that $a_{\nu(\xi)} < a_{\tau(0)}$ for $\nu(\xi) \geqslant \tau(1)$. By induction we can define an ω sequence, $\{a_{\tau(n)}\}$, with $a_{\tau(n)} < a_{\tau(n-1)}$, a well known impossibility. Thus $\lambda < \omega_\varrho$.

If $\sum\limits_{\xi<\omega_\varrho} a_\xi = k_1 + \omega^\delta$, then for some φ_0,

$$\sum_{\xi\geqslant\varphi\geqslant\varphi_0} a_\xi = \omega^\delta.$$

Let $\{b_\xi\}_{\xi<\omega_\varrho}$ be any permutation of the elements of $\{a_\xi\}$. As ω_ϱ is regular, we are able to repeat the procedure given in [1] and find a θ_0 so that

$$\sum_{\xi\geqslant\theta\geqslant\theta_0} b_\xi = \omega^\delta.$$

Thus the value of $\sum\limits_{\xi<\omega_\varrho} b_\xi$ is determined by calculating the value of $\sum\limits_{\xi<\theta_0} b_\xi$. For each θ_0 there are $\aleph_\varrho^{\overline{\overline{\theta_0}}}$ lifferent subsets of $\{a_\xi\}$ of power $\overline{\overline{\theta_0}}$ [2], and $\overline{\overline{\theta_0}}^{\overline{\overline{\theta_0}}}$ permutations of the elements of each set. Hence

$$N\Big(\sum_{\xi<\omega_\varrho} a_\xi\Big) \leqslant \sum_{\theta_0<\omega_\varrho} \aleph_\varrho^{\overline{\overline{\theta_0}}}\, \overline{\overline{\theta_0}}^{\overline{\overline{\theta_0}}} = \aleph_\varrho^{\aleph_\varrho}.$$

Suppose $\aleph_\varrho = \aleph_{\alpha+1}$. Let $a_\xi = \omega_{\alpha+1}$ for $\xi < \omega_\alpha$, and $a_\xi = 1$ for $\xi \geqslant \omega_\alpha$. Then $N(\Sigma a_\xi) = \aleph_{\alpha+1}$ and the theorem is proved.

Theorem 2. *If ω_ϱ is a regular ordinal and $\{a_\xi\}_{\xi<\omega_\varrho}$ is a non-decreasing transfinite sequence of ordinals, then $N(\Sigma a_\xi) = 1$.*

Proof. Let ω_ϱ be regular and $\{a_\xi\}$ non-decreasing. For $\{b_\xi\}_{\xi<\omega_\varrho}$, a permutation of the elements of $\{a_\xi\}$, let $b_{\tau(0)} = b_0 \geqslant a_0$ and suppose defined $\{b_{\tau(\xi)}\}_{\xi<\theta<\omega_\varrho}$. Let λ_θ be the smallest ordinal λ, $\lambda \geqslant \tau(\xi)$, $\xi < \theta$. Since ω_ϱ is regular, and since there are $\aleph_\varrho$ elements a_μ, $a_\mu \geqslant a_\theta$, we can find a $b_{\tau(\theta)} \geqslant a_\theta$, with $\tau(\theta) \geqslant \lambda_\theta$. Therefore

$$\Sigma a_\xi \leqslant \Sigma b_{\tau(\xi)} \leqslant \Sigma b_\xi.$$

A similar procedure yields $\Sigma b_\xi \leqslant \Sigma a_\xi$, so that the equality sign holds.

In conclusion we remark that if λ is a non-regular limit number, then there exists an increasing sequence $\{a_\xi\}_{\xi<\lambda}$ and a permutation $\{b_\xi\}_{\xi<\lambda}$ such that $\Sigma a_\xi < \Sigma b_\xi$.

Bibliography.

[1] W. Sierpiński, *Sur les séries infinies de nombres ordinaux*, Fund. Math. **36** (1949), p. 248-253.

[2] A. Tarski, *Sur les classes d'ensembles closes par rapport à certaines opérations élémentaires*, Fund. Math. **16** (1930), p. 181-304.

On models of axiomatic systems.

By

A. Mostowski (Warszawa).

This paper is devoted to a discussion of various notions of models which appear in the recent investigations of formal systems. The discussion will be applied to the study of the following problem: Given a formal system S based on an infinite number of axioms $A_1, A_2, A_3, \ldots$ is it possible to prove in S the consistency of the system based on a finite number $A_1, A_2, \ldots, A_n$ of these axioms?

1. Notations and definitions. We shall consider two systems S and s based on the functional calculus of the first order[1]). We shall not describe these systems in detail but give only some definitions which will be required later.

System s. We assume that the following symbols occur among the primitive signs of s:

1. Variables: $x_1, x_2, x_3, \ldots, x, y, z, \ldots$
2. Individual constants: $f_1, \ldots, f_\alpha$.
3. Functors (i. e. symbols for functions from individuals to individuals): $g_1, \ldots, g_\beta$. We denote by q_j the number of arguments of g_j $(j=1,\ldots,\beta)$.
4. Predicates (i. e. symbols for relations): $r_1, \ldots, r_\gamma$. We denote by p_i the number of arguments of r_i $(i=1,\ldots,\gamma)$.
5. Propositional connectives and quantifiers. We use the symbol $|$ for the "stroke function" and define other connectives in terms of the stroke. Quantifiers are denoted by symbols $(\exists x_i)$ and (x_i).

Among expressions which can be constructed from these signs we distinguish the following:

6. Terms. Variables and individual constants are terms. If $\Gamma_1, \ldots, \Gamma_{q_j}$ are terms, then so is $g_j(\Gamma_1, \ldots, \Gamma_{q_j})$, $j=1,\ldots,\beta$.

Terms will be denoted by the letters $\Gamma, \Gamma_1, \Gamma_2, \ldots$

[1]) For the functional calculus of the first order see *e. g.* Church [1], Chapter II.

7. Elementary formulas are expressions of the form $r_i(\Gamma_1,\ldots,\Gamma_{p_i})$ where $\Gamma_1,\ldots,\Gamma_{p_i}$ are terms. Elementary formulas will be denoted by the letters $E, E_1, E_2, \ldots$

8. Prime formulas are elementary formulas in which no variables occur. Prime formulas will be denoted by the letters $P, P_1, P_2, \ldots$

9. Matrices. Elementary formulas are matrices. If M_1 and M_2 are matrices, then so are $M_1|M_2$ and $(\exists x_i)M_1$ for $i=1,2,\ldots$ Matrices will be denoted by the letters $M, M_1, M_2, \ldots$

Note that matrices containing propositional connectives other than the stroke are easily definable by means of matrices containing only the stroke symbol. We shall occasionally use the following abbreviations:

$$M^0 \text{ for } M,$$
$$M^1 \text{ for } \sim M,$$
$$\sum_{i=1}^{\omega} M_i \text{ for } M_1 \vee \ldots \vee M_\omega,$$
$$\prod_{i=1}^{\omega} M_i \text{ for } M_1 \ldots M_\omega.$$

10. Free and bound variables. Substitution. The distinction between free and bound variables is assumed as known. The formula which results from a formula A by the substitution of the terms $\Gamma_1,\ldots,\Gamma_n$ for the variables $x_1,\ldots,x_n$ will be denoted by

$$Subst\ A(x_1/\Gamma_1,\ldots,x_n/\Gamma_n).$$

The operation of substitution is always performable when A is a term. If A is a matrix, it is sometimes necessary to re-name the bound variables occurring in A in order to make sure that the operation *Subst* can be performed. We shall always assume that the necessary changes in the bound variables of A have been performed before the operation *Subst* has been applied [2]).

11. Q-matrices. These are matrices in which no bound variables occur.

12. Axioms. We assume that the axioms of s are finite in number and have the form of Q-matrices in which no constants, functors, or predicates occur besides those which were enumerated in 2, 3, and 4. The axioms will be denoted by

$$a_i \quad \text{or} \quad a_i(x,y,\ldots,z), \quad i=1,\ldots,\delta.$$

[2]) An exact definition of the operation *Subst* is given in Church [1], pp. 56-58.

