For every $a \in M$ let $X_a \in \boldsymbol{X}$ be a fixed set such that[3]) $[X_a]=A_a$. Let $f=c(x)$ be the characteristic function[4]) of the family $\{X_a\}_{a \in M}$, that is, the mapping of $\mathfrak{X}$ into $C_{\mathfrak{m}}$ which associates with $x \in \mathfrak{X}$ an element $f \in C_{\mathfrak{m}}$ defined as follows: $f(a)=1$ if and only if $x \in X_a$. The mapping

$$h(F)=[c^{-1}(F)] \quad \text{for} \quad F \in \boldsymbol{F}_{\mathfrak{m},\mathfrak{n}}$$

is an $\mathfrak{n}$-additive homomorphism of $\boldsymbol{F}_{\mathfrak{m},\mathfrak{n}}$ into $\boldsymbol{X}/\boldsymbol{I}$ such that

$$h(C_{\mathfrak{m},a})=[c^{-1}(C_{\mathfrak{m},a})]=[X_a]=A_a, \qquad \text{q. e. d.}$$

Corollary 1 (Rieger's Theorem 6). *The σ-field $\boldsymbol{F}_{\mathfrak{m},\aleph_0}$ is the free Boolean σ-algebra with $\mathfrak{m}$ generators $C_{\mathfrak{m},a}$ $(a \in M)$*[5]).

This follows immediately from the fact that every Boolean σ-algebra is isomorphic to an $\aleph_0$-quotient algebra[6]).

Corollary 2. *Every $\mathfrak{n}$-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ with at most $\mathfrak{m}$ generators is isomorphic to an $\mathfrak{n}$-quotient algebra $\boldsymbol{F}_{\mathfrak{m},\mathfrak{n}}/\boldsymbol{J}$, where $\boldsymbol{J}$ is a suitable $\mathfrak{n}$-additive ideal.*

This is a generalization of Rieger's Theorem 4[7]).

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[3]) For $X \in \boldsymbol{X}$ the symbol $[X]$ will denote the element (coset) of $\boldsymbol{X}/\boldsymbol{I}$ determined by X.

[4]) M. H. Stone, *On Characteristic Functions of Families of Sets*, Fund. Math. **33** (1945), pp. 27-33. See also E. Marczewski, *The characteristic function of sets and some its applications*, Fund. Math. **31** (1938), pp. 207-223.

[5]) Another proof of this fact follows from Theorem VIII in my paper *On an analogy between measures and homomorphisms*, Annales Soc. Pol. Math. **23** (1950), pp. 1-20. That proof is based on Loomis's theorem for Boolean algebras with $\aleph_0$ generators only.

[6]) See L. H. Loomis, *On the representation of σ-complete Boolean algebras*, Bull. Am. Math. Soc. **53** (1947), pp. 757-760, and R. Sikorski, *On the representation of Boolean algebras as fields of sets*, Fund. Math. **35** (1948), pp. 247-258 (Theorem 5.3).

[7]) Loc. cit., p. 39.

Concerning the Cartesian product of Cantor-manifolds.

By

Karol Borsuk (Warszawa).

1. A set of points[1]) is called an *n-dimensional Cantor-manifold*[2]) if it is an n-dimensional compactum and it cannot be disconnected by a subset of dimension $\leqslant n-2$.

It is known[3]) that every n-dimensional Cantor-manifold is n-dimensional in every one of its points and that

(1) *If A and B are n-dimensional Cantor-manifolds and $\dim A \cdot B \geqslant n-1$, then $A+B$ is also an n-dimensional Cantor-manifold.*

We can easily see that if in the formula

$$C=A \times B \text{ [4])} \tag{2}$$

A and B are polytopes[5]) then C is a Cantor-manifold if and only if both polytopes A and B are Cantor-manifolds.

In this paper I shall show, by certain examples, that for arbitrary compacta there exists no relation between the Cantor-manifold property of A, B and C. Namely the following theorem holds:

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[1]) It is convenient to assume that all sets of points investigated in this paper are subsets of the Hilbert space.

[2]) P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fund. Math. **7** (1925), p. 124.

[3]) See for instance C. Kuratowski, *Topologie II*, Warszawa-Wrocław 1950, p. 106.

[4]) $A \times B$ denotes the Cartesian product of A and B.

[5]) By a *polytope* we understand a point-set contained in the Hilbert space and having a decomposition in a finite collection of geometrical (rectilinear) simplexes such that every face of each simplex of the collection belongs to the collection. This decomposition of a polytope is called its *triangulation*. Every set homeomorphic to a polytope is called a *curvilinear polytope*.

Theorem 1. *If we assume only that the sets A, B and C in formula* (2) *are locally connected continua then the supposition that some of them are Cantor-manifolds does not imply that any of the others is a Cantor-manifold.*

The proof of this theorem is given at the end of the paper. The main part of the paper concerns the investigation of the Cantor-manifold property of some sets, especially of a type of spaces called *approximative pseudo-manifolds*.

2. We have already observed that, for the polytopes, if A and B are Cantor-manifolds then C is also one, and if at least one of the polytopes A and B is not a Cantor-manifold, then C is also not a Cantor-manifold. Consequently to prove theorem 1 it only remains to give the following three examples:

Example 1 of two locally connected Cantor-manifolds A_1 and B_1 such that $C_1 = A_1 \times B_1$ is not a Cantor-manifold.

Example 2 of a locally connected Cantor-manifold A_2 and of a locally connected continuum B_2 which is not a Cantor-manifold, such that $C_2 = A_2 \times B_2$ is a Cantor-manifold.

Example 3 of two locally connected continua A_3 and B_3 which are not Cantor-manifolds and such that $C_3 = A_3 \times B_3$ is a Cantor-manifold.

3. Suitable examples will be constructed with the aid of the known surfaces of L. Pontrjagin[6]) showing the fallacy of the formula $\dim(A \times B) = \dim A + \dim B$.

First we establish some properties of the surfaces of Pontrjagin. Let S denote the circle composed by all complex numbers z with $|z| = 1$. Let I be the segment $0 \leqslant t \leqslant 1$. By a *Möbius band* mod m we understand the continuum M_m obtained from the product $S \times I$ by identifying on the circumference $S_0 = S \times (0)$ points corresponding to each other under the rotation of angle $2\pi/m$. In general M_m is a homogenously 2-dimensional curvilinear polytope, but we can realize it in the Hilbert space also as a rectilinear polytope. By the *boundary* of M_m we understand the circle $S_1 = S \times (1)$.

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[6]) L. Pontrjagin, *Sur une hypothèse fondamentale de la théorie de la dimension*, Comptes Rendus de l'Ac. des Sc. **190** (Paris 1930), p. 1105-1107.

