in L. R is a compact T_1-space. We have given a proof (omitted here) that M is compact for such spaces $R=L+a$.

I) Let L be discrete and non-denumerable. Then R is *Hausdorff* and *not* separable, and M is compact.

A special case of $R=L+a$ is the space R whose closed sets are R itself and its finite subsets.

II) Let R be non-denumerable. Then R is *not* Hausdorff, *not* separable, and M is compact.

III) Let R be denumerably infinite. Then R is *separable* and *not* Hausdorff, and M is compact.

Thus R Hausdorff and R separable are not necessary, either singly or together, for the compactness of M.

On Free $\aleph_\xi$-complete Boolean Algebras.

(With an Application to Logic).

By

Ladislav Rieger (Praha).

A Boolean algebra A is said to be *$\aleph_\xi$-complete* if any subset of elements of A the power of which does not exceed $\aleph_\xi$ has a g. l. b. and a l. u. b. in A. An $\aleph_\xi$-complete Boolean algebra $A_\mathfrak{m}^{\aleph_\xi}$ is said to be *free with* $\mathfrak{m}$ *free $\aleph_\xi$-generators* (where $\mathfrak{m}$ is any cardinal number) if there exists a subset $G \subset A_\mathfrak{m}^{\aleph_\xi}$ the power of which is $\mathfrak{m}$ so that G has the following properties:

(i) The only $\aleph_\xi$-complete subalgebra of $A_\mathfrak{m}^{\aleph_\xi}$ containing G is $A_\mathfrak{m}^{\aleph_\xi}$ itself. (We say that the elements of G $\aleph_\xi$-generate $A_\mathfrak{m}^{\aleph_\xi}$).

(ii) If φ is any mapping of G into another $\aleph_\xi$-complete algebra B then φ can be extended to a $\aleph_\xi$-complete [1]) homomorphic mapping of the whole algebra $A_\mathfrak{m}^{\aleph_\xi}$ into B.

Familiarity with these and other (better known) basic notions of the theory of Boolean algebras will be assumed. I refer to a brief exposition of these notions in R. Sikorski's papers [1] and [2] (this Fund. Math. 1948 and 1949). For a more extensive treatise, the monograph of G. Birkhoff [1] on Lattice Theory (sec. ed. 1948) is recommended.

Note that by an *α-ideal* (the symbol due to M. H. Stone), I understand what sometimes is called a *dual ideal*, i. e. a (nonvoid) subset I of the algebra A in question so that if $a, b \in I$ then $a \cap b \in I$ and if $a \subseteq b$, $a \in I$ then $b \in I$.

Of course, to each of the theorems of the present paper there is a dual one. The dualisation is left to the reader.

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[1]) Instead of $\aleph_\xi$-complete Boolean algebra and $\aleph_\xi$-complete homo(iso)-morphic(ism) and $\aleph_\xi$-complete ideal we simply say $\aleph_\xi$-algebra, $\aleph_\xi$-homo(iso)-morphic(ism), $\aleph_\xi$-ideal resp. Especially, a homomorphic mapping f is said to be *$\aleph_\xi$-homomorphic* if $f(\bigcup_{\iota\in I} x_\iota) = \bigcup_{\iota\in I} f(x_\iota)$ holds for any set I of indices with card $(I) \leqslant \aleph_\xi$. $\aleph_\xi$ is then said to be the *level of completeness*. Instead of the prefix $\aleph_0$- we use the more common symbol σ-($\aleph_0$-algebra $=\sigma$-algebra, $\aleph_0$-ideal $=\sigma$-ideal,...).

In this paper, I deal with some simple properties of free $\aleph_\xi$-algebras in general and of free $\aleph_0$-algebras (also called σ-algebras) especially, within various applications.

The main lemma (of part I) on the existance of a (not trivial) prime $a\sigma$-ideal P containing a given element $a \neq 0$ of a free σ-algebra $A_{\mathfrak{m}}^{\aleph_0}$ is proved in as constructive a manner as possible, by induction (transfinite, of course, and of the order ω_1) [2]). After being somewhat strengthened, this lemma produces, by a well known argument due to M. H. Stone, a σ-isomorphic representation of any free σ-algebra by a σ-field of sets. A corresponding theorem for free $\aleph_\xi$-algebras fails in the case $\aleph_\xi \geqslant 2^{\aleph_0}$ and hence for any uncountable level of completeness — whenever the (special) Continuum Hypothesis is assumed. (This follows at once from a result of R. Sikorski [1]).

As an application of the main theorem 3 (on the σ-isomorphic representation of free σ-algebras by σ-fields of sets) we immediately get a somewhat strengthened form of the known theorem of Loomis [1] (see also Sikorski [1] and Birkhoff [1]) on the representation of σ-algebras by σ-fields of sets. Further, we easily obtain a positive answer to the generalised problem Nr 79 of Birkhoff [1], asking (in its original form) whether the σ-field of all Borel subsets of a Cantor discontinuum is a free σ-algebra. A generalisation of a theorem of Sikorski [2] (his theorem 5.2) is an immediate consequence. Negative answers to further problems Nrs 78 and 80 of Birkhoff [1] are adjoined. A modification of problem Nr 78 (with affirmative answer) is mentioned.

In part II of the paper, I briefly trace an application of the present results to the Tarski-Lindenbaum algebra of the lower predicate calculus (of mathematical logic). I show that this algebra can be isomorphically immerged in the σ-field of Borel subsets of the Cantor discontinuum. The well known Gödel's ([1]) completeness theorem for the lower predicate calculus is an immediate consequence. A more carefull treatment of this subject (within further applications to logic) is planned to be published elsewhere [3]).

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[2]) ω_1 is the first uncountable ordinal, ω_ξ the first ordinal of power $\aleph_\xi$.

[3]) As a volume of the prepared series of special studies to be published by the Państwowy Instytut Matematyczny (Polish State Institute of Mathematics). — I presented the result of the subject of part II at prof. Mostowski's seminar of mathematical logic, in March and April 1950. It was also presented at the session of May 5, 1950, of the Polish Math. Soc., Wrocław Section.

Part I. Algebra.

Theorem 1 (Existence of free $\aleph_\xi$-algebras). *Let $\aleph_\xi$ be any infinite cardinal, $\mathfrak{m}$ another (not necessarily infinite) cardinal $\neq 0$. Then there exists a free $\aleph_\xi$-algebra $A_{\mathfrak{m}}^{\aleph_\xi}$ with $\mathfrak{m}$ free $\aleph_\xi$-generators.*

Proof. I. Let G be any set of the power $\mathfrak{m}$. Choose three fixed auxiliary elements denoted as $*$, σ, δ to be used in order to define complements, joins and meets respectively. Denoting by $\omega_{\xi+1}$ the first ordinal of the power $\aleph_{\xi+1}$ let us define what may be called (infinite) words, by a transfinite construction of the order $\omega_{\xi+1}$.

(i) Let Γ_0 contain all the $g \in G$ and all the ordered pairs $(g, *)$ denoted for convenience by the symbol g^*. We say Γ_0 contains the words of order 0.