13. The rules of proof admitted in s are the usual ones. We adjoin to them the rule of explicit definitions and the ε-rule [3]).

We shall add a few words to explain the ε-rule. To this end we define recursively the notions of *ε-terms* and *ε-matrices*.

The terms and matrices defined in 6 and 7 are ε-terms and ε-matrices. If $\Gamma, \Gamma_1, \ldots, \Gamma_k$ are ε-terms, then

$$Subst\ \Gamma(x_{i_1}/\Gamma_1,\ldots,x_{i_k}/\Gamma_k)$$

is an ε-term provided that (i) $x_{i_1},\ldots,x_{i_k}$ are the free variables of Γ; (ii) the bound variables of Γ are not free in $\Gamma_1,\ldots,\Gamma_k$. If $\Gamma_1,\ldots,\Gamma_{p_l}$ are ε-terms, then $r_l(\Gamma_1,\ldots,\Gamma_{p_l})$ is an ε-matrix ($l=1,\ldots,\gamma$). If M_1 and M_2 are ε-matrices, then so are $M_1|M_2$ and $(\exists x_i)M_1$. If M is an ε-matrix, then $(\varepsilon x_i)M$ is an ε-term.

The ε-rule states that for every ε-matrix M the matrix

$$M \supset Subst\ M(x_i/(\varepsilon x_i)M)$$

can be assumed as a theorem of s.

The question arises whether the assumptions concerning the form of axioms and rules of proof (cf. 12 and 13) are general enough to cover the cases of standard formal systems based on a finite number of axioms. The answer is affirmative. To see this we remark that the ε-rule enables us to get rid of quantifiers in the axioms provided that we introduce a sufficient number of ε-terms [4]). Since the explicit definitions are allowed in s, it follows that we can bring the axioms to the form of Q-matrices provided that we add a sufficient number of symbols to the symbols enumerated in 2 and 3. The resulting system then satisfies our assumptions and is equivalent to the given one provided that suitable definitions are introduced into the latter.

Example. Let one of the axioms have the form

(i) $$(x)(\exists y)M(x,y).$$

We introduce a new functor $g(x)$ and an axiom

(ii) $$M(x,g(x)).$$

Clearly (i) is deducible from (ii). Conversely (ii) can be obtained from (i) by means of the ε-rule and the explicit definition $g(x)=\varepsilon_y M(x,y)$.

[3]) The ordinary rules of proof for the functional calculus are given *e.g.* in Church [1], p. 40. For the ε-rule see Hilbert-Bernays [5], pp. 9-18.

[4]) See Hilbert-Bernays [5], pp. 16-17.

System S. The structure of S will be assumed similar to those of s. We shall, however, not assume that the number of axioms of S is finite. Furthermore we shall assume that S contains an arithmetic of integers to an extent which allows us to arithmetize in S the syntax of s and to prove in S the basic theorems of mathematical logic, *e. g.*, the completeness theorem of Gödel[5]). We shall use freely in S the arithmetical notions such as identity, sum, product etc.

14. Arithmetical counterparts of syntactical notions will be denoted by words printed in spaced italics. For instance *matrix* is short for a matrix of S with one free variable which is satisfied exclusively by the Gödel numbers of matrices of s. The actual construction of such a matrix is not needed; it suffices to know that it can be constructed.

To simplify the notation we shall often use in S the non-formal language and write, for example,

$$x \text{ is a } \textit{matrix} \text{ of } s$$

instead of *matrix* (x). Translations of such non-formal formulas to the official language of S will always be possible[6]).

The following constants will be used in S:

15. $\mathfrak{z}$ for the functor of S such that $\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \ldots$ are the *variables* of s. In other words $\mathfrak{z}_i$ is the Gödel number of the variable x_i $(i=1,2,\ldots)$.

16. $\mathfrak{f}_1, \ldots, \mathfrak{f}_\alpha$ for the *individual constants*, $\mathfrak{g}_1, \ldots, \mathfrak{g}_\beta$ for the *functors*, and $\mathfrak{r}_1, \ldots, \mathfrak{r}_\gamma$ for the *predicates* of s. In other words $\mathfrak{f}_i$ is the Gödel number of f_i, $\mathfrak{g}_j$ that of g_j, and $\mathfrak{r}_k$ that of r_k, $i=1,\ldots,\alpha$, $j=1,\ldots,\beta$, $k=1,\ldots,\gamma$.

17. Arbitrary *terms* will be denoted by letters $t, t_1, t_2, \ldots$, arbitrary *elementary formulas* by letters $e, e_1, e_2, \ldots$, arbitrary *prime formulas* by letters $p, p_1, p_2, \ldots$, and arbitrary *matrices* by letters $m, m_1, m_2, \ldots$

18. Let $\Gamma_1, \ldots, \Gamma_{q_j}$ be terms and $t_1, \ldots, t_{q_j}$ their Gödel numbers. The Gödel number of the term $g_j(\Gamma_1, \ldots, \Gamma_{q_j})$ will be denoted by $[\mathfrak{g}_j(t_1, \ldots, t_{q_j})]$. Thus the square brackets symbolize here a functor of S.

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[5]) Gödel [2].

[6]) More details concerning arithmetization are given in Gödel [3] and Hilbert-Bernays [5], § 4.

The square-brackets notation explained in the foregoing paragraph will be used consistently in many other similar situations. So *e. g.* if $m_{i,j}$ is the Gödel number of the matrix $M_{i,j}$, then $[\prod_{i=1}^{\omega} \sum_{j=1}^{\tau_i} m_{i,j}]$ is the Gödel number of the matrix $\prod_{i=1}^{\omega} \sum_{j=1}^{\tau_i} M_{i,j}$. If m is the Gödel number of M, then $[m^1]$ is the Gödel number of M^1 (i. e. of $\sim M$) and so on.

19. The following lemma is provable in S: In order that e be an *elementary formula* it is necessary and sufficient that e have the form $[\mathfrak{r}_i(t_1, \ldots, t_{p_i})]$ where $i \leqslant \gamma$ and $t_1, \ldots, t_{p_i}$ are *terms*. The integer i and *terms* $t_1, \ldots, t_{p_i}$ are determined by e. We put

$$i = Ind(e), \qquad t_j = Comp_j(e), \qquad j=1,\ldots,p_i.$$

Functors *Ind* and *Comp* are definable in S.

20. The arithmetical counterpart of the function *Subst* will be denoted by the symbol *Sb*. Thus if a is the Gödel number of an expression A, and $t_1, \ldots, t_n$ are the Gödel numbers of terms $\Gamma_1, \ldots, \Gamma_n$, then $Sb\, a(\mathfrak{z}_1/t_1, \ldots, \mathfrak{z}_n/t_n)$ is the Gödel number of the expression $Subst\, A(x_1/\Gamma_1, \ldots, x_n/\Gamma_n)$.

21. If $\Gamma, \Gamma_1, \ldots, \Gamma_{q_j}$ are terms of s, then the expressions

$$Subst\; g_j(\Gamma_1, \ldots, \Gamma_{q_j})(x_k/\Gamma)$$

and

$$g_j(Subst\; \Gamma_1(x_k/\Gamma), \ldots, Subst\; \Gamma_{q_j}(x_k/\Gamma))$$

are identical.

The equation

$$Sb[\mathfrak{g}_j(t_1, \ldots, t_{q_j})](\mathfrak{z}_k/t) = [\mathfrak{g}_j(Sb\; t_1(\mathfrak{z}_k/t), \ldots, Sb\; t_{q_j}(\mathfrak{z}_k/t))]$$

is provable in S.

22. The Gödel number of the i-th axiom of s (see 12) will be denoted by $[a_i]$ or by $[a_i(x, y, \ldots, z)]$.

23. Arithmetical sentences expressing the consistency of S and of s will be abbreviated as $\mathcal{N}(S)$ and $\mathcal{N}(s)$.

