omega,m)$ which are essentially non negative on a measurable subset $\sigma\subset\Omega$ and essentially non positive on $\tilde{\sigma}$. In this way to each face of the unit sphere can be associated a class of measurable subsets of Ω, each differing from the other on a set of measure zero on which the elements of the face F_σ are essentially non negative. The positive face F_Ω corresponds to Ω, and the negative face $F_0=-F_\Omega$ to the sets of measure zero. If $m(\sigma_1\cap\sigma_2)=0$, then $F_{\sigma_1}\cap F_{\sigma_2}\subset F_0$.

In this section the usual notations for a relation will indicate that the relation holds almost everywhere.

For any σ, let $\Gamma_\sigma = \bigcup_{x \in F_\sigma} (x + \mathfrak{C})$. If $\sigma_1 \subset \sigma_2$ for $x_1 \in F_{\sigma_1}$, there exists an $x_2 \in F_{\sigma_2}$ such that $x_1 \leqslant x_2$, and if $y_2 \in F_{\sigma_2}$ there exists a $y_1 \in F_1$ with $y_1 \leqslant y_2$.

Thus $\sigma_1 \subset \sigma_2$ if and only if $F_{\sigma_2} \subset \Gamma_{\sigma_1}$.

This ordering of the faces will be denoted by $F_{\sigma_1} \lesssim F_{\sigma_2}$.

If $\Gamma_\sigma^- = \bigcup_{x \in F} (x - \mathfrak{C})$, then $F_{\sigma_1} \lesssim F_{\sigma_2}$ is equivalent to the statement that $F_{\sigma_1} \subset \Gamma_{\sigma_2}^-$.

Consider the face $F_{\sigma_1 \cup \sigma_2}$. This set has the following properties:

(1) $\Gamma_{\sigma_1 \cup \sigma_2} \subset \Gamma_{\sigma_1} \cap \Gamma_{\sigma_2}$;

(2) If for some σ, $\Gamma_\sigma \supset \Gamma_{\sigma_1 \cup \sigma_2}$ and $\Gamma_\sigma \neq \Gamma_{\sigma_1 \cup \sigma_2}$, then $\Gamma_\sigma \not\subset \Gamma_{\sigma_1} \cap \Gamma_{\sigma_2}$.

Let $\Lambda_{\sigma_1,\sigma_2} = \Gamma_{\sigma_1} \cap \Gamma_{\sigma_2} \cap [\bigcap_\sigma \Gamma_\sigma^-]$ for σ such that $\Gamma_\sigma \subset \Gamma_{\sigma_1} \cap \Gamma_{\sigma_2}$. Then $F_{\sigma_1 \cup \sigma_2} \subset \Lambda_{\sigma_1,\sigma_2}$ and, in fact, $F_{\sigma_1 \cup \sigma_2} = \Lambda_{\sigma_1,\sigma_2} \cap S$, where S is the surface of the unit sphere.

To prove this, let $x \in \Lambda_{\sigma_1,\sigma_2}$, $\|x\| = 1$. Then, since $x \in \Gamma_{\sigma_1} \cup \Gamma_{\sigma_2}$, $x(t) \geqslant 0$ on $\sigma_1 \cup \sigma_2$. Since $x \in \Gamma_\sigma^-$ for each $\sigma \supset \sigma_1 \cup \sigma_2$, $x(t) \leqslant 0$ on the complement $\tilde{\sigma}$ of each such σ. Hence $x(t) \leqslant 0$ on $\tilde{\sigma}_1 \cap \tilde{\sigma}_2$. Thus $x \in F_{\sigma_1 \cup \sigma_2}$.

Similarly if $\{\sigma_\alpha\}$ is any family of measurable subsets of Ω, then

$$\Lambda_{\{\sigma_\alpha\}} = [\cap_\alpha \Gamma_{\sigma_\alpha}] \cap [\cap_\sigma \Gamma_\sigma^-], \quad F_{\cup \sigma_\alpha} = \Lambda_{\{\sigma_\alpha\}} \cap S,$$

where the second intersection in the first expression is taken over all sets such that $\Gamma_\sigma \subset \cap_\alpha \Gamma_{\sigma_\alpha}$.

Similarly $F_{\cap \sigma_\alpha} = [\cap_\alpha \Gamma_{\sigma_\alpha}^-] \cap [\cap_\sigma \Gamma_\sigma] \cap S$, where $\Gamma_\sigma \supset \cup_\alpha \Gamma_{\sigma_\alpha}$.