(ii) Let all the sets Γ_β of the words of orders at most β be defined for each $\beta \underset{(=)}{\leq} \alpha < \omega_{\xi+1}$ [4]). Suppose $\Gamma_\beta \subseteq \Gamma_{\beta'}$ for $\beta < \beta' \underset{(=)}{\leq} \alpha$. Let $S_{\alpha+1}$ be any subset of the set $\sum\limits_{\beta \underset{(=)}{\leq} \alpha} \Gamma_\beta$ (of the already defined words) with $2 \leqq \operatorname{card}(S_{\alpha+1}) \leqq \aleph_\xi$. Then the set $\Gamma_{\alpha+1}$ of words of order at most $\alpha+1$ consists of the words of $\sum\limits_{\beta \underset{(=)}{\leq} \alpha} \Gamma_\beta$ and of all the ordered pairs $(S_{\alpha+1}, \sigma)$ and $(S_{\alpha+1}, \delta)$ and of all the ordered triples $(S_{\alpha+1}, \sigma, *)$ and $(S_{\alpha+1}, \delta, *)$. We denote these pairs or respective triples (i. e. the words of order $\alpha+1$) for convenience by the symbols $S_{\alpha+1,\sigma}$, $S_{\alpha+1,\delta}$, $S^*_{\alpha+1,\sigma}$, $S^*_{\alpha+1,\delta}$ respectively, to show their dependence on the set $S_{\alpha+1}$. (Hence, the orders of words are non-limit ordinals by definition, which allows some formal simplifications). Clearly $\Gamma_\beta \subset \Gamma_{\beta'}$ for any two (non-limit) ordinals $\beta < \beta' < \omega_{\xi+1}$. We put $\Gamma = \sum\limits_{0 \leq \beta < \omega_{\xi+1}} \Gamma_\beta$ (β non-limit) as the resulting set of (infinite) words.

In the sequel, arbitrary words (i. e. elements of Γ) may also be denoted as $X, Y, Z, \ldots$ If X has one of the forms g, $S_{\beta,\sigma}$, $S_{\beta,\delta}$ then X^* means g^*, $S^*_{\beta,\sigma}$, $S^*_{\beta,\delta}$ respectively. If, on the contrary, X is of one of the later forms then X^* means the corresponding one of the former. In order to denote the assumed forming set S_β (and an assumed order β) of a word X explicitly, we write $X = S^{(*)}_{\beta,\sigma \atop \delta}$.

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[4]) The relation $\beta \underset{(=)}{\leq} \alpha$ (β, α ordinals) excludes identity if and only if α is a limit ordinal. Σ and Π denote set-sum and set-product.

II. Let φ be an arbitrary mapping of the set $G\subset\Gamma$ in a given $\aleph_\xi$-complete algebra B, $\varphi(g)\in B$ for $g\in G$. Extend φ to a mapping $\widetilde{\varphi}$ of the whole set Γ of words in B by the following induction of the order $\omega_{\xi+1}$:

(1) For the words or order 0, only $\widetilde{\varphi}(g^*)=\big(\varphi(g)\big)'$ remains to be defined [5]).

(2) If $\widetilde{\varphi}$ is defined in Γ_β (for words of order at most β), where $\beta \underset{(=)}{<} \alpha<\omega_{\xi+1}$, then put (for words of order $\alpha+1$):

$$\widetilde{\varphi}(S_{\alpha+1,\sigma})=\bigcup_{X\in S_{\alpha+1}}\widetilde{\varphi}(X),\quad \widetilde{\varphi}(S_{\alpha+1,\delta})=\bigcap_{X\in S_{\alpha+1}}\widetilde{\varphi}(X),\quad \widetilde{\varphi}(S^*_{\alpha+1,{\sigma\atop\delta}})=\big(\widetilde{\varphi}(S_{\alpha+1,{\sigma\atop\delta}})\big)';$$

where

$$S_{\alpha+1}\subseteq\sum_{\beta\underset{(=)}{<}\alpha}\Gamma_\beta,\quad 2\leqq\operatorname{card}(S_{\alpha+1})\leqq\aleph_\xi,$$

$\widetilde{\varphi}$ is defined for any $X\in\Gamma$ and it is $\widetilde{\varphi}(X)\in B$. We call $\widetilde{\varphi}$ an *evaluation* (with values in B).

Letting $\widetilde{\varphi}$ run over all evaluations and taking each of the different (in the sense of $\aleph_\xi$-isomorphism) $\aleph_\xi$-algebras of not more than $\mathfrak{m}^{\aleph_{\xi+1}}$ elements as a value algebra we finally say that *X and Y are equivalent words if $\widetilde{\varphi}(X)=\widetilde{\varphi}(Y)$ for each evaluation $\widetilde{\varphi}$.* Now, denoting by $[X],[Y],[Z],\ldots$ the corresponding classes of mutually equivalent words and by $A^{\aleph_\xi}_{\mathfrak{m}}$ the set of all these classes we consider $A^{\aleph_\xi}_{\mathfrak{m}}$ as a $\aleph_\xi$-algebra in the following (essentially well-known) sense:

(i) The complements in $A^{\aleph_\xi}_{\mathfrak{m}}$ are given by

$$[X]'=[X^*].$$

(ii) The $\aleph_\xi$-joins and $\aleph_\xi$-meets in $A^{\aleph_\xi}_{\mathfrak{m}}$ are given as follows:

Let S be any subset of elements of $A^{\aleph_\xi}_{\mathfrak{m}}$ with $2\leqq\operatorname{card}(S)\leqq\aleph_\xi$. For each $[X]\in S$ let us choose a fixed representative word $X\in[X]$. Then there exists exactly one ordinal $\overline{\alpha}<\omega_{\xi+1}$ which is the lowest upper bound of orders of the representative words X when $[X]\in S$. Denoting by $S_{\overline{\alpha}+1}$ the set of all these words we see (on account of the above construction) that $S_{\overline{\alpha}+1}\subset\sum_{\beta\underset{(=)}{<}\overline{\alpha}}\Gamma_\beta$ and hence $S_{\overline{\alpha}+1}$ give rise to the words $S_{\overline{\alpha}+1,\sigma}$, $S_{\overline{\alpha}+1,\delta}$ of order $\overline{\alpha}+1$. Therefore we can put

$$\bigcup_{[X]\in S}[X]=[S_{\overline{\alpha}+1,\sigma}]\in A^{\aleph_\xi}_{\mathfrak{m}},\qquad \bigcap_{[X]\in S}[X]=[S_{\overline{\alpha}+1,\delta}]\in A^{\aleph_\xi}_{\mathfrak{m}}$$

as the $\aleph_\xi$-join and as the $\aleph_\xi$-meet respectively, performed on elements of the set $S\subset A^{\aleph_\xi}_{\mathfrak{m}}$.

[5]) $'$ always denotes complement.

With these definitions, the verification of postulates of $\aleph_\xi$-algebras may be obvious if we observe the lattice ordering relation $[X]\subseteq[Y]$ given by $\widetilde{\varphi}(X)\subseteq\widetilde{\varphi}(Y)$ with each $\widetilde{\varphi}$ [6]). We can omit the details and only note that both the $\aleph_\xi$-infinite distributive laws

$$\Big(\bigcup_{[X]\in S}[X]\Big)\cap\Big(\bigcup_{[Y]\in V}[Y]\Big)=\bigcup_{([X],[Y])\in S\times V}\big([X]\cap[Y]\big)\qquad \Big(2\leqq\operatorname{card}{(S)\atop(V)}\leqq\aleph_\xi\Big)$$

— and dually — hold in $A^{\aleph_\xi}_{\mathfrak{m}}$ as well as in any $\aleph_\xi$-algebra (for this fact, see Birkhoff [1], p. 165).