2. Models of the first kind. Let $R_0(x)$, $R_i(x_1, \ldots, x_{p_i})$ be $\gamma+1$ matrices of S with the indicated number of free variables and let $T_j(x_1, \ldots, x_{q_j}, y)$ be β matrices of S such that matrices

$$T_j(x_1, \ldots, x_{q_j}, y') \cdot T_j(x_1, \ldots, x_{q_j}, y'') \supset y' = y'',$$

$$(\exists y)\; T_j(x_1, \ldots, x_{q_j}, y)$$

are provable in S. We define in S functors G (where $j=1,2,\ldots,\beta$) in the following way [7])

$$G_j(x_1,\ldots,x_{q_j})=(\iota y)\ T_j(x_1,\ldots,x_{q_j},y).$$

Finally let $F_1,\ldots,F_\alpha$ be α constants definable in S.

The $\alpha+\beta+\gamma+1$ tuple consisting of α constants F_i, of β functors G_i, and of $\gamma+1$ matrices R_k will be called a *pseudo-model of the first kind of s in S*.

In order to define when a pseudo-model is a real model we shall introduce some auxiliary definitions.

To every term Γ of s we let correspond a term T_Γ of S in the following way. If Γ is a variable, then $T_\Gamma=\Gamma$. If $\Gamma=f_i$, then $T_\Gamma=F_i$. Finally, if Γ has the form $g_j(\Gamma_1,\ldots,\Gamma_{q_j})$, then we put $T_\Gamma=G_j(T_{\Gamma_1},\ldots,T_{\Gamma_{q_j}})$.

To every elementary formula $E=r_l(\Gamma_1,\ldots,\Gamma_{p_l})$ of s we let correspond the matrix $E'=R_l(T_{\Gamma_1},\ldots,T_{\Gamma_{p_l}})$ of S. We extend this definition to all Q-matrices of s by putting $(M_1|M_2)'=M_1'|M_2'$. In particular we put $A_i=a_i'$ $(i=1,\ldots,\delta)$.

Definition. A pseudo-model

$$F_1,\ldots,F_\alpha,\quad G_1,\ldots,G_\beta,\quad R_0,R_1,\ldots,R_\gamma \tag{1}$$

is a *real model of the first kind of s in S* if the formulas

$$R_0(x)\cdot R_0(y)\cdot\ldots\cdot R_0(z)\supset A_i(x,y,\ldots,z),\quad i=1,2,\ldots,\delta,$$

are provable in S.

Models of the first kind are the ones with which one has to do in the usual proofs of consistency and of independence of axiomatic systems [8]). For comparison with other notions of models to be defined later, we shall note the following general facts concerning models of the first kind:

1. The general notion of models of the first kind is defined not in S but in the syntax of S.

2. Every particular pseudo-model is a finite set of matrices of S and can therefore be defined within S. The problem whether it is or is not a real model can be formulated and in particular cases also solved in S.

[7]) (ιx) [...x...] denotes the x which satisfies the condition ...x... Cf. Hilbert-Bernays [5], p. 381.

[8]) Cf. the independence and consistency proofs in [4], Chapter II.

3. If s contained an infinite number of axioms (independent of whether their set is or is not definable in S), then the problem whether an explicitly given pseudo-model is or is not a real model of s in S would be expressible in the syntax of S but not in S itself.

The following theorems concerning models of the first kind are well known but are given here for the sake of comparison with other notions of models [9]):

I. *If (1) is a real model of the first kind of s in S and if a Q-matrix $A(x,y,\ldots,z)$ is provable in s, then the matrix*

$$R_0(x)\cdot R_0(y)\cdot\ldots\cdot R_0(z)\supset A'(x,y,\ldots,z)$$

is provable in S.

II. *If (1) is a real model of the first kind of s in S and S is consistent, then so is s.*

Let us now assume that a real model of the first kind of s has been explicitly defined in S.

Theorems I and II are provable in the syntax of S, hence they are translatable into arithmetic and therefore into S. Denoting by $N(S)$ and $N(s)$ formulas of S corresponding (via arithmetization) to the syntactic statement: S *(or s) is self-consistent* (cf. section 1, definition 23), we obtain from II:

III. *The formula $N(S)\supset N(s)$ is provable in S.*

In spite of this result models of the first kind are of no use when one is examining the problem whether the formula $N(s)$ itself is or is not provable in S. Models of the second and third kinds which we shall discuss in the next sections will allow us to answer this question in many particular cases.

We note still the following theorem due to Wang [16]:

IV. *If the formula $N(s)$ is provable in S, then a model of the first kind can be defined explicitly in S.*

Indeed, the usual proofs of the completeness theorem of Gödel consist in exhibiting a model of the first kind of a (non-contradictory) first order system in the arithmetic of integers [10]). Taking s as this system and repeating the argument of Gödel in S (which is possible by our assumptions concerning S, cf. p. 136) we obtain the proof of theorem IV [11]).

[9]) Proofs of these theorems may be found *e.g.* in my book [7], Chapter XI.

[10]) Gödel [2] or Hilbert-Bernays [5], p. 185.

[11]) Wang [16], p. 287, gives a more detailed proof of this theorem.

3. Models of the second kind in the axiomatic theory of sets [12]). We assume in this section that S is an axiomatic system of set theory based *e. g.* on Zermelo's axioms.

The following definitions are to be thought of as belonging to S.

Let $\mathcal{M}$ be an arbitrary set and Z a finite set of positive integers. An $\mathcal{M}$-function with the set of arguments Z is defined as a set F of ordered pairs $\langle u,v\rangle$ such that $v\in\mathcal{M}$, u runs over all finite sequences [13]) satisfying the conditions

$$D(u)=Z,\quad D^*(u)\subset\mathcal{M},$$

and the following condition of single-valuedness holds:

$$\text{if}\quad \langle u,v'\rangle\in F\quad\text{and}\quad \langle u,v''\rangle\in F,\quad\text{then}\quad v'=v''.$$

The symbols $D(u)$ and $D^*(u)$ denote the domain and the counter-domain of u, i. e.

$$x\in D(u)\equiv(\exists y)[\langle x,y\rangle\in u],$$
$$x\in D^*(u)\equiv(\exists y)[\langle y,x\rangle\in u].$$

If u is a sequence satisfying the condition $D(u)=Z$ and Y is an arbitrary set of positive integers, then we denote by $u|Y$ the sequence u restricted to Y, i. e.

$$\langle i,a\rangle\in u|Y\equiv(\langle i,y\rangle\in u)\cdot(i\in Y).$$

An $\mathcal{M}$-relation with the set of arguments Z is defined as a set R of sequences u such that $D(u)=Z$ and $D^*(u)\subset\mathcal{M}$.

If Z consists of integers $i,j,k,\ldots$ and u is a sequence with the domain Z such that $\langle i,a\rangle$, $\langle j,b\rangle$, $\langle k,c\rangle$, ... are elements of u, then instead of $u\in R$ we write $R(a,b,c,\ldots)$ and say that R holds for the elements $a,b,c,\ldots$ Similarly, if H is an $\mathcal{M}$-function, then instead of $\langle u,v\rangle\in H$ we write $H(u)=v$ or $H(a,b,c,\ldots)=v$.

Let F_i be elements of $\mathcal{M}$ $(i=1,\ldots,a)$, let G_j be $\mathcal{M}$-functions with the sets of arguments $Z_j=\{1,2,\ldots,q_j\}$, $j=1,\ldots,\beta$, and let R_k be $\mathcal{M}$-relations with the sets of arguments $Z'_k=\{1,2,\ldots,p_k\}$, $k=1,2,\ldots\gamma$. The $a+\beta+\gamma+1$ tuple

$$(2)\qquad \mathcal{M},\quad F_1,\ldots,F_a,\quad G_1,\ldots,G_\beta,\quad R_1,\ldots,R_\gamma$$

will be called a *pseudo-model of the second kind of s in S.*

[12]) Results of this and the next section are due to Tarski [11] and [12]. and to Wang [15] and [17].