Lemma. *For every proper closed subset $A \supset S_1$ of M_m the circle S_1 is a retract*[7]) *of A.*

Proof. Consider a point $p_0 = (z_0, t_0) \in S \times I - S_0 - S_1 - A$. Manifestly there exists a retraction of $S \times I - (p_0)$ to the set $S_0 + S_1 + (z_1) \times I$, where $z_1 \in S - (z_0)$. To the set $S \times I - (p_0)$ corresponds in M_m the set $M_m - (p_0)$ and to the set $S_0 + S_1 + (z_1) \times I$ — a 1-dimensional subcontinuum N of M_m containing S_1. Obviously S_1 is retract of N. Hence S_1 is a retract of $M_m - (p_0)$ and also a retract of $A \subset M_m - (p_0)$.

4. Let γ_ν denote the 1-dimensional cycle of S defined by the formula

$$\gamma_\nu = \sum_{\mu=1}^{\nu} \left(e^{2\pi i \frac{\mu-1}{\nu}}, e^{2\pi i \frac{\mu}{\nu}}\right).$$

Evidently $\gamma = \{\gamma_\nu\}$ is a 1-dimensional true cycle[8]) in S (called the *basic cycle* of S). For every $m = 1, 2, \ldots$ it can be considered as a true cycle mod m in S_1 and then it is homologous to 0 in M_m, but totally unhomologous to 0 in S_1. On the other hand, for every closed proper subset $A \supset S_1$ of M_m the cycle γ is totally unhomologous to 0 in A. For otherwise there would exist a subsequence $\{\gamma_{i_\nu}\}$ of $\{\gamma_i\}$ homologous to 0 in A and the retraction of A to S_1 would transform the relation $\{\gamma_{i_\nu}\} \sim 0$ in A into the relation $\{\gamma_{i_\nu}\} \sim 0$ in S_1, which does not hold.

Let Δ be a triangle (closed) lying in the Euclidean 5-dimensional space E_5 and let Λ denote the interior of Δ. Consider a 5-di-

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[7]) A subset E_0 of a space E is called a *retract* of E if there exists a continuous mapping f *(retraction)* of E onto E_0 such that $f(x) = x$ for every $x \in E_0$.

[8]) Let E be a compactum and ε a positive number. By an *ε-simplex* of E we understand a finite subset of E with diameter $< \varepsilon$. In the known manner we introduce the notion of an oriented ε-simplex of E, of an ε-chain of E with arbitrarily given coefficients, and of an ε-cycle of E. An ε-cycle γ of E is said to be *ε-homologous in E* if there exists an ε-chain $\varkappa$ of E such that γ constitutes its boundary $\partial\varkappa$.

By a k-dimensional *true cycle* mod m of E one understands a sequence $\gamma = \{\gamma_i\}$ of k-dimensional ε_i-cycles mod m of E, where $\varepsilon_i \to 0$. A true cycle $\gamma = \{\gamma_i\}$ is *homologous to zero in E* (symbolically $\gamma \sim 0$ in E) whenever there exists a sequence $\{\eta_i\}$ of positive numbers convergent to zero and such that γ_i is η_i-homologous to zero in E. If there exists an $\varepsilon > 0$ such that no one of the cycles γ_i is ε-homologous to zero in E, then the true cycle $\gamma = \{\gamma_i\}$ is called *totally unhomologous to zero in E*.

mensional element[9]) $Q\subset E_5$ containing Δ in its interior. Evidently there exists a polytope M'_m homeomorphic to M_m such that

1° the boundary $\Gamma=\Delta-\Lambda$ of Δ constitutes the boundary of the strip of Möbius M'_m (mod m),

2° the set $M'_m-\Gamma$ lies in the interior of Q.

Clearly there is a mapping α of M'_m onto Δ which is the identity on the boundary Γ.

5. By the *surface of Pontrjagin* P_m we understand the topological limit[10]) of the sequence $\{P_{m,\nu}\}$ defined as follows:

$P_{m,1}$ is a triangle with the diameter 1, lying in E_5. By $\tau_{m,1}$ we denote the triangulation of $P_{m,1}$ consisting of one triangle $\Delta^1_{m,1}=P_{m,1}$. By $Q^1_{m,1}$ we denote a 5-dimensional convex element lying in E_5 and such that its diameter is equal to 1, its interior $R^1_{m,1}$ contains the interior $\Lambda^1_{m,1}$ of $\Delta^1_{m,1}$ and its boundary $Q^1_{m,1}-R^1_{m,1}$ contains the boundary $\Delta^1_{m,1}-\Lambda^1_{m,1}$ of $\Delta^1_{m,1}$. By $\varphi_{m,1}$ we denote a mapping retracting $Q_{m,1}$ to $P_{m,1}$.

Let us suppose that for some ν there is already defined a homogeneously 2-dimensional polytope $P_{m,\nu}\subset E_5$ and a triangulation $\tau_{m,\nu}$ of $P_{m,\nu}$ with the triangles $\Delta^i_{m,\nu}$ having diameters $\leqslant 2^{1-\nu}$. Moreover let us suppose that to every triangle $\Delta^i_{m,\nu}$ corresponds a 5-dimensional convex element $Q^i_{m,\nu}\subset E_5$ such that the interior $\Lambda^i_{m,\nu}$ of $\Delta^i_{m,\nu}$ lies in the interior $R^i_{m,\nu}$ of $Q^i_{m,\nu}$ and the boundary $\Gamma^i_{m,\nu}=\Delta^i_{m,\nu}-\Lambda^i_{m,\nu}$ of $\Delta^i_{m,\nu}$ lies on the boundary $Q^i_{m,\nu}-R^i_{m,\nu}$ of $Q^i_{m,\nu}$ and that the diameter of $Q^i_{m,\nu}$ is $\leqslant 2^{1-\nu}$. Furthermore we suppose that for every two triangles $\Delta^i_{m,\nu}$, $\Delta^j_{m,\nu}\in\tau_{m,\nu}$, $i\neq j$, it is $Q^i_{m,\nu}\cdot Q^j_{m,\nu}=\Delta^i_{m,\nu}\cdot\Delta^j_{m,\nu}$. By $\varphi_{m,\nu}$ we denote a mapping retracting the set $Q_{m,\nu}=\sum_i Q^i_{m,\nu}$ to $P_{m,\nu}$ in such a manner that $\varphi_{m,\nu}(Q^i_{m,\nu})=\Delta^i_{m,\nu}$ for every i.

We now replace every triangle $\Delta^i_{m,\nu}$ by a polytope $M^i_{m,\nu}$ such that

1° $M^i_{m,\nu}$ is homeomorphic to the Möbius band mod m,

2° $\Gamma^i_{m,\nu}$ is the boundary of $M^i_{m,\nu}$,

3° the set $M^i_{m,\nu}-\Gamma^i_{m,\nu}$ lies in the interior of $Q^i_{m,\nu}$.

Putting

$$P_{m,\nu+1}=\sum M^i_{m,\nu}$$

[9]) By an *n-dimensional element* we understand a set homeomorphic to an n-dimensional (closed) simplex.