It remains to be shown that the postulational properties (i) and (ii) (of the Introduction) of free $\aleph_\xi$-algebras are satisfied by $A^{\aleph_\xi}_{\mathfrak{m}}$, with the set of free $\aleph_\xi$-generators $[g]$ ($g\in G$), this set of evidently different one element classes identified with G, if desired.

For (i), it is sufficient to point out that to any $[X]\in A^{\aleph_\xi}_{\mathfrak{m}}$ we have a suitable subset G_X of the set G of free $\aleph_\xi$-generators so that $\operatorname{card}(G_X)\leqq\aleph_\xi$ and the elements of G_X concide to build $[X]$ by means of repeated $\aleph_\xi$-operations of $A^{\aleph_\xi}_{\mathfrak{m}}$.

Indeed, if X is word of order 0 or 1 then our assertion is trivially true. Hence suppose it is true for words of an order not exceeding β, $\beta<\omega_{\xi+1}$. Since any word $Y=S^{(*)}_{\beta+1,{\sigma\atop\delta}}$ of the order $\beta+1$ is formed by a set $S_{\beta+1}$ of not more than $\aleph_\xi$ words of orders at most β (no matter whether β is a limit ordinal or not) and since each of such words (elements of Γ) is constructed by means of not more than $\aleph_\xi$ free $\aleph_\xi$-generators (on account of the inductive assumption) hence our assertion becomes true for $\beta+1$ too, by the equality $\aleph_\xi^2=\aleph_\xi$, i. e. it is true in general.

For (ii), it suffices to note that the above construction itself, by any evaluation $\widetilde{\varphi}$ represents exactly one desired $\aleph_\xi$-complete homomorphic extension $\overline{\varphi}$ of a mapping φ (of G in B) simply given by $\overline{\varphi}(g)=\varphi(g)$ (on G) and by $\overline{\varphi}([X])=\widetilde{\varphi}(X)$ (on the whole $A^{\aleph_\xi}_{\mathfrak{m}}$), q. e. d.

[6]) For defining lattice operations by means of the lattice odering see Birkhoff [1].

Theorem 2. *Let $A_{\mathfrak{m}}^{\aleph_\xi}$ be a free $\aleph_\xi$-algebra of theorem 1 and G be the set of its free $\aleph_\xi$-generators. Let ψ be any mapping of G in an $\aleph_\xi$-algebra B.*

Then the extended $\aleph_\xi$-homomorphic mapping $\overline{\psi}$ of the whole $A_{\mathfrak{m}}^{\aleph_\xi}$ in B is uniquely determined. If the set $\psi(G)\subset B$ $\aleph_\xi$-generates B then B is a $\aleph_\xi$-homomorphic image of $A_{\mathfrak{m}}^{\aleph_\xi}$ and we have an $\aleph_\xi$-isomorphism

$$A_{\mathfrak{m}}^{\aleph_\xi}/\overline{\psi}^{-1}(1)\cong B,$$

where $I=\overline{\psi}^{-1}(1)$ is the $a\aleph_\xi$-ideal of all the elements of $A_{\mathfrak{m}}^{\aleph_\xi}$ the image of which (under $\overline{\psi}$) is the unit 1 of the algebra B.

Proof. The first assertion is clear. Indeed, in general, if a homomorphic mapping of any algebraic system in another system of the same kind exists and has preassigned values on a set of generators of the first system, then there exists only one such homomorphic mapping. (A rigorous inductive elaboration of this known argument in our case is clear).

The second assertion including the so-called first lemma on isomorphism (in our case) may obviously be proved mutatis mutandi of the well known argument.

Corollary 1 (The uniqueness of $A_{\mathfrak{m}}^{\aleph_\xi}$). *A free $\aleph_\xi$-complete algebra is (up to $\aleph_\xi$-isomorphisms) uniquely determined by the cardinal number $\mathfrak{m}$ of its free $\aleph_\xi$-generators and by the level $\aleph_\xi$ of its completeness.*

Because if G_1 and G are sets of free $\aleph_\xi$ generators (of the free $\aleph_\xi$-algebras A and $A_{\mathfrak{m}}^{\aleph_\xi}$ with card $(G_1)=\mathfrak{m}$) then the one-one mapping φ of G onto G_1 can be transfinitely extended to a $\aleph_\xi$-homomorphic mapping $\overline{\varphi}$ of $A_{\mathfrak{m}}^{\aleph_\xi}$ onto A. If this $\aleph_\xi$-homomorphism $\overline{\varphi}$ were not, in fact, an $\aleph_\xi$-isomorphism, then the inverse mapping φ^{-1} of G_1 onto G could not be extended to an $\aleph_\xi$-homomorphic mapping of A onto $A_{\mathfrak{m}}^{\aleph_\xi}$ which contradicts the fact that A it free. (The plurality of counter-images of the unit 1 of A by $\overline{\varphi}$ implies that in extending φ^{-1} one must get an ambiguity of obtained values of $\overline{\varphi}^{-1}$ at a certain stage of the induction).

Corollary 2 (The universality of $A_{\mathfrak{m}}^{\aleph_\xi}$). *Any $\aleph_\xi$-algebra B which has a set H of $\aleph_\xi$-generators with* card $(H)\leqq\mathfrak{m}$ *is an $\aleph_\xi$-homomorphic image of the free $\aleph_\xi$-complete algebra $A_{\mathfrak{m}}^{\aleph_\xi}$ and hence $B\cong A_{\mathfrak{m}}^{\aleph_\xi}/I$ with a suitable $a\aleph_\xi$-ideal I.*

This follows immediately from theorem 2 by a suitable mapping ψ with $\psi(G)=H$.

Note that the universality property of corollary 2 does not suffice to give a characterisation of $A_{\mathfrak{m}}^{\aleph_\xi}$. Indeed, one can easily prove that the direct product $A_{\mathfrak{m}}^{\aleph_\xi}\times(0,1)$ is not free although it has, of course, the universality property in question. On the other hand, one cannot assert that $A_{\mathfrak{m}}^{\aleph_\xi}$ is directly irreducible which is clear from the (trivial finite) example of $A_m^{\aleph_\xi}\times A_m^{\aleph_\xi}=A_{m+1}^{\aleph_\xi}$ (m positive integer) caused by $2^{2^m}\cdot 2^{2^m}=2^{2^{m+1}}$.

There remains a quite natural question, namely that of *whether there is an infinite free complete algebra* in the sense of being a free $\aleph_\xi$-algebra with respect to any level $\aleph_\xi$ of completeness.

It seems plausible that such an algebra does not exist (see the impossibility of its construction by direct application of the method of proof of theorem 1) but I have not been able to prove this.

In the sequel, we shall be interested almost exclusively in $\aleph_0$-complete free algebras (free σ-algebras) which represent by far the most important case for their application. Further justification of this limitation of the subject will appear from theorem 7.