[13]) Sequences are defined as functions (many-one relations) with domains contained in the set of positive integers. Cf. Tarski [11], p. 287.

We shall now explain when a pseudo-model is a real model. As in section 2 we need some auxiliary definitions.

We shall denote by $B(t)$ the set of *free variables* which *occur* in a *term* t and by $B(m)$ the set of *free variables* which *occur* in a *matrix* m of s.

With these definitions it is not difficult to prove the existence and uniqueness of a function $H_t(u)$ and a relation *Stsf* which are of fundamental importance in the investigations of the semantic of s. The exact definitions of the function H and the relation *Stsf* are given below in lemmas 1 and 2 together with the proofs of their existence and uniqueness. To facilitate our exposition we explain informally the intuitive meaning of these concepts.

Let t be a *term*, m a *matrix* of s, and let u be a sequence $\{\langle i,a\rangle,\langle j,b\rangle,\langle k,c\rangle,\ldots\}$ where $i,j,k,\ldots$ are the *free variables* of t or of m. Then $H_t(u)=H_t(a,b,c,\ldots)$ is what is usually called the value of t for the values $a,b,c,\ldots$ given respectively to the *free variables* of t. The relation u *Stsf* m holds if and only if the elements $a,b,c,\ldots$ satisfy the matrix m in the domain $\mathcal{M}$ of individuals.

Lemma 1. *There exists exactly one function $H(t,u)=H_t(u)$ such that*

1⁰ *t runs over terms of s;*

2⁰ *$H_t(u)$ considered as a function of u alone is an $\mathcal{M}$-function with the set of arguments $B(t)$;*

3⁰ *if t is a variable, then $H_t(\{\langle t,x\rangle\})=x$;*

4⁰ *if $t=\mathfrak{f}_i$, then $H_t(u)=F_i$, $i=1,\ldots,a$;*

5⁰ *if $t=[\mathfrak{g}_j(t_1,\ldots,t_{q_j})]$, then $H_t(u)=G_j(H_{t_1}(u_1),\ldots,H_{t_{q_j}}(u_{q_j}))$ where $u_n=u|B(t_n)$, $n=1,\ldots,q_j$, $j=1,\ldots,\beta$.*

Lemma 2. *There exists exactly one binary relation Stsf such that*

1⁰ *the counterdomain of Stsf consists of matrices of s;*

2⁰ *for a fixed matrix m the set $E_u[u \text{ Stsf } m]$ is an $\mathcal{M}$-relation with the set of arguments $B(m)$;*

3⁰ *if m is the elementary formula $[\mathfrak{r}_i(t_1,\ldots,t_{p_i})]$, then*

$$u\ \mathit{Stsf}\ m\equiv R_i(H_{t_1}(u_1),\ldots,H_{t_{p_i}}(u_{p_i}))$$

where $u_n=u|B(t_n)$ for $n=1,\ldots,p_i$, $i=1,\ldots,\gamma$;

4⁰ *u Stsf $(m_1|m_2)\equiv u_1$ non-Stsf m_1 or u_2 non-Stsf m_2 where $u_i=u|B(m_i)$ for $i=1,2$;*

5^0 *if* $m=[(\exists\mathfrak{z}_i)m_1]$ *and the* **variable** $\mathfrak{z}_i$ *is not free in* m_1, *then* u *Stsf* $m\equiv u$ *Stsf* m_1. *If however* $\mathfrak{z}_i$ *is free in* m_1, *then*

$$u \text{ Stsf } m \equiv \text{there is an element } a\in\mathfrak{M} \text{ such that } u+\{\langle\mathfrak{z}_i,a\rangle\} \text{ Stsf } m_1.$$

Note that both lemmas are provable in S.

Definition [14]. A pseudo-model (2) is a *real model of the second kind of s in S* if for every *axiom* $[a_i]$ of s and for every sequence u satisfying the conditions $D(u)=B([a_i])$ and $D^*(u)\subset\mathfrak{M}$ the following formula holds

$$u \text{ Stsf } [a_i], \quad i=1,\ldots,\delta.$$

Using this definition one can prove the following theorems:

V [15]. *If (2) is a real model of the second kind of s in S and if m is a* **matrix** *provable in s, then u Stsf m for an arbitrary sequence u satisfying the conditions* $D(u)=B(m)$ *and* $D^*(u)\subset\mathfrak{M}$.

VI [16]. *If at least one real model of the second kind of s in S exists, then* $N(s)$.

Theorem V and VI as well as all the previous definitions and theorems belong to the system S.

We now abandon S and pass to its syntax. We can then formulate the following statements concerning models of the second kind. These statements should be compared with statements 1-3 of section 2, pp. 138-139:

1. The general notion of models of the second kind is definable within S.

2. Every individual model of the second kind is an element of the universe of discourse of S.

3. Models of the second kind can also be defined in cases where the number of axioms of s is infinite and statements 1 and 2 above also remain valid.

From the circumstance that theorem VI has been proved in S we infer that the following theorem holds:

VII. *If the existence of a real model of the second kind of s is provable in S, then so is the formula N(s) expressing the consistency of s.*

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[14]) Cf. Tarski [12], p. 8.
[15]) Cf. Tarski [11], p. 317, theorem 5, and p. 358.
[16]) Cf. Tarski [11], p. 318, theorem 7, and p. 359.

Hence models of the second kind enable us to obtain absolute consistency proofs whereas models of the first kind yield merely relative consistency proofs.

We note finally that just as in section 2 we can derive from the completness theorem of Gödel the following theorem which is the converse of VII:

VIII. *If the sentence N(s) is provable in S, then so is the sentence stating the existence of at least one model of the second kind of s in S.*

4. Impossibility of a finite axiomatization of set-theory. Let us assume as in section 3 that S is an axiomatic system of set-theory and let s be a system based on a finite number of axioms of S. Since we assume the ε-rule both in S and in s, we can assume that the axioms of s contain no quantifiers and that S contains all functors occurring in the axioms of s.

From now on until the formulation of theorem IX we again assume that our discussion takes place in system S.

Let $\mathfrak{M}$ be an arbitrary non-void set such that

$$f_1,\ldots,f_\alpha\in\mathfrak{M},$$
$$\text{if } m_1,\ldots,m_{q_j}\in\mathfrak{M}, \text{ then } g_j(m_1,\ldots,m_{q_j})\in\mathfrak{M}, \qquad j=1,\ldots,\beta.$$

Put $F_1=f_1,\ldots,F_\alpha=f_\alpha$ and define the $\mathfrak{M}$-functions G_j $(j=1,\ldots,\beta)$ as sets of pairs

$$\langle\{\langle 1,m_1\rangle,\ldots,\langle q_j,m_{q_j}\rangle\},\, g_j(m_1,\ldots,m_{q_j})\rangle,$$

where $m_1,\ldots,m_{q_j}$ vary independently in $\mathfrak{M}$. Further let R_k $(k=1,\ldots,\gamma)$ be $\mathfrak{M}$-relations defined by the equivalence

$$\{\langle 1,m_1\rangle,\ldots,\langle p_k,m_{p_k}\rangle\}\in R_k\equiv r_k(m_1,\ldots,m_{p_k}).$$

The $\alpha+\beta+\gamma+1$ tuple

$$\mathfrak{M},\quad F_1,\ldots,F_\alpha,\quad G_1,\ldots,G_\beta,\quad R_1,\ldots,R_\gamma \tag{3}$$

constitutes a pseudo-model of the second kind of s in S.