Hence under the ordering as defined, the faces of the unit sphere form a lattice.

Since $m(\Omega) = 1$, by the Radon-Nikodym theorem, the space $L(\Omega)$ can be represented as the space of all completely additive, absolutely continuous set functions defined over measurable subsets of Ω. Hence X is equivalent to the space of all real functions $\{\Phi\}$ defined over the $\{F_\sigma\}$ such that

(1) $\Phi(F_0) = 0$,

(2) if $\{F_{\sigma_i}\}$ is a sequence of faces such that $F_{\sigma_i} \cap F_{\sigma_j} \subset F_0$ for $i \neq j$, then

$$\sum_1^\infty \Phi(F_{\sigma_i}) = \Phi(F_{\cup \sigma_i}).$$

5. Characterization of the space l. The space l is an L space determined over a completely atomic measure space where $m(\Omega) < \infty$. A completely atomic measure space is a Borel field of measurable sets such that each set of positive measure contains a subset of minimal positive measure.

Theorem 3. *Necessary and sufficient conditions that a Banach space $\mathfrak{X}$ be equivalent to the space l are that the conditions of Theorem 1 be satisfied and in addition that there exist a countable set of extreme points $\{v_i\}$, $\|v_i\| = 1$, $i = 1, 2, 3, \ldots$, such that the unit sphere in $\mathfrak{X}$ is the closed convex set determined by the $\{\pm v_i\}$.*

Proof. Let $\mathfrak{X} = l$ and let $v_i = (0, 0, \ldots, 0, 1, 0, \ldots)$ with 1 in the i-th place and 0 elsewhere. It can easily be shown that the set F of theorem 1 is the convex closure of the $\{\pm v_i\}$, and hence that the conditions are necessary.

To prove the sufficiency of the conditions, suppose that the unit sphere in the space is the smallest closed convex set determined by the $\{\pm v_i\}$. Since these are extreme points, they lie in F or $-F$. Assume that all $v_i \in F$. Also $v_i \wedge v_j = \theta$ for $i \neq j$, since a geometric argument shows that $v_j \bar{\in} x + \mathfrak{C}$ for any $x \neq \lambda v_j$, $x \in \mathfrak{C}$.

In fact, if π and π' are two disjoint sets of integers and $\{\xi_i\}$, $\{\xi_j\}$, $i \in \pi$, $j \in \pi'$, are sets of non negative numbers, since the $\{v_i\}$ are integrable functions, it is possible to deduce from this that

$$\Big(\sum_{i \in \pi} \xi_i v_i\Big) \wedge \Big(\sum_{j \in \pi'} \xi_j v_j\Big) = \theta.$$

Hence by K 9

$$\Big\|\sum_{i \in \pi} \xi_i v_i + \sum_{j \in \pi'} \xi_j v_j\Big\| = \Big\|\sum_{i \in \pi} \xi_i v_i - \sum_{j \in \pi'} \xi_j v_j\Big\|.$$

Consider two disjoint sets of integers π and π' and let $\{\alpha_i\}$, $\{\beta_j\}$ be finite sets of non negative numbers, where $i \in \pi$ and $j \in \pi'$ with

$$\sum_{i \in \pi} \alpha_i + \sum_{j \in \pi'} \beta_j = 1,$$

and if $S_{\pi,\pi'}$ is the closure of the convex set determined by $\{v_i\}$, $i \in \pi$; $\{-v_j\}$, $j \in \pi'$, then $S_{\pi,\pi'}$ consists of all elements of the form

$$\sum_{i \in \pi} \alpha_i v_i - \sum_{j \in \pi'} \beta_j v_j$$

plus all limit points of such elements. However

$$\Big\|\sum_{i\in\pi}\alpha_i v_i-\sum_{j\in\pi'}\beta_j v_j\Big\|=\Big\|\sum_{i\in\pi}\alpha_i v_i+\sum_{j\in\pi'}\beta_j v_j\Big\|=1$$

and the same holds for the limit points. Hence each $S_{\pi,\pi'}$ lies on the surface of the unit sphere. If $\pi\cup\pi'$ is all the positive integers, then $S_{\pi,\pi'}$ is a face of the unit sphere. Also every point of unit norm is in such a set, for otherwise it would be in a set $S_{\pi,\pi'}$ with $\pi\cap\pi'$ non empty and could be approximated by a point of the form