[10]) See for instance C. Kuratowski, *Topologie I*, Warszawa-Wrocław 1948, p. 245.

consider a triangulation $\tau_{m,\nu+1}$ of $P_{m,\nu+1}$ such that every triangle $\Delta^j_{m,\nu+1}\in\tau_{m,\nu+1}$ has the diameter $\leqslant 2^{-\nu}$ and that the 1-dimensional skeleton $T_{m,\nu}$[11]) of $\tau_{m,\nu}$ is contained in the 1-dimensional skeleton $T_{m,\nu+1}$ of $\tau_{m,\nu+1}$. It follows that for every triangle $\Delta^j_{m,\nu+1}\in\tau_{m,\nu+1}$ there exists a triangle $\Delta^i_{m,\nu}\in\tau_{m,\nu}$ such that $\Delta^j_{m,\nu+1}\subset Q^i_{m,\nu}$. We infer that there exists a 5-dimensional convex element $Q^j_{m,\nu+1}\subset Q^i_{m,\nu}$ such that the interior $\Lambda^j_{m,\nu+1}$ of $\Delta^j_{m,\nu+1}$ is contained in the interior $R^j_{m,\nu+1}$ of $Q^j_{m,\nu+1}$, the boundary $\Gamma^j_{m,\nu+1}=\Delta^j_{m,\nu+1}-\Lambda^j_{m,\nu+1}$ of $\Delta^j_{m,\nu+1}$ lies on the boundary $Q^j_{m,\nu+1}-R^j_{m,\nu+1}$ of $Q^j_{m,\nu+1}$, and the diameter of $Q^j_{m,\nu+1}$ is $\leqslant 2^{-\nu}$. We can easily see that the elements $Q^j_{m,\nu+1}$ can be chosen in such a manner that

$$Q^j_{m,\nu+1}\cdot Q^{j'}_{m,\nu+1}=\Delta^j_{m,\nu+1}\cdot\Delta^{j'}_{m,\nu+1},$$

for every two indices $j\neq j'$.

Obviously there exists a mapping $\varphi_{m,\nu+1}$ retracting $Q_{m,\nu+1}=\sum_j Q^j_{m,\nu+1}$ to $P_{m,\nu+1}$ in such a manner that $\varphi_{m,\nu+1}(Q^j_{m,\nu+1})=\Delta^j_{m,\nu+1}$ for every index j. Hence

(3) $$\varrho(\varphi_{m,\nu+1}(x),x)\leqslant 2^{-\nu}\quad\text{for every}\quad x\in Q_{m,\nu+1}.$$

6. Let us observe that

(4) *S_1 is a retract of every proper closed subset $A\supset S_1$ of $P_{m,\nu}$.*

Statement (4) is true for $\nu=1$. Assume it for an ν and suppose that $A\supset S_1$ is a proper closed subset of $P_{m,\nu+1}$. Then there exists a triangle $\Delta^i_{m,\nu}\in\tau_{m,\nu}$ such that the Möbius band M^i_m obtained from it by the construction of $P_{m,\nu+1}$ is not contained in A. It follows that there exists a retraction r_i of the set

$$A\cdot M^i_m+\Gamma^i_{m,\nu}$$

to the boundary $\Gamma^i_{m,\nu}$ of $\Delta^i_{m,\nu}$. Putting

$$\varphi(x)=\varphi_{m,\nu}(x)\quad\text{for every}\quad x\in P_{m,\nu+1}-M^i_m,$$
$$\varphi(x)=r_i(x)\quad\text{for every}\quad x\in A\cdot M^i_m,$$

we obtain a continuous mapping of $A+(P_{m,\nu+1}-M^i_m)$ into $P_{m,\nu}-\Lambda^i_{m,\nu}$. But by the hypothesis of the induction there exists a retraction r' of $P_{m,\nu}-\Lambda^i_{m,\nu}$ to S_1. It suffices to put

$$r(x)=r'\varphi(x)\quad\text{for every}\quad x\in A$$

in order to obtain a retraction of A to S_1.

[11]) By the *k-dimensional skeleton* of a triangulation τ we understand the polytope built of all simplexes of τ of dimension $\leqslant k$.

Furthermore let us observe that

(5) *For every* ν *the polytope* $P_{2,\nu}$ *is a 2-dimensional pseudo-manifold* [12]) *with the boundary* S_1.

To prove it let us observe that $P_{2,1}$ is a 2-dimensional pseudo-manifold with the boundary S_1 and that the construction of $P_{2,\nu+1}$ is such that the pseudo-manifold property of $P_{2,\nu}$ implies the pseudo-manifold property of $P_{2,\nu+1}$ with unchanged boundary.

7. The polytopes $P_{m,\nu}$ converge to a continuum P_m called the *surface of Pontrjagin* mod m. It is clear that

(6) $$P_m=\prod_{\nu=1}^{\infty} Q_{m,\nu}.$$

and that P_m contains the 1-dimensional skeleton of $P_{m,\nu}$, for every $\nu=1,2,\ldots$ In particular

(7) $$S_1\subset P_m.$$

Moreover let us observe that the common part of the surface P_m and of the element $Q^i_{m,\nu}$ is connected (even homeomorphic to P_m). Since the diameter of $Q^i_{m,\nu}$ is $\leqslant 2^{1-\nu}$ we infer that P_m is the sum of a finite number of connected sets each of diameter arbitrarily small. It follows [13]) that

(8) P_m *is a locally connected continuum.*

In particular [14])

(9) P_m *is arcwise connected.*

[12]) By an *n-dimensional pseudo-manifold* we understand here always a bounded n-dimensional pseudo-manifold, that is an n-dimensional polytope M which is a Cantor-manifold and has a triangulation τ in which every $(n-1)$-dimensional simplex is a face of one or two n-dimensional simplexes of τ. The sum N of all $(n-1)$-dimensional simplexes of τ which are faces of precisely one n-dimensional simplex of τ is called the *boundary* of the n-dimensional pseudo-manifold M. Every set homeomorphic to an n-dimensional pseudo-manifold will be called a *curvilinear pseudo-manifold*. To the boundary of the pseudo-manifold corresponds by the homeomorphism the boundary of the curvilinear pseudo-manifold.

[13]) W. Sierpiński, *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fund. Math. **1** (1920), p. 44.

[14]) S. Mazurkiewicz, *Sur les lignes de Jordan*, Fund. Math. **1** (1920), p. 201.

The mapping $\varphi_{m,\nu}$ retracting $Q_{m,\nu}$ to $P_{m,\nu}$ is defined on the set $P_m\subset Q_{m,\nu}$ and it maps P_m into $P_{m,\nu}$. For every x belonging to the 1-dimensional skeleton of $P_{m,\nu}$ it is

$$\varphi_{m,\nu}(x)=x.$$

We conclude by (3) that

(10) *The set* P_m *is* $2^{1-\nu}$*-deformable into the 2-dimensional polytope* $P_{m,\nu}$ *in such a manner that* S_1 *is carried into itself.*

We infer [15]) by (10) that the dimension of P_m is $\leqslant 2$.

Consider now the basic cycle γ mod m of S_1. Obviously γ is totally unhomologous to 0 in S_1 and homologous to zero in $P_{m,\nu}$ for every $\nu=1,2,\ldots$ Hence

(11) *There exists a 1-dimensional true cycle mod* m *of* S_1 *totally unhomologous to zero in* S_1, *but homologous to zero in* P_m.