To show that this pseudo-model is a real model we remark that if $[a_i]$ is an *axiom* of s with the **free variables** $\mathfrak{z}_1,\ldots,\mathfrak{z}_k$ and if u is any sequence $\{\langle\mathfrak{z}_1,u_1\rangle,\ldots,\langle\mathfrak{z}_k,u_k\rangle\}$ with $u_1,\ldots,u_k\in\mathfrak{M}$, then

$$u \text{ Stsf } [a_i]\equiv a_i(u_1,\ldots,u_k) \tag{4}$$

(cf. "convention W" in Tarski [11], p. 305). Since the right side of (4) is an axiom of S, we obtain $u\, Stsf\, [a_i]$. This formula being provable in S, we infer that (3) is a real model of s in S.

The construction carried out above is expressible in S; therefore on using theorem VII we obtain:

IX. *If s is a finitely axiomatizable sub-system of S, then the sentence $N(s)$ is provable in S.*

Remarks. 1. The assumption made above that S and s both contain the ε-rule is not essential for the validity of theorem IX. Indeed, let s' and S' be systems without the ε-rule which become equivalent to s and S after adjunction of that rule and explicit definitions. It is evident that $N(s')$ is not stronger than $N(s)$ and hence provable in S. Since $N(s')$ is an arithmetical sentence in which the ε-symbol does not occur, it follows from the second ε-theorem of Hilbert and Bernays that $N(s')$ is provable without the ε-rule, i. e. in S'.

2. One might ask why our construction breaks down when s contains infinitely many axioms, *e. g.* when $s=S$. To answer this question we recall that equivalences of the form (4) are provable in S for each a_i separately. There are no means by which to express in S anything which could serve as a logical product of infinitely many such equivalences.

The following theorems are easy corollaries of IX:

X. *If S is self-consistent, it is not finitely axiomatizable* [17]).

Proof. First of all we remark that there exists a finitely axiomatizable sub-system s_0 of S which is at least as strong as the arithmetic of integers based on Peano's axioms with the axiom of mathematical induction (conceived as an axiom-schema).

To prove this we remark that this system of arithmetic is equivalent to the system (Z) of Hilbert-Bernays [5], p. 384. It has been shown by Novak and Wang [8], p. 90, that upon extending (Z) by the introduction of a new primitive notion and suitable axioms we obtain a system which is finitely axiomatizable. The new primitive notion is that of a predicative class of integers. The resulting system s_0 is therefore certainly weaker than S since in S we have at our disposal the general notion of classes which satisfies all the axioms formulated by Novak and Wang in their construction. Hence s_0 is a sub-system of S.

The existence of s_0 being proved, we proceed as follows.

According to a theorem of Gödel [18]) the sentence $N(s)$ is provable in no self-consistent system s which contains (Z). Hence if S is self-consistent and s is an axiomatizable sub-system of S which contains s_0, then $N(s)$ is not provable in s. Since $N(s)$ is provable in S according to IX, we infer that systems s and S are not equivalent.

Theorem X is thus proved.

XI. *If S is self-consistent, it is ω-incomplete* [19]).

Proof. Let S' be a system equivalent to S but without the ε-rule and with a fixed set of primitive functors and predicates (i. e. explicit definitions are not allowed in S'). Let $a_1, a_2, \dots$ be axioms of S'. We can assume that no a_i contains free variables. Put $m(i)=[a_1\cdot a_2\cdot\ldots\cdot a_i \supset a_1\cdot\sim a_1]$, $i=1,2,\ldots$, and let $\Theta(x)$ be the matrix

$m(x)$ is unprovable in the first order functional calculus.

It is easy to see that this matrix can be written in purely arithmetical terms and is therefore a matrix of S.

According to IX, sentences $\Theta(1), \Theta(2), \ldots$ are all provable in S whereas the general statement $(x)\Theta(x)$ is equivalent to $N(S)$ and hence unprovable in S unless S is inconsistent.

XII [20]). *If S is self-consistent, then there exist consistent but ω-inconsistent sets of arithmetical sentences.*

Indeed, sentences $\sim N(S), \Theta(1), \Theta(2), \Theta(3) \ldots$ form such a set.

5. Models of the second kind in the axiomatic theory of real numbers.

Almost all we have said in sections 3 and 4 can be repeated when S is an axiomatic theory of real numbers. When speaking of the arithmetic of real numbers, we have in mind systems in which the class of integers as well as the development of any real number into decimal (or other) fractions is definable and can be proved to exist.

[17]) This theorem has been first proved by Wang [15]. A critique of this proof by Rosser [9] appears unfounded; cf Wang [17].

[18]) Gödel [3], theorem XI, p. 196.

[19]) For the notion of ω-completeness see Tarski [13]. The result obtained in theorem XI is of course not new.

[20]) Of course this result is not new either. See Gödel [3], p. 190 and Tarski [13], p. 108.

Note that the arithmetic of real numbers in its usual formulation is based on an infinite number of axioms because the axiom of continuity cannot be expressed otherwise as a schema.

The notions of functions and relations do not occur explicitly among the primitive notions of arithmetic. Some particular cases of these notions, however, are definable in arithmetic and these particular cases are general enough to enable us to carry over the proofs given in sections 4 and 5 from set-theory to arithmetic.

The procedure is as follows.

First, we define one-to-one correspondences between integers and finite sequences of k integers, $k=1,2,\ldots$ If an integer n is made to correspond with a k-tuple $(n_1,\ldots,n_k)$ and p_k is the k-th prime, then we shall identify the integer p_k^n with the k-tuple $(n_1,\ldots,n_k)$. In this way we obtain an arithmetical substitute for the notion of a finite sequence of integers.

It is well known that we can effectively establish a one-to-one correspondence between real numbers and sets of integers. In other words we can find a matrix $\Phi(x,n)$ such that the following formulas are provable in S:

$$\Phi(x,n)\supset n \text{ is an integer},$$
$$x'=x''\equiv(n)[\Phi(x',n)\equiv\Phi(x'',n)].$$

A real number x will be called a *k-termed relation* if $(n)[\Phi(x,n)\supset(\exists m)n=p_k^m]$. Integers $n_1,\ldots,n_k$ are said *to be in relation* x if the integer n corresponding to the sequence $(n_1,\ldots,n_k)$ satisfies the condition $\Phi(x,p_k^n)$. In this case we write $x(n_1,\ldots,n_k)$.

A real number x is called a *function with k arguments* if it satisfies the following conditions:

$$x \text{ is a binary relation},$$
$$x(n_1,n_2)\supset(\exists m)(n_1=p_k^m),$$
$$(m)(\exists n_2)x(p_k^m,n_2),$$
$$x(n_1,n_2')\cdot x(n_1,n_2'')\supset n_2'=n_2''.$$

The value of the function x for the arguments $n_1,\ldots,n_k$ is defined as $(\iota n_2)x(p_k^m,n_2)$ where m is the integer corresponding to the sequence $n_1,\ldots,n_k$.

Having defined the notions of functions and relations, we can reconstruct without difficulty all the definitions and proofs which were given in sections 3 and 4. In this way we arrive at the following results:

XIII. *If the system S of the arithmetic of real numbers is self-consistent and s is a finitely axiomatizable sub-system of S, then the sentence $X(s)$ is provable in S.*

XIV. *The arithmetic of real numbers is not finitely axiomatizable.*

6. Models of the third kind. The method described in sections 3-5 does not apply in the case in which S is the system of the arithmetic of positive integers based on Peano's postulates. The failure of the method is caused by the fact that no model of the second kind is definable in the arithmetic of positive integers for a system s in which the existence of infinitely many individuals is provable.

Models of the third kind which we shall discuss presently will enable us to prove theorems similar to IX-XII for the case in which S is the system of the arithmetic of integers. These models were first defined by Hilbert and Bernays who stated their definitions in a non-formal language[21]). We shall present here an arithmetical counterpart of the Hilbert-Bernays definition in order to discuss the possibility of its use in a formal system S.