Lemma 1 (The main lemma on prime $a\sigma$-ideals in free σ-algebras). *Let $A_{\mathfrak{m}}^{\aleph_0}$ be the free σ-algebra* (from theorem 1). *Let G with* card $(G)=\mathfrak{m}$ *be a set of free σ-generators of $A_{\mathfrak{m}}^{\aleph_0}$. Let $[X]\in A_{\mathfrak{m}}^{\aleph_0}$ be any given element different from the zero 0 of $A_{\mathfrak{m}}^{\aleph_0}$.*

Then there exists (and, moreover, can be constructed) *a prime $a\sigma$-ideal P of $A_{\mathfrak{m}}^{\aleph_0}$ so that $[X]\in P\neq A_{\mathfrak{m}}^{\aleph_0}$.*

Proof. (For the notation, see proof of theorem 1). Choose a fixed representative word X in the class $[X]$ being the given nonzero element of $A_{\mathfrak{m}}^{\aleph_0}$. It can be assumed that X is not of the order 0. Indeed, if $X=g$ then the lemma is trivially proved with the prime $a\sigma$-ideal $P=\overline{\varphi}^{-1}(1)$ determined by the correspondence $\varphi(g)=1$, $\varphi(\tilde{g})=0$ [for any $\tilde{g}\neq g$ (where φ maps G onto the value algebra $B=(0,1)$), in the sense of the proofs of theorems 1 and 2. And if $X=g^*$ then the same is true with $\varphi(g)=0$ and $\varphi(\tilde{g})=1$ for $\tilde{g}\neq g$. Hence it can be assumed that $X=S_{\alpha,\delta}^{(*)\sigma}$, where $1\leqq\alpha<\omega_1$, α is a non-limit countable ordinal, i. e. the order of X.

First, let us determine what can be called a subword of the word X.

(I) A word Y is a *subword* of a word Z, Z being of order 0 exactly if $Y=Z$ in the case $Z=g$ and if $Y=Z$ or $Y=g$ in the case $Z=g^*$. A word Y is a *subword* of the word $Z=S_{1,\delta}^{(*)\sigma}$ (of order 1) if either $Y=g\in S_1$ respectively $Y=g^*\in S_1$, or $Y=S_{1,\substack{\sigma\\ \delta}}$, or $Y=Z$. The set of all subwords of a word Z may be (for a moment) denoted by Γ_Z.

(II) A word $Z=S^{(*)}_{\alpha+1,\substack{\sigma\\ \delta}}$ (of order $\alpha+1>1$) possesses the set Γ_Z of subwords formed by the set sum $S_{\alpha+1}+\sum_{Y\in S_\beta}\Gamma_Y$, where β with $0<\beta\underset{(=)}{\leq}\alpha$ [7]) is determined by $S^{(*)}_{\beta,\substack{\sigma\\ \delta}}\in S_{\alpha+1}$ (and enlarged by adjoining Z itself).

By induction, we easily see that $\operatorname{card}(\Gamma_Z)\leqq\aleph_0$ for any word $Z\in\Gamma$.

Now, let us form the set $\Gamma_X+\Gamma_{X^*}$, i. e. let us adjoin to Γ_X every Y^* with $Y\in\Gamma_X$, as it is easily seen from the definition, or, if one wishes, from the de Morgan laws. We can and will assume $\Gamma_X+\Gamma_{X^*}$ to be arranged in a fixed finite or infinite sequence

$$X=X_1,X_2,\dots,X_k,\dots$$

(k is a positive integer) which will be the subject of our further considerations.

Our next step is to construct an auxiliary increasing sequence of $\alpha\sigma$-ideals $I_0\subseteq I_1\subseteq\dots\subseteq I_n\subseteq\dots$ (n is a non-negative integer) in the algebra $A^{\aleph_0}_{\mathfrak{m}}$, by complete induction.

(i) Set as I_0 the principal α-ideal generated by the element $[X]$, $0\neq[X]=[X_1]\in A^{\aleph_0}_{\mathfrak{m}}$. I_0 is moreover a non-trivial $\alpha\sigma$-ideal containing $[X]$.

(ii) Let I_n be defined as a non-trivial $\alpha\sigma$-ideal so that $I_l\subseteq I_n$ for $0\leqq l<n$. Let I_n have the following property (pn):

If a word X_i (of the designed sequence) with $i\leqq n$ is of the form $X_i=S_{\beta,\sigma}$ (of course, $\beta\leqq\alpha$) and if $[X_i]\in I_n$ then there exists a suitable subword $X_j\in S_\beta$ of X_i so that $[X_j]\in I_n$.

Note that I_0 enjoys the property $(p0)$ with respect to the void subset of the X_i's with $i\leqq 0$.

From I_n, we proceed to I_{n+1} as follows.

Let I^0_{n+1} be the α-ideal formed by I_n and by the element $[X_n]\in A^{\aleph_0}_{\mathfrak{m}}$, whenever, of course, this ideal does not contain 0. In the contrary case (that the ideal to be formed is trivial) put $I^0_{n+1}=I$. I^0_{n+1} is an (non-trivial) $\alpha\sigma$-ideal, which immediately follows by the (countably) infinite distributive law (see the proof of theorem 1).

[7]) $\beta\underset{(=)}{\leq}\alpha$ always denote that equality is excluded for a limit α.

I^0_{n+1} being defined, let us consider all the words X_j, with $1\leqq j\leqq n+1$, such that $X_j=S_{\gamma,\sigma}$ (with a suitable $\gamma<\alpha$), and that $[X_j]\in I^0_{n+1}$.

Let us prove that to any such X_j there is a word $X_s\in S_\gamma$ (in our sequence) so that the α-ideal formed by I^0_{n+1} and by $[X_s]$ does not contain 0.

Indeed, suppose on the contrary that under the above assumption (on X_j) to any $Y_l=X_{k_l}\in S_\gamma$ $(l=1,2,\dots)$ there is in I^0_{n+1} an element $[Z_l]$ so that $[Y_l]\cap[Z_l]=0$. Then $[Z_l]\subseteq[Y_l]'$ whence $[Y_l]'\in I^0_{n+1}$ for each integer $l=1,2,\dots$ Since I^0_{n+1} is an $\alpha\sigma$-ideal we get $\bigcap_{l=1}^{\infty}[Y_l]'=(\bigcup_{l=1}^{\infty}[Y_l])'\in I^0_{n+1}$. But because $[X_j]=[S_{\gamma,\sigma}]=\bigcup_{l=1}^{\infty}[Y_l]\in I^0_{n+1}$, by hypothesis, we get $0\in I^0_{n+1}$ which is excluded.

Hence we can and will form the $\alpha\sigma$-ideal I^1_{n+1} by the $\alpha\sigma$-ideal I^0_{n+1} and by the element $[X_s]$ (s is a suitable positive integer), where $[X_s]=[Y_l]$ is one of the factors of the join $\bigcup_{Y_l\in S_\gamma}[Y_l]=[X_j]=[S_{\gamma,\sigma}]$, whenever our assumption holds for X_j.

Repeating this adjoining process as many times as possible (i. e. at least $n+1$ times) we get I_{n+1} by the last step. We have $I_n\subseteq I_{n+1}$, the $\alpha\sigma$-ideal I_{n+1} does not contain 0 and it obviously has the property $(p\,n+1)$.

Finally, put $I=\sum_{k=0}^{\infty}I_k$ as the result of our auxiliary construction.

We point out the following needed properties of I:

(a) I is an α-ideal not containing 0 (I is non-trivial).