It follows [16]) that $\dim P_m\geqslant 2$. Thus we have shown that

(12) $$\dim P_m=2.$$

Moreover (11) implies that

(13) S_1 *is not a retract of* P_m.

Let us observe that

(14) S_1 *is a retract of every proper closed subset* $A\supset S_1$ *of* P_m.

In fact, for ν sufficiently large, $\varphi_{m,\nu}(A)$ is a closed proper subset of $P_{m,\nu}$ and $S_1\subset\varphi_{m,\nu}(A)$. By (4) there exists a mapping $r_\nu(x)$ retracting the set $\varphi_{m,\nu}(A)$ to S_1.

Putting

$$r(x)=r_\nu\varphi_{m,\nu}(x) \quad \text{for every} \quad x\in A$$

we obtain a retraction of A to S_1.

Thus we have shown by (13) and (14) that the mapping defined in S_1 as the identity cannot be extended over P_m, but it can be extended over every closed proper subset $A\supset S_1$ of P_m. It follows [17]) that

(15) P_m *is a 2-dimensional Cantor-manifold.*

[15]) By Alexandroff's theorem on approximation to compacta by polytopes. See for instance, W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton 1941, Princeton University Press, p. 72.

[16]) See, for instance, W. Hurewicz and H. Wallman, l. c., p. 151.

[17]) W. Hurewicz and H. Wallman, l. c., p. 95.

8. The most important property of the surfaces of Pontrjagin is that the dimension of $P_m \times P_n$ is equal to 3 if $(m,n)=1$ and equal to 4 if $(m,n)>1$. The proof of this property is only sketched in the paper of Pontrjagin [18]). For the sake of completeness I shall give here the detailed proof elaborated by R. Sikorski.

Firstly let us show that

(16) *If* $(m,n)=1$, *then* $P_{m,\nu+1}\times P_{n,\nu+1}$ *is* $2^{2-\nu}$*-deformable into the 3-dimensional polytope* $T_{m,\nu}\times P_{n,\nu}+P_{m,\nu}\times T_{n,\nu}$.

Let $B_{i,j}$ be the boundary of $M^i_{m,\nu}\times M^j_{n,\nu}$, i. e.

$$B_{i,j}=\Gamma^i_{m,\nu}\times M^j_{n,\nu}+M^i_{m,\nu}\times\Gamma^j_{n,\nu},$$

and let $S_{i,j}$ be the boundary of the 4-dimensional cube $\Delta^i_{m,\nu}\times\Delta^j_{n,\nu}$ i. e.

$$S_{i,j}=\Gamma^i_{m,\nu}\times\Delta^j_{n,\nu}+\Delta^i_{m,\nu}\times\Gamma^j_{n,\nu}.$$

Evidently $S_{i,j}$ is a 3-dimensional sphere.

As it was remarked at the end of Nr. **4**, there is a continuous mapping of $M^i_{m,\nu}$ onto $\Delta^i_{m,\nu}$ which is the identity on the common boundary $\Gamma^i_{m,\nu}$. Consequently there exists a mapping α of $P_{m,\nu+1}$ onto $P_{m,\nu}$ such that

$$\alpha(M^i_{m,\nu})=\Delta^i_{m,\nu}\quad\text{and}\quad\alpha(\Gamma^i_{m,\nu})=\Gamma^i_{m,\nu}.$$

Analogously there exists a mapping β of $P_{n,\nu+1}$ onto $P_{n,\nu}$ such that

$$\beta(M^j_{n,\nu})=\Delta^j_{n,\nu}\quad\text{and}\quad\beta(\Gamma^j_{n,\nu})=\Gamma^j_{n,\nu}.$$

The transformation $\gamma(x,y)=(\alpha(x),\beta(y))$, where $x\in P_{m,\nu+1}$, $y\in P_{n,\nu+1}$ maps $P_{m,\nu+1}\times P_{n,\nu+1}$ into $P_{m,\nu}\times P_{n,\nu}$ so that

$$\gamma(M^i_{m,\nu}\times M^j_{n,\nu})\subset\Delta^i_{m,\nu}\times\Delta^j_{n,\nu}\quad\text{and}\quad\gamma(B_{i,j})\subset S_{i,j}.$$

Let $\gamma_{i,j}$ denote the mapping γ restricted to the set $M^i_{m,\nu}\times M^j_{n,\nu}$. Since $(m,n)=1$, the polytope $M^i_{m,\nu}\times M^j_{n,\nu}$ contains no one 4-dimensional relative cycle mod $B_{i,j}$ [19]). By Hopf's theorem [20]), there exists a mapping $\varkappa_{i,j}$ of $M^i_{m,\nu}\times M^j_{n,\nu}$ into $S_{i,j}$ such that

$$\varkappa_{i,j}(x,y)=\gamma_{i,j}(x,y)=\gamma(x,y)\quad\text{for}\quad(x,y)\in B_{i,j}.$$

If $(i,j)\neq(i',j')$ and $(x,y)\in(M^i_{m,\nu}\times M^j_{n,\nu})\cdot(M^{i'}_{m,\nu}\times M^{j'}_{n,\nu})$ then $(x,y)\in B_{i,j}\cdot B_{i',j'}$. Hence

$$\varkappa_{i,j}(x,y)=\gamma(x,y)=\varkappa_{i',j'}(x,y).$$

[18]) See footnote [6]).

[19]) See P. Alexandroff and H. Hopf, *Topologie I*, Berlin 1935, p. 310.

[20]) See P. Alexandroff and H. Hopf, l. c., p. 501.

The union $\varkappa$ of all mappings $\varkappa_{i,j}$ is a transformation of the set

$$\sum_{i,j}M^i_{m,\nu}\times M^j_{n,\nu}=P_{m,\nu+1}\times P_{n,\nu+1}$$

into the set

$$\sum_{i,j}S_{i,j}=T_{m,\nu}\times P_{n,\nu}+P_{m,\nu}\times T_{n,\nu}.$$

The mapping $\varkappa$ is a $2^{2-\nu}$-deformation, since

$$\varkappa(M^i_{m,\nu}\times M^j_{n,\nu})\subset S_{i,j}\subset Q^i_{m,\nu}\times Q^j_{n,\nu},\qquad M^i_{m,\nu}\times M^j_{n,\nu}\subset Q^i_{m,\nu}\times Q^j_{n,\nu}$$

and the diameter of $Q^i_{m,\nu}\times Q^j_{n,\nu}$ is $\leqslant 2^{2-\nu}$.

By (16) and (10)

$$\dim P_m\times P_n\leqslant 3.$$

The converse inequality being evidently also true [21]), we infer:

(17) *If* $(m,n)=1$ *then* $\dim P_m\times P_n=3$.