We shall make the same assumptions concerning the systems S and s as in section 1. The system S will however be slightly enlarged by the adjunction of the symbols $\wedge$ and $\vee$, denoting the Boolean zero and the Boolean unit. Boolean addition and multiplication will be denoted by $+$ and $\cdot$ and, when many summands or factors are present, by the Σ- and Π-symbols. Boolean complementation will be denoted by an upper index 1. For symmetry we put $\wedge^0=\wedge$ and $\vee^0=\vee$.

A term Γ of s will be called a *constant term* if no variable occurs in it. It is easy to construct in arithmetic (and hence in S) a functor T with one free variable such that the following formulas are provable in S:

if y is an integer, then $T(y)$ is a constant term;
if x is a constant term, then $x=T(y)$ for some y.

We shall now construct in S a function $S(x,m)$ which enumerates all *matrices* that can be obtained from a Q-*matrix* m by all possible *substitutions* of *constant terms* for the *free variables* of m.

We introduce first the auxiliary functors σ, S, and μ.

[21]) See Hilbert-Bernays [5], pp. 33-36.

Let σ be a functor of S with two free variables such that the formula

$$x = 2^{\sigma(1,x)-1} \cdot 3^{\sigma(2,x)-1} \cdot \ldots \cdot p_k^{\sigma(k,x)-1} \cdot \ldots$$

is provable in S. In other words the definition of $\sigma(y,x)$ is obtained by expressing in S the following definition:

$\sigma(y,x) = 1 +$ *(the exponent of the y-th prime in the development of x into the product of primes).*

For any *term* t we put

$$S(t,x,1) = Sb\, t(\mathfrak{z}_1 / T(\sigma(1,x))),$$
$$S(t,x,y+1) = Sb\, S(t,x,y)(\mathfrak{z}_{y+1} / T(\sigma(y+1,x))).$$

This is clearly an inductive definition of the type which can be represented in S by a single functor. Hence $S(t,x,y)$ is a functor of S.

Let $\mu(t)$ be the functor of S defined as the largest integer y such that $\mathfrak{z}_y$ *occurs* in t. We put

$$S(t,x) = S(t,x,\mu(t))$$

and call $S(t,x)$ the *x-th substitution of t*. $S(t,x)$ is clearly a functor of S.

It can be proved in S that

$$(x,t)\{S(t,x) \text{ is a constant term}\}.$$

If $e = [\mathfrak{r}_l(t_1, \ldots, t_{p_l})]$ is an *elementary formula*, then we put

$$S(e,x) = [\mathfrak{r}_l(S(t_1,x), \ldots, S(t_{p_l},x))]$$

and call $S(e,x)$ the *x-th substitution of e*. The following statement is provable in S:

$$(x,e)\{S(e,x) \text{ is a prime formula}\}$$

(cf. section 1, definition 8).

We define now the *x-th substitution* of an arbitrary *Q-matrix* m. The definition proceeds by induction. If $m = e$, then $S(e,x)$ has been defined above. If $m = [m_1 | m_2]$, then we put $S(m,x) = [S(m_1,x) | S(m_2,x)]$. By a standard procedure we transform this inductive definition into an explicit one whin can be expressed in S.

We shall call a *pseudo-model of the third kind* or briefly a *valuation*, a functor Φ or S with one free variable such that the following formula is provable in S:

if p is a prime formula, then $\Phi(p) = \vee$ or $\Phi(p) = \wedge$.

Let Φ be an arbitrary valuation. We consider a functor $Val_\Phi(m,x)$ of S with two free variables satisfying the following conditions: a) the first argument of Val_Φ runs over *Q-matrices* and the second over arbitrary positive integers; b) the following statements (5) and (6) are provable in S:

(5) $$Val_\Phi(e,x) = \Phi(S(e,x)),$$

(6) $$Val_\Phi([m_1 | m_2], x) = \{Val_\Phi(m_1,x) \cdot Val_\Phi(m_2,x)\}^1$$

(the upper index 1 denotes here the Boolean complementation).

It is easy to construct effectively a functor Val_Φ which satisfies the above conditions. All we have to do is to remark that conditions (5) and (6) can be considered as an inductive definition of Val_Φ and that inductive definitions of this kind can be transformed into ordinary definitions which are expressible in S.

Definition. A pseudo-model Φ is called a *real model of the third kind of s in S* if the following formulas are provable in S:

$$(x)\{Val_\Phi([a_i],x) = \vee\}, \quad i = 1, \ldots, \delta.$$

Models of the third kind are in some respects similar to models of the first kind. Indeed, it follows from the definition that

1. The general notion of models of the third kind is defined not in S but in its syntax (since Φ was defined as a functor of S);

2. Every particular pseudo-model of the third kind can be defined within S (since each particular Φ can be written down by means of symbols allowed in S). The problem whether this pseudo-model is or is not a real model of s can be formulated and in particular cases solved in S.

On the other hand

3. The notion of models of the third kind retains its meaning also for cases in which s contains an infinite number of axioms.

Indeed, Φ is a real model if the formula

$$(m)[(m \text{ is an axiom of } s) \supset (x)\{Val_\Phi(m,x) = \vee\}]$$

is provable in S. This definition is meaningful not only when the *axioms* of s are finite in number but more generally when their set is definable in S.

In this respect there is an analogy between models of the third and second kinds.

We shall now investigate the problem whether models of the third kind can be used to obtain absolute proofs of consistency.

Following Hilbert-Bernays[22]) we shall call a *Q-matrix* m *verifiable* if $(x)Val_\Phi(m,x)=\vee$. The following theorem can then be proved:

XV. *The formula*

$(m)\{(m$ *is a Q-matrix provable in* $s)\supset(m$ *is verifiable*$)\}$ *is provable in* S.

The proof of this theorem has been given by Hilbert-Bernays [5], pp. 33-36. We note that this proof is straightforward for the case in which m can be obtained from the *axioms* by the elementary calculus with free variables[23]). Hence the essential step in the proof of XV consists in proving in S the implication:

$(m$ *is provable in* $s)\supset(m$ *can be obtained from the axioms of* s *by means of the elementary calculus with free variables*$)$.

This implication is established by Hilbert and Bernays with the help of the first ε-theorem.

Another proof of XV has been given by Łoś[24]). His proof is not finitary but can be translated in S along with the Hilbert-Bernays proof.

As a corollary from XV we obtain

XVI[25]). *The following implication is provable in* S: *if* Φ *is a real model of* s *in* S, *then* $N(s)$.

It follows from XVI that models of the third kind can be used when an absolute proof of consistency is desired.

7. Impossibility of a finite axiomatization of arithmetic[26]). We assume in this section that S is the system of arithmethic of positive integers based on Peano's postulates and that s is a finitely axiomatizable sub-system of S. We can assume that axioms of s have been brought to the normal form

$$a_i=\prod_{\nu=1}^{\omega(i)}\sum_{\xi=1}^{\tau(\nu,i)}E_{\nu,\xi}^{\lambda_{\nu,\xi,i}}(i),\quad \lambda_{\nu,\xi,i}=0\quad\text{or}\quad\lambda_{\nu,\xi,i}=1,$$

where

$$E_{\nu,\xi}(i)=r_{\pi(\nu,\xi,i)}(\Gamma_{\nu,\xi,i,1},\ldots,\Gamma_{\nu,\xi,i,p_{\pi(\nu,\xi,i)}}).$$

---

22) Hilbert-Bernays [5], p. 36.

23) Hilbert-Bernays [5], p. 380.

24) Łoś [6], p. 36, theorem 34.

25) Hilbert-Bernays [5], p. 36, call XVI the "Wf-theorem".

26) The impossibility of a finite axiomatization of the arithmetic of positive integers was first shown by Ryll-Nardzewski in [10]. Theorem XVIII of the present paper is however slightly stronger than the result of Ryll-Nardzewski.

Further we can assume that constants, functors, and predicates of s occur also in S.

We shall denote by $e_{\nu,\xi}(i)$ the Gödel number of $E_{\nu,\xi}(i)$.