$$Z=\sum_{i\in\pi}\alpha_i v_i-\sum_{j\in\pi'}\beta_j v_j$$
$$=\sum_{i\in\pi-(\pi\cap\pi')}\alpha_i v_i+\sum_{k\in\pi\cap\pi'}(\alpha_k-\beta_k)v_k-\sum_{j\in\pi'-(\pi\cap\pi')}\beta_j v_j$$

with $\{\alpha_i\}$, $\{\beta_j\}$, finite sets of non negative numbers such that

$$\sum_i\alpha_i+\sum_j\beta_j=1.$$

By properties K 8 and K 9 it can be seen that $\|z\|<1$. Hence the $S_{\pi,\pi'}$ with $\pi\cap\pi'$ empty and $\pi\cup\pi'=1$, consist of all the faces of the unit sphere.

The faces S_k determined by v_k and $\{-v_j\}$, $j\neq k$, represent subsets σ_k of minimal positive measure, since the only face for which $F\lessdot S_k$ is F_0. If S_π is any face, then $S_k\lessdot S_\pi$ for all k for which $k\in\pi$.

Thus every measurable subset contains an atomic subset, and since the space Ω is of finite measure, the space of integrable functions represented is the space l.

University of Wisconsin.

References.

[1] James A. Clarkson, *A Characterization of C-spaces*, Annals of Math., vol. **48**, N° 4 (1947), pp. 845-850.

[2] Shizuo Kakutani, *Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Annals of Math., vol. **42**, N° 2 (1941), pp. 523-537.

[3] — *Concrete Representations of Abstract (M) Spaces*, Annals of Math., vol. **42**, N° 4 (1941), pp. 994-1024.

[4] Samuel Eilenberg, *Banach Space Methods in Topology*, Trans. Amer. Math. Soc., vol. **41**, N° 3 (1937), pp. 374-481.

[5] M. G. Krein and M. A. Rutman, *Linear Operators leaving invariant a cone in a Banach Space*, Uspehi Matem. Nauk (N. S.) **3**, N° 1 (23) (1948), pp. 3-95.

Undecidability of Some Topological Theories.

By

Andrzej Grzegorczyk (Warszawa).

The purpose of the present paper is to prove the essential undecidability of some elementary theories of closure algebra[1]), of Brouwerian algebra, of the algebra of bodies, of the algebra of convexity and of the semi-projective algebra. This is attained by means of the method of interpretation based on the general theorems of Tarski[2]) and on the theorems of Mostowski, Tarski and Robinson[3]) concerning the essential undecidability of a finitely axiomatizable arithmetic. The main idea of the proof is that this arithmetic can be interpreted as an arithmetic of finite sets. Hence each theory in which we can define the class of finite sets and some operations on the finite sets is undecidable.

The elementary theories $\mathcal{T}_1, \mathcal{T}_2, \ldots$ discussed herein are assumed to be formalized in the lower functional calculus, they all have the same logical constants (connectives, quantifiers, identity symbol), logical axiom schemes and rules of inference. Each theory $\mathcal{T}_n\langle\mathfrak{X},\mathfrak{O}_1,\mathfrak{O}_2,\mathfrak{O}_3\rangle$ is determined by its non-logical (primitive) constants: $\mathfrak{X},\mathfrak{O}_1,\mathfrak{O}_2,\mathfrak{O}_3$ and non-logical axioms. The terms: $\mathfrak{X},\mathfrak{O}_1,\mathfrak{O}_2,\mathfrak{O}_3$ denote „univers du discours", and certain relations. The non-elementary theories $\mathcal{S}_1,\mathcal{S}_2,\ldots$ of order ω contain the simple theory of (finite) types. (I. e. each type of the order $n+1$ is the class of all subclasses of the types of order n). The non-logical (primitive) constants: $\mathfrak{X}',\mathfrak{O}_1',\mathfrak{O}_2',\mathfrak{O}_3'$ of a non-elementary theory $\mathcal{S}_n\langle\mathfrak{X}',\mathfrak{O}_1',\mathfrak{O}_2',\mathfrak{O}_3'\rangle$ denote only the relations and operations defined over the individuals of the lowest type. The variable letters $\mathcal{A},\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ will be used to designate theories which can be $\mathcal{T}_n$ or $\mathcal{S}_n$. $\mathcal{A}'$ will be

---

[1]) The undecidability of the closure algebra has been proved in another way by Stanisław Jaśkowski in 1939. See [2].

[2]) See Tarski [11] and Robinson [10].

[3]) See Mostowski [9].