(b) If $X_j=S_{\beta,\sigma}$ and $[X_j]\in I$ (with any fixed j) then at least one $X_i\in S_\beta$ satisfies $[X_i]\in I$.

(This is a weakened property of being $\alpha\sigma$-prime of I with respect to „components" of $[X]$).

(c) $[X_i]\in I$ if and only if $[X_i]'=[X_i^*]$ non $\in I$.

(This is another weakened property of being prime of I).

Since (a) and (b) follow directly from the above construction of I, we prove only (c). Indeed, $[X_j]\in I$ immediately implies $[X_j]'$ non $\in I$ by (a). On the other hand, suppose $[X_j]$ non $\in I$, i. e. $[X_j]$ non $\in I_j$. This can only be caused by a suitable $[Z]\in I_{j-1}$ such that $[X_j]\cap[Z]=0$, on account of the above construction of I. But this means $[Z]\subseteq[X_j]'\in I_{j-1}\subseteq I$, which proves property (c).

Remembering the proof of theorem 1, let us put $\varphi(g)=1$ whenever the word g or g^* is a member of the sequence $X=X_1,X_2,\dots$ and either $g=[g]\in I$ or $g'=[g]'$ non $\in I$ holds. Otherwise, put $\varphi(g)=0$.

With the help of φ determine the unique σ-homomorphic mapping $\overline{\varphi}$ of $A_{\mathfrak{m}}^{\aleph_0}$ onto the value algebra $B=(0,1)$, in the sense of the proof of theorem 1, so that $\overline{\varphi}(g)=\varphi(g)$, $\overline{\varphi}(g')=\varphi(g^*)$.

We shall prove that if $[X_k]\in I$ then $\overline{\varphi}([X_k])=1$.

Since this is true by definition for each X_k of the order 0, suppose, on the contrary, that there is an X_i of the lowest possible order $\beta>0$ so that $[X_i]\in I$, yet $\overline{\varphi}([X_i])=0$.

(1) If $X_i=S_{\beta,\sigma}$ then by the property (b) of I there is an $[X_j]\in I$ so that $X_j\in S_\beta$ and $[X_i]=[S_{\beta,\sigma}]=\bigcup_{X_\Gamma\in S_\beta}[X_\Gamma]$. By inductive assumption and since X is of an order lower than β, $[X_j]\in I$ implies $\overline{\varphi}([X_j])=1$ and hence $0=\overline{\varphi}([X_i])=\bigcup_{X_\Gamma\in S_\beta}\overline{\varphi}([X_\Gamma])\supseteq\overline{\varphi}([X_j])=1$, which is impossible.

(2) Let $X_i=S_{\beta,\delta}$. Since $\overline{\varphi}([X_i])=\bigcap_{X_j\in S_\beta}\overline{\varphi}([X_j])=0$ there is at least one $X_k\in S_\beta$ (of order lower than β) so that $\overline{\varphi}([X_k])=0$. Hence, by inductive assumption, $[X_k]$ non $\in I$ and therefore $[X_k^*]=[X_k]'\in I$ by the property (c) of I. But since $[X_i]=\bigcap_{X_j\in S_\beta}[X_j]\subseteq[X_i]\in I$ we get the impossible result $[X_i]\cap[X_i]'=0\in I$. Because the remaining cases (3) of $X_i=S_{\beta,\sigma}^*$ and (4) of $X_i=S_{\beta,\delta}^*$ reduce to the former by de Morgan's laws, we finally conclude:

$[X]=[X_1]$ being itself contained in I we have $\overline{\varphi}([X])=1$. Now, $\overline{\varphi}^{-1}(1)=P$ is the desired prime $a\sigma$-ideal (compare with theorem 2, if desired), which completes the proof of lemma 1.

A methodological remark on the proof of the lemma 1.

It is perhaps not without interest to notice that the basic suggestion of the inductive construction of an auxiliary ideal such as I above, goes back to mathematical logic, namely to the use of Hilberts ε-symbol [8]). (See Hilbert-Bernays, *Grundlagen der Mathematik* I and esp. II). A similarly developed argument was used by L. Henkin [1] [8]) to prove Gödel's [1] theorem on the completeness of lower predicate calculus. Since lemma 1 leads almost directly to Loomis theorem (see theorem 5 below) on the representation of σ-algebras by σ-fields of sets (see Loomis [1]), and since we can prove [9]) that Gödels completeness theorem can easily be deduced with the help of the theorem of Loomis as the only essential lemma, one surprisingly concludes that Gödel's theorem and Loomis' theorem are equivalent in a certain methodological sense (each of them dealing, of course, with quite different subjects). The existence of the prime ideal P of lemma 1 can be (perhaps in a somewhat shorter way) inferred in a nonconstructive manner from Sikorski's (see [1]) use of Baire's theorem concerning sets of the first category in M. H. Stone's representative space of the algebra in question. (Sikorski gave a simple proof in [1] of Loomis' theorem by this method). Another method of proof of lemma 1 directly uses Loomis' theorem (compare Birkhoff [1], ex. 3 (b), p. 168). For methodological reasons which seem to be quite natural, I have chosen the present order of ideas.

[8]) I owe this remark to A. Mostowski who also called my attention to L. Henkin's paper.

[9]) See author's paper *On countable generalized σ-algebras, with a proof of Gödel's completeness theorem*, to appear in Čas. mat. fys. 1951.

Lemma 2. *To any two elements $[X]$ and $[Y]$ of the free σ-algebra $A_{\mathfrak{m}}^{\aleph_0}$ such that the contrary of $[X]\subseteq[Y]$ is true, there is a prime $a\sigma$-ideal P such that $[X]\in P$ and $[Y]$ non $\in P$.*

Proof. Since $[X]\cap[Y]'\neq 0$ by supposition, apply lemma 1 to the element $[X]\cap[Y]'$. Of course, the resulting P contains $[X]$ but it cannot contain $[Y]$ because it contains $[Y]'$ and is not trivial.

By a well known argument of M. H. Stone (cf. [1], see also Birkhoff [1]) we can prove without difficulty, on account of the lemma 2, the main

Theorem 3. *The free σ-algebra $A_{\mathfrak{m}}^{\aleph_0}$ with any cardinal number $\mathfrak{m}$ of free σ-generators can be σ-isomorphically represented by a σ-field $F_{\mathfrak{m}}^{\aleph_0}$ of subsets of the set $\mathfrak{S}$ of prime $a\sigma$-ideals $P\neq A_{\mathfrak{m}}^{\aleph_0}$ when any $[X]\in A_{\mathfrak{m}}^{\aleph_0}$ corresponds one-one to the set $\mathfrak{S}([X])\subseteq\mathfrak{S}$ of prime $a\sigma$-ideals $P\neq A_{\mathfrak{m}}^{\aleph_0}$ containing $[X]$.*

Another formulation: *In order to construct $A_{\mathfrak{m}}^{\aleph_0}$ by the proof of theorem* 1, ***it is sufficient to use*** *$B=(0,1)$ as the only value σ-algebra.*

An immediate consequence of theorems 3 and 2 is the following somewhat strengthened form of the theorem of Loomis:

Theorem 4. *An arbitrary σ-algebra B is a σ-homomorphic image of any σ-field $F_{\mathfrak{m}}^{\aleph_0}$ of sets of theorem 3, whenever the cardinal number $\mathfrak{m}$ is not exceeded by the lowest cardinal number of any set H of σ-generators of B. Any σ-homomorphism $\overline{\varphi}$ of $F_{\mathfrak{m}}^{\aleph_0}$ onto B is given by a mapping φ of any family G of sets being free generators of $F_{\mathfrak{m}}^{\aleph_0}$ onto any H, in the sense of the constructions of theorems 1 and 2. Each $\overline{\varphi}$ induces the σ-isomorphism of B with the quotient algebra $F_{\mathfrak{m}}^{\aleph_0}/I_\varphi$, where $I_\varphi=\overline{\varphi}^{-1}(1)$ is the $a\sigma$-ideal of all sets of $F_{\mathfrak{m}}^{\aleph_0}$ whose image is the unit 1 of B.*

Notice that a further topological strengthening of the theorem is presented in theorem 6 below.