Lemma 3. *If* Φ *is an arbitrary valuation, then the following formula is provable in* S:

$$Val_\Phi([a_i],x)=\prod_{\nu=1}^{\omega(i)}\sum_{\xi=1}^{\tau(\nu,i)}(Val_\Phi(e_{\nu,\xi}(i),x))^{\lambda_{\nu,\xi,i}}$$

(exponents λ as well as the symbols Π and Σ have here the Boolean meaning).

Proof. Immediate from (5) and (6).

Lemma 4. *Under the assumptions of lemma* 3 *the following equivalence is provable in* S:

$$\{Val_\Phi([a_i],x)=\vee\}\equiv$$
$$(\nu)\{\nu\leqslant\omega(i)\supset(\exists\,\xi)[(\xi\leqslant\tau(\nu,i))\cdot(Val_\Phi(e_{\nu,\xi}(i),x)=\vee^{\lambda_{\nu,\xi,i}})]\}.$$

Proof. Immediate from lemma 3.

Let t run over *constant terms* of s. We consider a functor Θ of S such that the following equations be provable in S:

$$\Theta(\mathfrak{f}_i)=f_i,\quad i=1,\ldots,\alpha,\tag{7}$$

$$\Theta([\mathfrak{g}_j(t_1,\ldots,t_{q_j})])=g_j(\Theta(t_1),\ldots,\Theta(t_{q_j})),\quad j=1,\ldots,\beta.\tag{8}$$

It is easy to construct explicitly a functor satisfying these conditions. Indeed, (7) and (8) contain an inductive definition which can be transformed into an explicit one and the definiens of the explicit definition thus obtained is the required Θ.

The intuitive meaning of the functor Θ can be explained as follows: Consider an arbitrary constant term Γ of s; of course it denotes an integer. Let t be the Gödel number of Γ. Then $\Theta(t)$ is the integer denoted by Γ.

We put $\Theta(T(x))=\vartheta(x)$. Note that ϑ is a term of S with one free variable.

$\vartheta(x)$ is of course the integer denoted by the x-th constant term (in the enumeration of constant terms given by the function T).

Lemma 5. *Let* Γ *be a term of* s, t *its Gödel number and* h *the largest integer such that* x_h *occurs in* Γ. *Then the following equation is provable in* S:

$$\Theta(S(t,x))=Subst\,\Gamma(x_1/\vartheta(\sigma(1,x)),\ldots,x_h/\vartheta(\sigma(h,x))).\tag{9}$$

Proof. If $\Gamma = x_i$, then $t = \mathfrak{z}_i$, $S(t,x) = T(\sigma(i,x))$ and hence $\Theta(S(t,x)) = \vartheta(\sigma(i,x))$. On the other hand the right side of (9) is $\vartheta(\sigma(i,x))$. Hence the lemma is true in this case.

If $\Gamma = f_i$, then $t = \mathfrak{f}_i$ and the left and right sides of (9) can easily be shown to be equal to f_i.

Let us assume that the lemma is proved for terms $\Gamma_1, \ldots, \Gamma_{q_j}$ with the Gödel numbers $t_1, \ldots, t_{q_j}$. Let Γ be the term $g_j(\Gamma_1, \ldots, \Gamma_{q_j})$; the Gödel number of Γ is $t = [\mathfrak{g}_j(t_1, \ldots, t_{q_j})]$. It follows from lemma 21 in section 1 and from (8) that equations

$$
\begin{aligned}
& S(t,x) = [g_j(S(t_1,x), \ldots, S(t_{q_j},x))], \\
(10) \qquad & \Theta(S(t,x)) = g_j(\Theta(S(t_1,x)), \ldots, \Theta(S(t_{q_j},x)))
\end{aligned}
$$

are provable in S. On the other hand if we put

$$\Gamma_i^{(x)} = \mathit{Subst}\, \Gamma_i(x_1/\vartheta(\sigma(1,x)), \ldots, x_h/\vartheta(\sigma(h,x))),$$

we obtain from lemma 21 in section 1 the equation

$$(11) \qquad \mathit{Subst}\, \Gamma(x_1/\vartheta(\sigma(1,x)), \ldots, x_h/\vartheta(\sigma(h,x))) = g_j(\Gamma_1^{(x)}, \ldots, \Gamma_{q_j}^{(x)})$$

provable in S. Since by the inductive assumption equations

$$\Theta(S(t_i,x)) = \Gamma_i^{(x)}$$

are provable in S, we obtain the desired result by comparing formulas (10) and (11).

Lemma 5 is thus proved. Observe that this lemma is not a theorem of S but a theorem-schema. The statement of this lemma must be proved separately for every Γ.

Definition. If Γ is a term and M a Q-matrix of s and if h is the largest integer such that x_h occurs in Γ or in M, then we put

$$
\begin{aligned}
\Gamma^{(x)} &= \mathit{Subst}\, \Gamma(x_1/\vartheta(\sigma(1,x)), \ldots, x_h/\vartheta(\sigma(h,x))), \\
M^{(x)} &= \mathit{Subst}\, M(x_1/\vartheta(\sigma(1,x)), \ldots, x_h/\vartheta(\sigma(h,x))).
\end{aligned}
$$

Note that $\Gamma^{(x)}$ is a term of S with one free variable x. Similarly $M^{(x)}$ is a matrix of S with one free variable x.

Lemma 6. *If* $M = \sum_{\nu=1}^{\omega} \prod_{\xi=1}^{\tau(\nu)} E_{\nu,\xi}^{\lambda_{\nu,\xi}}$ *where the* $E_{\nu,\xi}$ *are elementary formulas and the* $\lambda_{\nu,\xi}$ *are equal to* 0 *or* 1, *then*

$$M^{(x)} = \sum_{\nu=1}^{\omega} \prod_{\xi=1}^{\tau(\nu)} (E_{\nu,\xi}^{(x)})^{\lambda_{\nu,\xi}}.$$

The proof is obvious.

We shall now define a valuation Φ of which we shall show later that it is a real model of s in S.

$$
\begin{aligned}
& \{\Phi(p) = s\} \equiv \\
& \sum_{j=1}^{7} \{(\mathit{Ind}\,(p) = j) \cdot (r_j(\Theta(\mathit{Comp}_1\,(p)), \ldots, \Theta(\mathit{Comp}_{p_j}(p))) \cdot (s = \vee) \\
& \qquad \vee \sim r_j(\Theta(\mathit{Comp}_1\,(p)), \ldots, \Theta(\mathit{Comp}_{p_j}(p))) \cdot (s = \wedge))\}.
\end{aligned}
$$

(Cf. section 1, def. 19, p. 137, for the definition of the functions *Ind* and *Comp*).

It is evident that Φ is a functor of S. The formula

$$\Phi(p) = \vee \quad or \quad \Phi(p) = \wedge$$

is provable in S. Indeed, it can be proved in S that if p is a *prime formula*, then $\mathit{Comp}_j(p)$ is a *constant term* and hence $\Theta(\mathit{Comp}_j(p))$ is a perfectly defined term of S.

Lemma 7. *Let* $r_i(\Gamma_1, \ldots, \Gamma_{p_i})$ *be an elementary formula and* $e = [\mathfrak{r}_i(t_1, \ldots, t_{p_i})]$ *its Gödel number. The following equivalences are then provable in* S *for* $\lambda = 0, 1$:

$$\Phi(S(e,x)) = \vee^{\lambda} \equiv r_i^{\lambda}(\Gamma_1^{(x)}, \ldots, \Gamma_{p_i}^{(x)}).$$

Proof. By definition $S(e,x) = [\mathfrak{r}_i(S(t_1,x), \ldots, S(t_{p_i},x))]$ whence it follows that equations

$$
\begin{aligned}
& \mathit{Ind}\,(S(e,x)) = i, \\
& \mathit{Comp}_1(S(e,x)) = S(t_1,x), \\
& \ldots\ldots\ldots\ldots \\
& \mathit{Comp}_{p_i}(S(e,x)) = S(t_{p_i},x)
\end{aligned}
$$

are provable in S. Now observe that the formula

$$S(e,x) \text{ is a } \textit{prime formula}$$

is provable in S. Upon using the definition of Φ we obtain therefore the formula provable in S

$$\Phi(S(e,x)) = \vee^{\lambda} \equiv r_i^{\lambda}(\Theta(S(t_1,x)), \ldots, \Theta(S(t_{p_i},x))).$$

Since equations $\Theta(S(t_j,x)) = \Gamma_j^{(x)}$ are provable in S (cf. lemma 5), we get the desired result directly from the last formula.