Now let us observe that if $(m,n)>1$ and k is a prime common factor of m and n and $\gamma=\{\gamma_i\}$ denotes (as in Nr **4**) the basic cycle of S_1 then γ can be considered as a true cycle mod k totally unhomologous to 0 in S_1. Then $\gamma\times\gamma=\{\gamma_i\times\gamma_i\}$ [22]) is a 2-dimensional true cycle mod k totally unhomologous to 0 in $S_1\times S_1$ but homologous to 0 in $S_1\times P_n$ and also in $P_m\times S_1$. Hence there exists in $S_1\times P_n$ a sequence $\{\varkappa_i\}$ of chains mod k with the diameter of simplexes convergent to 0 such that $\partial\varkappa_i=\gamma_i$ for $i=1,2,\ldots$ Similarly there exists in $P_m\times S_1$ a sequence $\{\lambda_i\}$ of chains mod k with the diameter of simplexes convergent to 0 such that $\partial\lambda_i=\gamma_i$ for $i=1,2,\ldots$ It follows that putting

$$\gamma_i^*=\varkappa_i-\lambda_i\qquad\text{for every } i=1,2,\ldots$$

we obtain a 3-dimensional true cycle mod k in $P_m\times P_n$. If we observe that $\gamma\times\gamma$ is totally unhomologous to 0 in $S_1\times S_1$ and that $(P_m\times S_1)\cdot(S_1\times P_n)=S_1\times S_1$ we infer by the known theorem of Phragmen-Brouwer [23]) that the 3-dimensional true cycle $\{\gamma_i^*\}$ is

[21]) W. Hurewicz, *Über den sogenannten Produktsatz der Dimensionstheorie*, Math. Ann. **102** (1929), p. 306.

[22]) $\gamma_i\times\varkappa_i$ denotes the Cartesian product of the chains γ_i and $\varkappa_i$. See P. Alexandroff and H. Hopf, l. c., p. 302 and S. Lefschetz, *Algebraic Topology*, New York 1942, p. 138. Also K. Borsuk, *On the Decomposition of Manifolds into Products of Curves and Surfaces*, Fund. Math. **33** (1945), p. 280.

[23]) See P. Alexandroff, *Dimensionstheorie. Ein Beitrag zur Geometrie der abgeschlossenen Mengen*, Math. Ann. **106** (1932), p. 186. Also K. Borsuk, *Über sphäroidale und H-sphäroidale Räume*, Recueil Mathématique I (43), Moscou 1936, p. 646.

totally unhomologous to 0 in $P_m \times S_1 + S_1 \times P_n$. On the other hand, $\{\gamma_i^*\}$ is evidently homologous to 0 in $P_m \times P_n$. Hence $\dim(P_m \times P_n) \geqslant 4$. The inverse inequality is also true, because [24] $\dim P_m \times P_n \leqslant \dim P_m + \dim P_n = 4$. Consequently

(18) *If* $(m,n) > 1$ *then* $\dim P_m \times P_n = 4$.

9. Let M be an n-dimensional pseudo-manifold with the boundary $N \neq 0$. If τ denotes a triangulation of M, then the $(n-1)$-dimensional chain mod 2 consisting of all $(n-1)$-dimensional simplexes lying on N with coefficients equal to 1 is a cycle mod 2 homologous to zero on M and not homologous to zero on N. Evidently the last two properties characterize this cycle among all $(n-1)$-dimensional cycles mod 2 on N [25]. It follows that if γ is an $(n-1)$-dimensional true cycle mod 2 on N homologous to zero on M but totally unhomologous to zero on N then γ is homologous on N to the true cycle $\overline{\gamma} = \{\overline{\gamma}_i\}$ in which $\overline{\gamma}_i$ denotes the $(n-1)$-dimensional cycle mod 2 consisting of all $(n-1)$-dimensional simplexes of the i-th barycentric subdivision of an arbitrarily given triangulation of the polytope N with the coefficients equal to 1.

Lemma. *Let N be the boundary of an n-dimensional pseudo-manifold M and let A be a closed proper subset of M. Then every continuous mapping f of N into the $(n-1)$-dimensional sphere S_{n-1} has a continuous extension over $A+N$.*

Proof. This statement is a simple consequence of Hopf's well known extension theorem [26]. But it is also easy to give an elementary proof of it. It suffices to observe that for every point $a \in M - N - A$ there exists a mapping $r(x)$ retracting $M - (a)$ to a closed subset E of M such that

$$N \subset E \quad \text{and} \quad \dim E = n-1.$$

The last condition implies that f has a continuous extension $\bar{f}$ on E with values belonging to S_{n-1}. Putting

$$f^*(x) = \bar{f}r(x) \quad \text{for every} \quad x \in A+N$$

we obtain the required extension of f over $A+N$.

[24] W. Hurewicz and H. Wallman, l. c., p. 33.
[25] See, for instance, H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Chalsea Publishing Company, New York 1947, p. 91.
[26] See, for instance, W. Hurewicz and H. Wallman, l. c., p. 147.

10. We shall say that a compactum C is an *approximative n-pseudo-manifold* with the boundary C_0 if the following conditions are satisfied:

1^0 *C_0 is a closed subset of C.*

2^0 *There exists in C_0 an $(n-1)$-dimensional true cycle γ mod 2 homologous to zero on C, but totally unhomologous to zero on C_0.*

3^0 *For every $\varepsilon > 0$ there exists an ε-mapping* [27] *f_ε of C onto an n-dimensional pseudo-manifold M such that $f_\varepsilon(C_0)$ is a subset of the boundary N of M.*

By a general theorem [28] the condition 3^0 is equivalent to the condition

$3^{0\prime}$ *For every $\varepsilon > 0$ there exists an ε-deformation* [29] *f_ε of C onto an n-dimensional curvilinear pseudo-manifold M such that $f_\varepsilon(C_0)$ is a subset of the boundary N of M.*

Examples. Every pseudo-manifold (with boundary) is an approximative pseudo-manifold, but the reciprocal assertion is not true. Namely let us observe that

(19) *P_2 is an approximative 2-pseudo-manifold with the boundary S_1.*

In fact condition 2^0 follows by (11), and condition $3^{0\prime}$ is a consequence of (5) and (10).

11. Let us consider some elementary properties of the approximative pseudo-manifolds:

(20) *If C is an approximative n-pseudo-manifold then* $\dim C = n$.

Proof. By the condition $3^{0\prime}$ an approximative n-pseudo-manifold C is ε-deformable (for every $\varepsilon > 0$) into an n-dimensional polytope. Hence [30] $\dim C \leqslant n$. On the other hand the existence in C_0 of an $(n-1)$-dimensional true cycle γ homologous to zero in C, but totally unhomologous to zero in C_0 (assured by the condition 2^0) implies that the dimension of C is $\geqslant n$. Hence (20) is true.

[27] We say that f_ε is a *ε-mapping* of C if f_ε is continuous and the inverse image of every point $y \in f_\varepsilon(C)$ has a diameter less than ε.
[28] C. Kuratowski, *Topologie II*, Warszawa-Wrocław 1950, p. 18.
[29] We say that f_ε is an *ε-deformation* if f_ε is continuous and for every point $x \in C$ it is $\varrho(x, f_\varepsilon(x)) < \varepsilon$.
[30] See footnote [15].