Let us return, for a moment, to free $\aleph_\xi$-complete algebras in general.

Theorem 5. *For any $\aleph_\xi \geqq 2^{\aleph_0}$ (and hence for any uncountable $\aleph_\xi$, by the Continuum Hypothesis) the free $\aleph_\xi$-algebra $A_{\mathfrak{m}}^{\aleph_\xi}$ with $\mathfrak{m} \geqq \aleph_0$ cannot be $\aleph_\xi$-isomorphically represented by an $\aleph_\xi$-additive field of sets.*

Proof. Let L/L_0 be the quotient algebra of Lebesgue measurable subsets of the interval $(0,1)$ taken modulo the ideal L_0 of subsets of measure zero. As is well known (see Wecken [1]), L/L_0 is complete (in the sense of any cardinal level $\aleph_\xi$). Now, Sikorski ([1], p. 257) has shown that, for $\aleph_\xi \geqq 2^{\aleph_0}$, L/L_0 cannot be isomorphic to a quotient algebra A/J, where A is an $\aleph_\xi$-additive (i. e. $\aleph_\xi$-complete) field of sets and J is an $\alpha\aleph_\xi$-ideal in A. (Strictly speaking Sikorski shows the dual of this fact). Hence if $A_{\mathfrak{m}}^{\aleph_\xi}$ with $\mathfrak{m} \geqq \aleph_0$ and $\aleph_\xi \geqq 2^{\aleph_0}$ could be represented as an $\aleph_\xi$-additive field of sets, then we should obtain a contradiction in applying theorem 2, corollary (2), with $B = L/L_0$.

Notice that we can prove theorem 5 without using Sikorski's or Wecken's result, although assuming the general Continuum Hypothesis.

Theorem 5, of course, makes free $\aleph_\xi$-algebras of little importance.

It is time to call attention to the fact that the *mentioned result of Sikorski directly yields the impossibility of a (positive) solution of the problem Nr 80 of Birkhoff* [1], p. 168. This problem consists in seeking for a generalisation of Loomis' theorem (theorem 14, p. 168, in Birkhoff [1]) to cardinals (i. e. levels of completeness) other than $\aleph_0$. (Strictly speaking, without the Continuum Hypothesis one gets the impossibility of such generalisation for ordinals at least $2^{\aleph_0}$).

For free $\aleph_\xi$-algebras, one has, by theorem 6 below and by the proof of theorem 1, a weak surrogate of the representability by fields of sets:

Every free $\aleph_\xi$-algebra $A_{\mathfrak{m}}^{\aleph_\xi}$ can be taken as an extension of a σ-field $F_{\mathfrak{m}}^{\aleph_0}$ of sets.

Let us return to topological [10]) representations of free σ-algebras by Borel subsets of a generalised Cantor discontinuum.

We recall that by a generalised Cantor discontinuum $\mathfrak{C}_{\mathfrak{m}}$, determined by the infinite cardinal $\mathfrak{m}$ only, we mean, as is usual in topology, the topological (combinatorial) product of an infinity of $\mathfrak{m}$ spaces each consisting of only two points, say 0 and 1. Points of $\mathfrak{C}_{\mathfrak{m}}$ are then functions φ defined on an abstract set G of the power $\mathfrak{m}$, taking only the values 0 and 1. The open basis of $\mathfrak{C}_{\mathfrak{m}}$ consists of open-closed subsets of $\mathfrak{C}_{\mathfrak{m}}$ each such set being formed by functions $\varphi \in \mathfrak{C}_{\mathfrak{m}}$ so that $\varphi(g) = 0$ or $\varphi(g) = 1$ for a finite set of arguments $g \in G$. It is well known that $\mathfrak{C}_{\mathfrak{m}}$ is a (bi)compact Hausdorff space which is totally disconnected, i. e. zerodimensional.

10) For basic topological notions used in the sequel see Kuratowski, *Topologie I*, Warszawa-Wrocław 1948, or Alexandroff-Hopf, *Topologie I*, Berlin, Springer 1935.

Consider a free σ-algebra $A_{\mathfrak{m}}^{\aleph_0}$ (of theorem 1). Then we observe at once the one-one correspondence (see proof of theorem 1 and theorem 2) between prime $\alpha\sigma$-ideals P_φ in $A_{\mathfrak{m}}^{\aleph_0}$ and points φ of $\mathfrak{C}_{\mathfrak{m}}$, given by $P_\varphi = \overline{\varphi}^{-1}(1)$; $B = (0,1)$.

Now, $A_{\mathfrak{m}}^{\aleph_0}$ contains the free (finitely additive) algebra $A_{\mathfrak{m}}$ with $\mathfrak{m}$ free generators as a subalgebra. It is well known (see Stone [1] and compare Birkhoff [1]) that the above mentioned open-closed basis of $\mathfrak{C}_{\mathfrak{m}}$ forms a finitely additive field $F_{\mathfrak{m}}$ of sets, $F_{\mathfrak{m}}$ being isomorphic with $A_{\mathfrak{m}}$ and this isomorphism is given by the one-one correspondence $\varphi \leftrightarrow Q_\varphi = A_{\mathfrak{m}} P_\varphi$ of α-prime ideals Q_φ of $A_{\mathfrak{m}}$ with points φ of $\mathfrak{C}_{\mathfrak{m}}$. But it is evident (by the construction in the proof of theorem 1) that any prime α-ideal $Q_\varphi = A_{\mathfrak{m}} P_\varphi$ of $A_{\mathfrak{m}}$ can be extended to exactly one prime $\alpha\sigma$-ideal P_φ of $A_{\mathfrak{m}}^{\aleph_0} \supseteq A_{\mathfrak{m}}$, the converse of this one-one correspondence $P_\varphi \leftrightarrow Q_\varphi$ being obvious.

Now, in the σ-isomorphic representation of the free σ-algebra $A_{\mathfrak{m}}^{\aleph_0}$ by the σ-field $F_{\mathfrak{m}}^{\aleph_0}$ of sets of prime $\alpha\sigma$-ideals in theorem 4, the points may be taken as points of $\mathfrak{C}_{\mathfrak{m}}$. Since by this interpretation the free finitary subalgebra $A_{\mathfrak{m}}$ of $A_{\mathfrak{m}}^{\aleph_0}$ isomorphically goes into the open-closed basis $F_{\mathfrak{m}}$ of $\mathfrak{C}_{\mathfrak{m}}$ and since the whole isomorphic image of $A_{\mathfrak{m}}^{\aleph_0}$, the σ-field of (certain) subsets of $\mathfrak{C}_{\mathfrak{m}}$, is generated by its subfield $F_{\mathfrak{m}}$ hence $F_{\mathfrak{m}}^{\aleph_0}$ is contained in the σ-field of Borel's subsets of $\mathfrak{C}_{\mathfrak{m}}$ (by the postulational property (i) of free algebras).