(21) *If C_0 is the boundary of an approximative n-pseudo-manifold C and A is a closed proper subset of C, then every continuous mapping φ of C_0 into the $(n-1)$-dimensional sphere S_{n-1} has a continuous extension over $A+C_0$.*

Proof. It is known [31]) that there exists a neighbourhood U of C_0 (in the Hilbert space [32])) and a continuous extension $\overline{\varphi}$ of φ over U with values belonging to S_{n-1}. Let f_ε denote the ε-deformation considered in condition $3^{0\prime}$. If ε is sufficiently small then

$$f_\varepsilon(C_0)\subset U \quad \text{and} \quad f_\varepsilon(A) \subsetneqq M.$$

Consider the mapping $\overline{\varphi}$ only in the set $f_\varepsilon(C_0)$. By the lemma of Nr **9** there exists a continuous extension ψ of $\overline{\varphi}$ over the set $f_\varepsilon(A)$ with values belonging to S_{n-1}. Let us put

$$\varphi_\varepsilon(x)=\overline{\varphi}f_\varepsilon(x) \quad \text{for every} \quad x\in C_0,$$
$$\psi_\varepsilon(x)=\psi f_\varepsilon(x) \quad \text{for every} \quad x\in A+C_0.$$

We see at once that for ε sufficiently small the mapping φ_ε is on C_0 arbitrarily near to the mapping φ and ψ_ε constitutes a con tinuous extension of φ_ε over the set $A+C_0$. It follows [33]) that the mapping φ also has a continuous extension over $A+C_0$ with values belonging to S_{n-1}.

(22) *Every approximative pseudo-manifold is a Cantor-manifold.*

Proof. Let C be an approximative n-pseudo-manifold with the boundary C_0 and let γ denote the $(n-1)$-dimensional true cycle satisfying the condition 2^0. Let f_ε be an ε-deformation of C satisfying the condition $3^{0\prime}$. If ε is sufficiently small then f_ε carries γ into an $(n-1)$-dimensional true cycle $\gamma_\varepsilon=\{\gamma_{\varepsilon,i}\}$ of N homologous to zero on M, but totally unhomologous to zero on N. As we have observed in Nr **9**, the true cycle γ_ε is homologous in N to the true cycle $\overline{\gamma}=\{\overline{\gamma}_i\}$ in which $\overline{\gamma}_i$ denotes the cycle mod 2 consisting of all $(n-1)$-dimensional simplexes of the i-th barycentric subdivision of an arbitrarily given triangulation of the polytope N with coefficients equal to 1.

[31]) See, for instance, W. Hurewicz and H. Wallman, l. c., p. 82.

[32]) See footnote [1]).

[33]) See K. Borsuk, *Sur un espace des transformations continues et ses applications topologiques*, Monatsh. f. Math. u. Phys. **38** (1931), p. 382.

By Hopf's extension theorem [34]) there exists a continuous mapping φ of N into S_{n-1} carrying the true cycle $\overline{\gamma}$ into a true cycle $\overline{\gamma}_\varphi$ totally unhomologous to zero in S_{n-1}. The mapping φf_ε transforms C_0 into S_{n-1} in such a manner that the true cycle γ is carried by it onto a true cycle γ_φ homologous in S_{n-1} to the true cycle $\overline{\gamma}_\varphi$. We infer that φf_ε is not extendable over C (with respect to S_{n-1}). But, by (21), the mapping φf_ε is extendable over every closed proper subset of C containing C_0. It follows [35]) that C is an n-dimensional Cantor-manifold.

Remark. We can easily see that not every n-dimensional Cantor-manifold is an approximative pseudo-manifold. For instance the continuum composed of 3 segments having one end-point in common is evidently a 1-dimensional Cantor-manifold, but it is not an approximative pseudo-manifold.

12. ***Theorem 2.*** *If C is an approximative n-pseudo-manifold with the boundary C_0 and C' is an approximative n'-pseudo-manifold with the boundary C_0', then $D=C\times C'$ is an approximative $(n+n')$-pseudo-manifold with the boundary $D_0=C_0\times C'+C'\times C_0'$.*

Proof. Let γ be an $(n-1)$-dimensional true cycle mod 2 in C_0 satisfying the condition 2^0 and let γ' be an $(n'-1)$-dimensional true cycle satisfying the analogous condition for C_0' and C'. Then there exists a sequence $\{\varepsilon_i\}$ of positive numbers convergent to zero and two sequences $\varkappa=\{\varkappa_i\}$ composed of n-dimensional ε_i-chains $\varkappa_i$ mod 2 in C such that $\partial\varkappa_i=\gamma_i$, and $\varkappa'=\{\varkappa_i'\}$ composed of n'-dimensional ε_i-chains $\varkappa_i'$ mod 2 in C' such that $\partial\varkappa_i'=\gamma_i'$. Putting

$$\chi=\gamma\times\varkappa'+\varkappa\times\gamma'=\{\gamma_i\times\varkappa_i'+\varkappa_i\times\gamma_i'\}$$

we obtain an $(n+n'-1)$-dimensional true cycle mod 2 in D_0.

Let f_ε denote an ε-deformation of C into an n-dimensional curvilinear pseudo-manifold M with the boundary N satisfying condition $3^{0\prime}$ and f_ε' and ε-deformation of C' onto an n'-dimensional curvilinear pseudo-manifold M_n' with the boundary N' satisfying a condition analogous to $3^{0\prime}$. Putting

$$g_\varepsilon(x,y)=(f_\varepsilon(x), f_\varepsilon'(y)) \quad \text{for every} \quad (x,y)\in D$$

[34]) See footnote [26]).

[35]) See footnote [17]).

we obtain an $\varepsilon\cdot\sqrt{2}$-deformation of D into the $(n+n')$-dimensional curvilinear pseudo-manifold $M\times M'$. Evidently g_ε maps the set D_0 into the boundary $M\times N'+N\times M'$ of the pseudo-manifold $M\times M'$. Thus the conditions 1^0 and $3^{0'}$ are satisfied.

The true cycle χ bounds in D the infinite chain

$$\varkappa\times\varkappa'=\{\varkappa_i\times\varkappa_i'\}.$$

Hence our theorem will be proved if we show that the true cycle χ is totally unhomologous to zero in D_0.