The least σ-field containing all open and closed subsets of a Boolean topological space C will be called the *minimal σ-field* of (Borel) subsets of C.

Theorem 6. *The free σ-algebra $A_{\mathfrak{m}}^{\aleph_0}$ with $\mathfrak{m}$ generators is σ-isomorphically represented by the minimal σ-field of Borel subsets of the generalised Cantor discontinuum $\mathfrak{C}_{\mathfrak{m}}$.*

Corollary (1). *The free σ-algebra $A_{\aleph_0}^{\aleph_0}$ with countably many free σ-generators is σ-isomorphic with the σ-field of Borel's subsets of Cantor's discontinuum $\mathfrak{C} = \mathfrak{C}_{\aleph_0}$.*

This is the positive solution of problem Nr 79 of Birkhoff [1], whereas theorem 6 is its generalisation.

Note that the usual representation of points of $\mathfrak{C}_{\aleph_0}$ as real numbers c can be obtained by the well known correspondence

$$c(\varphi)=2\sum_{n=1}^{\infty}\frac{\varphi(n)}{3^n} \quad \text{with} \quad G=(1,2,\ldots,n,\ldots)$$

Corollary (2). *Any σ-algebra B generated by $\mathfrak{m}$ generators is a σ-homomorphic image of the minimal σ-field $F_{\mathfrak{m}}^{\aleph_0}$ of Borel's subsets of the generalised Cantor discontinuum $\mathfrak{C}_{\mathfrak{m}}$.*

(Proof by theorem 2).

This is a generalisation of Sikorski's theorem 5.2 of [2], p. 20.

It is noteworthy that though the minimal σ-field of Borel's subsets of the generalised Cantor discontinuum $\mathfrak{C}_{\mathfrak{m}}$ enjoys the property of being *algebraically* universal for any cardinal $\mathfrak{m}$, the *topological* universality of $\mathfrak{C}=\mathfrak{C}_{\aleph_0}$ has no full analogue in $\mathfrak{C}_{\mathfrak{m}}$ with $\mathfrak{m}>\aleph_0$, is well known.

Let us turn to the discussion of problem Nr 78 of Birkhoff [1], p. 168, reprinted as follows:

Prove (or disprove) that if a Boolean σ-algebra A is σ-generated by a subset G then every $a\neq 0$ in A contains some finite or infinite countable meet $\bigcap g_i\neq 0$ of elements of G.

First, the disproving is almost trivially accomplished by finite free σ-algebras with n-free generators $g_1,g_2,\ldots,g_n\in G$ and $a=g_1'$. Of course, $g_1'=a\neq 0$ and $\bigcap_{1\leqslant i\leqslant n} g_i\neq 0$. The evaluation $\varphi(g_1)=\varphi(g_2)=\ldots=\varphi(g_n)=1$ implies $\varphi(g_1')=0$ whereas $\varphi(\bigcap_{i=1}^{n} g_i)=1$ which shows that $\bigcap g_i \subsetneqq g_1'$ is impossible. But since the equality is obviously impossible too, the counterexample is clear. — An infinite counterexample of the other kind is that of A being the σ-field of Borel's subsets of Cantor discontinuum with G consisting of the family of open-closed subsets each being formed by the reals $c=2\sum_{n=1}^{\infty}\varphi(n)/3^n$ with $\varphi(n)=1$ under fixed n, and finally with a being the one-point Borel subset $(0)=a$ containing the number 0 only.

Hence a stronger modification of the problem may be desirable. Such a question could be, e. g., the following one: Does there exist, for any $a\neq 0$ in any *free* σ-algebra $A_{\mathfrak{m}}^{\aleph_0}$ a set of free generators G so what $0\neq\bigcap g_i\subseteq a$, $g_i\in G$? (A modification of this kind without limiting the set of generators to free generators would be trivial, of course).

The answer now is *yes*. The proof of this fact is not difficult, on account of theorem 6.

Part II. Application to logic.

The reader's familiarity with basic notions of the lower predicate calculus [11]) will be assumed.

We lay down the following convenient and usual notations:

Capital Latin letters such as $X,Y,Z,\ldots$ (eventually with indices) denote propositional variables.

Small Latin letters such as $x,y,z,\ldots$ denote individual variables (eventually with indices).

By symbols such as $F(\cdot),G(\cdot,\cdot),H(\cdot,\cdot,\cdot),\ldots$ we denote predicate variables with $1,2,3,\ldots$ arguments resp.

The logical junctives and quantifiers and brackets are denoted by the symbols $\&$, $\vee$, $\rightarrow$, $\sim$, $\forall$, $\exists$, (,).

We assume the well known concept of a *formula* [12]) as a recursively defined kind of certain finite sequences formed by the already enumerated elements. Formulae will be denoted by the letters $\mathfrak{A},\mathfrak{B},\ldots$

We assume a system of axioms of the two-valued logic and the well known rules of inference: modus ponens, the substitution rule and the quantifier rules. The definition of a *provable formula* (= identical formula) is clear.

If we identify two formulae $\mathfrak{A}$ and $\mathfrak{B}$ whenever $(\mathfrak{A}\rightarrow\mathfrak{B})\&(\mathfrak{L}\rightarrow\mathfrak{A})$ is provable, then we obtain a Boolean algebra, called the *Tarski-Lindenbaum algebra* of the lower predicate calculus and denoted by *TL*. The element of *TL* which is determined by a formula $\mathfrak{A}$, will be denoted by $[\mathfrak{A}]$. The following equations define the Boolean operations and the elements 0 and 1 in *TL*:

$$[\mathfrak{A}]\cup[\mathfrak{B}]=[(\mathfrak{A})\vee(\mathfrak{B})], \quad [\mathfrak{A}]\cap[\mathfrak{B}]=[(\mathfrak{A})\,\&\,(\mathfrak{B})],$$
$$[\mathfrak{A}]'=[\sim(\mathfrak{A})], \quad 0=[(\mathfrak{A})\,\&\,(\sim(\mathfrak{A}))], \quad 1=[(\mathfrak{A})\rightarrow(\mathfrak{A})].$$

The Boolean interpretation of implication is the following:

$$[(\mathfrak{A})\rightarrow(\mathfrak{B})]=[\mathfrak{A}]'\cup[\mathfrak{B}].$$

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[11]) See D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, Bd. II Sup. I. A. or D. Hilbert and A. Ackermann, *Grundzüge der theoretischen Logik*, Zw. Aufl.

[12]) Formel, well formed formula by certain logicians.

Moreover, besides these finitary operations, we get a countable set of some meets and joins of a countable infinity of elements of TL[13]), as well defined by the equations

$$\bigcap_{x\in I}[\mathfrak{A}(x)] = [\forall y\, \mathfrak{A}(y)], \quad \bigcup_{x\in I}[\mathfrak{A}(x)] = [\exists y\, \mathfrak{A}(y)]$$

(where I denotes the set of individual variables, $\mathfrak{A}$ contains x freely, $\mathfrak{A}$ not contains y)[14]). It is to be noted that the special countably infinite meets and joins, though infinite operations (in the obvious sense of Boolean algebras), they are in each case given by a finite number of inference steps, in the sense of the calculus.

Let us also mention that TL is (up to isomorphism) uniquely determined by the numbers of different variables of any one kind, hence there are many types of TL.

Now we shall apply to the Boolean algebra LT the results already stated in part I of this paper.

First, we extend the notion of a free σ-algebra (free σ-generator, σ-ideal, σ-homomorphism, a. s. o.) *in relativising them to σ-operations* (i. e. countably infinite joins and meets) of a *certain defined kind* (an algebraic characterisation of the kind of σ-operations considered in the Lindenbaum-Tarski algebra may be elaborated elsewhere; see the last footnote of the Introduction). This will be done by simply adjoining to the points (i) and (ii) of the Introduction the condition that in the (in this generalised sense) σ-free algebra (to be defined) the σ-operations in question are to be limited to the preassigned kind. (This preassigned kind of operations will be assumed, of course, to include the usual finite operations). *Second*, we prove without great difficulty that the Tarski-Lindenbaum algebra of the lower predicate calculus *is σ-free in this extended sense, with respect to the above mentioned σ-operations* (given by quantifiers) the free generators (in the extended sense) being the classes $[F(x,y,\ldots,z)], \ldots$ and $[X], \ldots$ On the other hand, as we have seen, the σ-field of Borel's subsets of Cantor's discontinuum is a σ-complete free algebra. Hence any of its subalgebras containing all the open-closed sets is σ-free even in the mentioned extended sense, with respect to a suitable limitation in performing countably infinite joins and meets.

Now, remembering that the Tarski-Lindenbaum algebra is, in fact, an extension of the free (usual) Boolean algebra of the propositional calculus, we can conclude with:

Theorem 7. *The Tarski-Lindenbaum algebra TL of the lower predicate calculus can be isomorphically represented (with respect to all the finite and the defined infinite operations) by a subalgebra of the σ-field $F^{\aleph_0}_{\aleph_0}$ of Borel subsets of the Cantor discontinuum $\mathfrak{C}$. A representation is executed by any one-one correspondence between the countably infinite set of the σ-free generators of TL (in the extended sense) of the form $[F(x)],[G(x,y)],\ldots$ or $[X],[Y],\ldots$ — and the countably infinite set of free σ-generators of $F^{\aleph_0}_{\aleph_0}$, represented by the open-closed sets of Cantors numbers c with 1 at the fixed k-th place in the expansion*

$$c = 2\sum_{k=1}^{\infty}\frac{\varphi(k)}{3^k}.$$

It is to be noted that the Borel subsets of $\mathfrak{C}$ which can occur in a representation of the Tarski-Lindenbaum algebra TL are of a finitary Borel class (with a finite repetition of the indices σ, δ) and that each such „finitary" Borel subset can be used to represent a class of mutually equivalent formulae in a suitable representation of TL.

As an easy corrollary, we get the Gödel completeness theorem, in the following algebraical form:

To any element $[\mathfrak{A}]\neq 0$ of the Tarski-Lindenbaum algebra TL of the lower predicate calculus, there exists a σ-homomorphic mapping (in the extended sense of defined σ-operations) $\overline{\varphi}$ of TL onto the algebra $B=(0,1)$ of „true" $=1$ and „false" $=0$, so that $\overline{\varphi}([\mathfrak{A}])=1$.

Choose a fixed point c in the nonvoid Borel subset (of $\mathfrak{C}$) representing $[\mathfrak{A}]\neq 0$. Put $\overline{\varphi}([\mathfrak{C}])=1$ or 0 according to whether the Borel subset corresponding to $[\mathfrak{C}]$ contains c or not. Then this homomorphic mapping $\overline{\varphi}$ of TL onto $(0,1)$ is nothing more than a logical evaluation, and hence it defines — by ascribing either the „true" or the „false" to any „atomic formula" — $F(x,y,\ldots,z)$ individual predicates (fulfilling the formula $\mathfrak{A}$ in question) if individual variables are taken for individua. (Indeed, $\overline{\varphi}$ gives more,

[13]) The idea of interpreting quantifiers as infinite algebraic operations is, of course, not new. (For an interesting recent conception of this idea, see F. I. Mautner, *Logic as Invariant Theory, an Extension of Klein's Erlanger Program,* Amer. J. Math. **68** (1946), pp. 345-386.

[14]) Strictly speaking, there is a little formal complication: For the possibility of substituting any free individual variable x in $\mathfrak{A}(\ )$, sometimes a suitable „change of nomination" („Umbenennung") of variables in $\mathfrak{A}(\ )$ is needed.

i. e. a simultaneous interpretation of *all* predicate variables as individual predicates. An analogous situation is in Henkin [1]. For a detailed discussion of the already sketched application of part I to logic, see my paper announced in the footnote [3]) of the end of Introduction.

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(For further literature, see notes and footnotes of the text).

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A Note to Rieger's Paper „On Free $\aleph_\xi$-complete Boolean Algebras" [1]).

By

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The subject of this note is a simple proof of Rieger's Theorem 6 [2]).

Let M be an abstract set with cardinal $\mathfrak{m}$ (finite or infinite), and let $C_\mathfrak{m}$ denote the set of all functions f on M, the values of which are the numbers 0 and 1 only. ($C_\mathfrak{m}$ is the so-called *generalized Cantor discontinuum*, i. e. the Cartesian product of $\mathfrak{m}$ spaces, each of which is composed of the numbers 0 and 1 only).

For $a \epsilon M$ let $C_{\mathfrak{m},a}$ denote the set of all $f \epsilon C_\mathfrak{m}$ such that $f(a)=1$. For every (infinite) cardinal $\mathfrak{n}$ let $F_{\mathfrak{m},\mathfrak{n}}$ denote the least $\mathfrak{n}$-additive field of subsets of $C_\mathfrak{m}$ containing all the sets $C_{\mathfrak{m},a}$ $(a \epsilon M)$.

If $\boldsymbol{X}$ is an $\mathfrak{n}$-additive field of subsets of a set $\mathfrak{X}$, and if $\boldsymbol{I}$ is an $\mathfrak{n}$-additive ideal of $\boldsymbol{X}$, then the $\mathfrak{n}$-complete Boolean algebra $\boldsymbol{X}/\boldsymbol{I}$ is called an $\mathfrak{n}$-*quotient algebra*. In particular, every $\mathfrak{n}$-additive field of sets is also an $\mathfrak{n}$-quotient algebra (the ideal $\boldsymbol{I}$ then contains only the empty set).

Theorem. *$F_{\mathfrak{m},\mathfrak{n}}$ is the free $\mathfrak{n}$-quotient algebra with $\mathfrak{m}$ generators $C_{\mathfrak{m},a}$ $(a \epsilon M)$.*

This means:

For every family $\{A_a\}_{a \epsilon M}$ of elements of any $\mathfrak{n}$-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ there exists an $\mathfrak{n}$-additive homomorphism h of $\boldsymbol{F}_{\mathfrak{m},\mathfrak{n}}$ into $\boldsymbol{X}/\boldsymbol{I}$ such that $h(C_{\mathfrak{m},a})=A_a$ $(a \epsilon M)$.

[1]) This volume, pp. 29-46.

[2]) Loc. cit., p. 41.