Suppose, on the contrary that χ is not totally unhomologous to zero in D_0. Then there exists an increasing sequence of indices $\{i_\nu\}$ such that the true cycle

$$\chi'=\{\gamma_{i_\nu}\times\varkappa'_{i_\nu}+\varkappa_{i_\nu}\times\gamma'_{i_\nu}\}$$

is homologous to zero in D_0. This means that there exists a sequence $\{\lambda_\nu\}$ such that λ_ν is an $(n+n')$-dimensional chain mod 2 in D_0 with the diameter of simplexes $\leqslant\eta_\nu$, where $\eta_\nu\to 0$ and with

$$\partial\lambda_\nu=\gamma_{i_\nu}\times\varkappa'_{i_\nu}+\varkappa_{i_\nu}\times\gamma'_{i_\nu}.$$

Applying a suitable dislocation of vertices of λ_ν, we may assume that every simplex of λ_ν either lies in one of the sets $C_0\times C'$ and $C\times C_0'$ or is disjoint with it. Let $\bar\lambda_\nu$ denote the chain mod 2 composed by all simplexes of λ_ν lying in $C_0\times C'$, and let $\bar\lambda'_\nu=\lambda_\nu+\bar\lambda_\nu$. Then

$$\lambda_\nu=\bar\lambda_\nu+\bar\lambda'_\nu \quad\text{and}\quad \partial\lambda_\nu=\gamma_{i_\nu}\times\varkappa'_{i_\nu}+\varkappa_{i_\nu}\times\gamma'_{i_\nu}=\partial\bar\lambda_\nu+\partial\bar\lambda'_\nu.$$

It follows that

$$\partial\bar\lambda_\nu+\gamma_{i_\nu}\times\varkappa'_{i_\nu}=\partial\bar\lambda'_\nu+\varkappa_{i_\nu}\times\gamma'_{i_\nu}.$$

But the chain on the left side lies in $C_0\times C'$ and the chain on the right side lies in $C\times C_0'$. Consequently every one of them lies in $(C_0\times C')\cdot(C\times C_0')=C_0\times C_0'$ and

$$\partial(\partial\bar\lambda_\nu+\gamma_{i_\nu}\times\varkappa'_{i_\nu})=\gamma_{i_\nu}\times\partial\varkappa'_{i_\nu}=\gamma_{i_\nu}\times\gamma'_{i_\nu}.$$

Hence $\gamma_{i_\nu}\times\gamma'_{i_\nu}$ is η_i-homologous to zero in $C_0\times C_0'$, that is the true cycle $\{\gamma_{i_\nu}\times\gamma'_{i_\nu}\}$ is homologous to zero in $C_0\times C_0'$. But this is impossible, because the suppositions that $\{\gamma_i\}$ is totally unhomologous to zero in C_0 and $\{\gamma_i'\}$ totally unhomologous to zero in C_0' imply[36]) that $\{\gamma_i\times\gamma_i'\}$ is totally unhomologous to zero in $C_0\times C_0'$.

[36]) See for instance K. Borsuk, *On the Decomposition of Manifolds into Products of Curves and Surfaces*, Fund. Math. **33** (1945), p. 282.

Corollary 1. *For approximative pseudo-manifolds the formula* $\dim(X\times Y)=\dim X+\dim Y$ *holds.*

Proof. This is an application of theorem 2 and of (20).

Corollary 2. *If C is an approximative n-pseudo-manifold and K a polytope which is an m-dimensional Cantor-manifold, then $C\times K$ is an $(n+m)$-dimensional Cantor-manifold.*

Proof. The m-dimensional simplexes of an arbitrarily given triangulation of K may be ordered in a finite sequence $\Delta_1,\Delta_2,\dots,\Delta_k$ such that every one of the sets

$$\Delta_{i+1}\cdot(\Delta_1+\Delta_2+\dots+\Delta_i)\quad\text{for}\quad i=1,2,\dots,(k-1)$$

contains an $(m-1)$-dimensional simplex Δ_i^*. Let us put

$$K_i=\Delta_1+\Delta_2+\dots+\Delta_i\quad\text{for}\quad i=1,2,\dots,k.$$

By theorem 2 the set $C\times K_1=C\times\Delta_1$ is an approximative $(n+m)$-pseudo-manifold. It follows by (20) and (22) that $C\times K_1$ is an $(n+m)$-dimensional Cantor-manifold. Let us assume that for an $i<k-1$ the set $C\times K_i$ is an $(n+m)$-dimensional Cantor-manifold. Applying theorem 2 to the sets C and Δ_{i+1} and to the sets C and Δ_i^* we infer by (1) that $C\times K_{i+1}$ is an $(n+m)$-dimensional Cantor-manifold. This proves that the set $C\times K_k=C\times K$ is also an $(n+m)$-dimensional Cantor-manifold.

13. ***Theorem 3.*** *If E is a compactum of dimension $\leqslant n$ and there exists a sequence $\{E_k\}$ of n-dimensional Cantor-manifolds such that*

$$E_k\subset E,\quad \lim_{k=\infty}E_k=E,$$

then E is an n-dimensional Cantor-manifold.

Proof. Consider a decomposition of E into two closed proper subsets E' and E''. Then there exist two points $a'\in E'-E''$ and $a''\in E''-E'$. Let ε be a positive number such that

$$\varepsilon<\operatorname{Min}(\varrho(a',E''),\varrho(a'',E')).$$

By our hypothesis, for k sufficiently large they are

$$\varrho(a',E_k)<\varepsilon\quad\text{and}\quad\varrho(a'',E_k)<\varepsilon.$$

It follows that there exist two points $b'\in E_k-E''$ and $b''\in E_k-E'$. Evidently the set $E_k\cdot E'\cdot E''$ cuts E_k between b' and b''. But E_k is an n-dimensional Cantor-manifold. Consequently $\dim(E_k\cdot E'\cdot E'')\geqslant n-1$ and also $\dim(E'\cdot E'')\geqslant n-1$. Hence E is an n-dimensional Cantor-manifold.

Corollary 1. *If C is an approximative n-pseudo-manifold and E an arcwise connected curve, then $C\times E$ is an $(n+1)$-dimensional Cantor-manifold.*

Proof. Let $\{a_k\}$ be a countable dense subset of E. Because of the arcwise connectedness of E there exists a sequence of curvilinear 1-dimensional connected polytopes $\{E_k\}$ such that $E_k\subset E$ and $a_i\in E_k$ for every $i=1,2,...,k$. Then $\lim_{k=\infty} E_k=E$ and also $\lim_{k=\infty}(C\times E_k)=C\times E$. Moreover it is $\dim C\times E\leqslant n+1$. But corollary 2 of Nr **12** implies that the sets $C\times E_k$ are $(n+1)$-dimensional Cantor-manifolds. Hence our statement is a consequence of theorem 3.

Corollary 2. *The set $P_2\times P_3$ is a 3-dimensional Cantor-manifold.*

Proof. By (17) it is $\dim P_2\times P_3=3$. Let $\{a_k\}$ be a countable dense subset of P_3. By (9) there exists a sequence of curvilinear 1-dimensional connected polytopes $\{E_k\}$ such that $E_k\subset P_3$ and $a_i\in E_k$ for every $i=1,2,...,k$. Then $\lim_{k=\infty} E_k=P_3$ and also $\lim_{k=\infty}(P_2\times E_k)=P_2\times P_3$. But (19) and the corollary 2 of Nr **12** imply that the sets $P_2\times E_k$ are 3-dimensional Cantor-manifolds. Applying theorem 3 we obtain the required statement.

Corollary 3. *The set $P_2\times P_2\times P_3\times P_3$ is a 6-dimensional Cantor-manifold.*

Proof. The set $P_2\times P_2\times P_3\times P_3$ is homeomorphic to the set $(P_2\times P_3)\times(P_2\times P_3)$. But $\dim(P_2\times P_3)=3$, hence[37])

$$\dim(P_2\times P_2\times P_3\times P_3)\leqslant 6.$$

Consider now, as in the proof of corollary 2, a sequence $\{E_k\}$ of curvilinear 1-dimensional connected polytopes such that $E_k\subset P_3$ and $\lim_{k=\infty} E_k=P_3$. It follows that the curvilinear 2-dimensional connected polytopes $E_k\times E_k$ are 2-dimensional Cantor-manifolds such that $\lim_{k=\infty}(E_k\times E_k)=P_3\times P_3$ and $\lim_{k=\infty}(P_2\times P_2\times E_k\times E_k)=P_2\times P_2\times P_3\times P_3$. Applying (19), theorem 2 and corollary 2 of Nr **12** we conclude that the sets $P_2\times P_2\times E_k\times E_k$ are 6-dimensional Cantor-manifolds. We infer, by theorem 3, that $P_2\times P_2\times P_3\times P_3$ is also a 6-dimensional Cantor-manifold.

[37]) See footnote [24]).

14. Now it is easy to finish the proof of theorem 1. As we have observed in Nr **2** it remains to give the three examples enumerated there:

Example 1. Let Q be a 2-dimensional element such that the part common to Q and P_3 is a simple arc. Putting

$$A_1=P_2 \quad\text{and}\quad B_1=P_3+Q$$

we obtain, by (15), (8) and (1) two 2-dimensional locally connected Cantor-manifolds. But the Cartesian product

$$C_1=A_1\times B_1=P_2\times P_3+P_2\times Q$$

is not a Cantor-manifold because it is 3-dimensional at every point of $P_2\times(P_3-Q)$ and is 4-dimensional in every point of $P_2\times(Q-P_3)$.

Example 2. Let L be a simple arc such that $L\cdot P_3$ contains only one point. Putting

$$A_2=P_2 \quad\text{and}\quad B_2=L+P_3$$

we obtain by (8) two locally connected continua such that A_2 is by (15) a 2-dimensional Cantor-manifold and B_2 is not a Cantor-manifold, because it is 1-dimensional at every point of $L-P_3$ and 2-dimensional at every point of P_3-L. The Cartesian product

$$C_2=A_2\times B_2=P_2\times L+P_2\times P_3$$

is however a 3-dimensional Cantor-manifold, because $P_2\times L$ is by corollary 2 of Nr **12**, a 3-dimensional Cantor-manifold, $P_2\times P_3$ is, by corollary 2 of Nr **13**, a 3-dimensional Cantor-manifold and the set $(P_2\times L)\cdot(P_2\times P_3)=P_2\times(L\cdot P_3)$, as homeomorphic to P_2, is 2-dimensional.

Example 3. Besides the surfaces of Pontrjagin P_2 and P_3, consider two others, copies of analogous surfaces P_2' and P_3' such that every one of the sets $P_i\cdot P_i'$ is a simple arc L_i, for $i=2,3$. Putting

$$A_3=(P_2\times P_2)+(P_2'\times P_2'),$$
$$B_3=(P_3\times P_3)+(P_3'\times P_3')$$

we obtain two locally connected 4-dimensional continua which are not Cantorian-manifolds, because the common part of the sets $P_i\times P_i$ and $P_i'\times P_i'$ where $i=2,3$, is equal to $L_i\times L_i$ hence it is 2-dimensional. But the Cartesian product

$$C_3=A_3\times B_3=(P_2\times P_2)\times(P_3\times P_3)+(P_2\times P_2)\times(P_3'\times P_3')+$$
$$+(P_2'\times P_2')\times(P_3\times P_3)+(P_2'\times P_2')\times(P_3'\times P_3')$$

is a 6-dimensional Cantor-manifold. In fact by corollary 3 of Nr **13**, every one of the four summands is a 6-dimensional Cantor-manifold and the common part of two successive summands is the Cartesian product of the 4-dimensional (by (18)) set homeomorphic to $P_2 \times P_2$ or to $P_3 \times P_3$ and of a 2-dimensional element, hence [38]) it is also 6-dimensional.

15. ***Problems.*** Is the Cartesian product of an n-dimensional Cantor-manifold and a 1-dimensional continuum always an $(n+1)$-dimensional Cantor-manifold?

Is the Cartesian product of two locally contractible Cantor-manifolds always a Cantor-manifold?

If $A \times B$ is a locally contractible Cantor-manifold is it true that A and B are also Cantor-manifolds?

If $A \times B$ is an approximative pseudo-manifold is it true that A and B are also approximative pseudo-manifolds?

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[38]) See footnote [31]).

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Measures in Fully Normal Spaces.

By

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The present note contains two decomposition theorems concerning Borel measures in fully normal (i. e. paracompact) spaces. These theorems are closely related to the results of E. Marczewski and R. Sikorski [5] on Borel measures in metric spaces. The third theorem, proved by similar methods, asserts that every fully normal space is a Q-space, in the sense of E. Hewitt [2], unless some of its closed discrete subspaces are not so. It may be noticed that it is possible to deduce this result from the decomposition theorems of the present note and E. Hewitt's results [1]) concerning measures in Q-spaces.

All spaces considered are completely regular [2]) topological spaces.

The following notations are used: if P is a space, then $\boldsymbol{F}(P)$, $\boldsymbol{G}(P)$, $\boldsymbol{F}^*(P)$, $\boldsymbol{G}^*(P)$ denote, respectively, the family of all closed sets, the family of all open sets, the family of all sets of the form $f^{-1}(M)$, f continuous real-valued, M closed (or, equivalently, of the form $f^{-1}(0)$, f continuous real-valued), and the family of complements of sets from $\boldsymbol{F}^*(P)$. The meaning of $\boldsymbol{F}_\sigma(P)$, $\boldsymbol{F}^*_\sigma(P)$, $\boldsymbol{G}_\delta(P)$, $\boldsymbol{G}^*_\delta(P)$ is clear. $\boldsymbol{B}(P)$ or $\boldsymbol{B}^*(P)$ denotes the least σ-field containing $\boldsymbol{F}(P)$ or $\boldsymbol{F}^*(P)$ respectively. The sets belonging to $\boldsymbol{B}(P)$ will be called *Borel sets* (relative to P); those belonging to $\boldsymbol{B}^*(P)$ will be called *Baire sets* (relative to P).

Clearly, we always have $\boldsymbol{B}^*(P) \subset \boldsymbol{B}(P)$. If P is perfectly normal [3]), then $\boldsymbol{F}^*(P) = \boldsymbol{F}(P)$ (see e. g. [9]) and therefore $\boldsymbol{B}^*(P) = \boldsymbol{B}(P)$.

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[1]) See [2a], Theorem 16.

[2]) A Hausdorff space P is called *completely regular* if, for any closed set $A \subset P$ and any $x \epsilon P - A$, there exists a real-valued continuous function f in P such that $f(x) = 1$, $f(A) = 0$.

[3]) A normal space P is called *perfectly normal* if $\boldsymbol{F}(P) \subset \boldsymbol{G}_\delta(P)$.