Observe that lemma 7 is not a single theorem but a theorem-schema of S.

XVII. *Φ as defined on* p. 153 *is a real model of s in S.*

Proof. Let $e_{\nu,\xi}(i)$ occurring in the statement of lemma 3 be the Gödel number of the elementary matrix

$$E_{\nu,\xi}(i)=r_{\pi(\nu,\xi,i)}(\Gamma_{\nu,\xi,i,1},\ldots,\Gamma_{\nu,\xi,i,p_{\pi(\nu,\xi,i)}})$$

and let $t_{\nu,\xi,i,\varrho}$ be the Gödel number of $\Gamma_{\nu,\xi,i,\varrho}$. According to lemma 7 we can prove in S the formula

$$Val_{\Phi}(e_{\nu,\xi}(i),x)=\vee^{\lambda_{\nu,\xi,i}}\equiv r^{\lambda_{\nu,\xi,i}}_{\pi(\nu,\xi,i)}(\Gamma^{(x)}_{\nu,\xi,i,1},\ldots,\Gamma^{(x)}_{\nu,\xi,i,p_{\pi(\nu,\xi,i)}}).$$

Using lemma 4 we infer that the formula

$$Val_{\Phi}([a_i],x)=\vee\equiv$$
$$(\nu)\{\nu\leqslant\omega(i)\supset(\exists\xi)[(\xi\leqslant\tau(\nu,i))\cdot r^{\lambda_{\nu,\xi,i}}_{\pi(\nu,\xi,i)}(\Gamma^{(x)}_{\nu,\xi,i,1},\ldots,\Gamma^{(x)}_{\nu,\xi,i,p_{\pi(\nu,\xi,i)}})]\}$$

is provable in S. Now we have

$$r^{\lambda_{\nu,\xi,i}}_{\pi(\nu,\xi,i)}(\Gamma^{(x)}_{\nu,\xi,i,1},\ldots,\Gamma^{(x)}_{\nu,\xi,i,p_{\pi(\nu,\xi,i)}})=(E^{(x)}_{\nu,\xi}(i))^{\lambda_{\nu,\xi,i}}$$

and hence the equivalence

$$Val_{\Phi}([a_i],x)=\vee\equiv\prod_{\nu=1}^{\omega(i)}\sum_{\xi=1}^{\tau(\nu,i)}(E^{(x)}_{\nu,\xi}(i))^{\lambda_{\nu,\xi,i}}$$

is provable in S. By lemma 6 the right side of the equivalence can be replaced by

$$\prod_{\nu=1}^{\omega(i)}\sum_{\xi=1}^{\tau(\nu,i)}(E^{\lambda_{\nu,\xi,i}}_{\nu,\xi}(i))^{(x)},$$

i. e. by $a_i^{(x)}$. It follows that the formula

$$Val_{\Phi}([a_i],x)=\vee\equiv a_i^{(x)}$$

is provable in S. Since $a_i^{(x)}$ is a substitution of an axiom of s, it is provable in S. Hence the equation

$$Val_{\Phi}([a_i],x)=\vee$$

is provable in S and theorem XVII is proved.

From theorems XVI and XVII we obtain

XVIII. *If s is a finitely axiomatizable sub-system of S, then the formula $N(s)$ is provable in S.*

XIX [27]). *S is not finitely axiomatizable.*

[27]) This theorem is due to Ryll-Nardzewski who obtained it in [10] by a different method.

To prove this theorem we need the following

Lemma 8. *There exists a finitely axiomatizable sub-system s_0 of S such that if s is a finitely axiomatizable sub-system of S which contains s_0, then the sentence $N(s)$ is not provable in s.*

We shall content ourselves with a sketch of the proof.

We shall take for granted the arithmetization of the system S along the lines indicated by Gödel [3]. The arithmetical counterparts of the metamathematical notions will be denoted by symbols used by Gödel although strictly speaking the symbols should be modified because the system arithmetized by Gödel is different from S.

Let F be a primitive recursive function such that $F(m,n,p)=0$ if and only if n, p are *sentences* of S and m is a *proof* of the *implication* p Imp n in the functional calculus of the first order.

Let F' be defined as follows

$$F'(m,n,p)=F\left(m,\mathrm{Sb}\,n\binom{19}{Z(n)},p\right). \tag{12}$$

Let f' be the Gödel number of the equation $F'(m,n,p)=0$. Hilbert and Bernays [5], pp. 310-323, have shown that the following implication is provable in S:

$$F'(m,n,p)=0\supset(\exists x)\,xB\,\mathrm{Sb}\,f'\begin{pmatrix}17 & 19 & 23\\ Z(m) & Z(n) & Z(p)\end{pmatrix}. \tag{13}$$

Analysing their proof we find that 1^0 the x whose existence is stated in (13) is a primitive recursive function Q of m, n, p; 2^0 the axioms of S which occur in the proof with the Gödel number $Q(m,n,p)$ are finite in number and independent of m, n, and p. If we denote by $A_1,\ldots,A_k$ these axioms and by ax the Gödel number of their conjunction, we obtain instead of (13) the following implication provable in S:

$$F'(m,n,p)=0\supset F\left(Q(m,n,p),\mathrm{Sb}\,f'\begin{pmatrix}17 & 19 & 23\\ Z(m) & Z(n) & Z(p)\end{pmatrix},ax\right)=0. \tag{14}$$

Hilbert and Bernays have further shown (cf. [5], pp. 307-308) that the implication

$$(\exists x)F'(x,17\ \mathrm{Gen}\ u,p)=0\supset(q)(\exists x)F'\left(x,\mathrm{Sb}\,u\binom{19}{Z(q)},p\right)=0 \tag{15}$$

is provable in S.

Now let s_0 be a sub-system of S based on the axioms $A_1, \ldots, A_k$ and on those axioms of S which are necessary to prove implications (14) and (15) as well as the recursion-equations for the functions F, Sb, Imp, Neg, Z, and Q.

We shall show that s_0 has the property stated in the lemma.

Indeed, let s be a finitely axiomatizable sub-system of S containing s_0 and denote by Ax the Gödel number of the conjunction of the axioms of s. It follows that (14) and (15) (when expressed in the symbols of s) are theorems of s.

From (14) we infer that the following implication is provable in s:

$$F'(m,n,p)=0 \supset F\left(Q(m,n,p), \mathrm{Sb}\, f'\left(\begin{matrix}17 & 19 & 23\\ Z(m) & Z(n) & Z(p)\end{matrix}\right), Ax\right)=0. \tag{16}$$

We put

$$\beta=\mathrm{Sb}\, f'\left(\begin{matrix}23\\ Z(Ax)\end{matrix}\right), \qquad \gamma=17\ \mathrm{Gen}\ \mathrm{Neg}\ \beta, \qquad \delta=\mathrm{Sb}\ \gamma\left(\begin{matrix}19\\ Z(\gamma)\end{matrix}\right).$$

It is easy to see that

$$\delta=17\ \mathrm{Gen}\ \mathrm{Neg}\ \mathrm{Sb}\ \beta\left(\begin{matrix}19\\ Z(\gamma)\end{matrix}\right).$$

Repeating the argument of Gödel [3], pp. 187-189, we can prove that δ is *unprovable* in s, provided that s is consistent, i. e. that

$$N(s) \supset (x) F(x, \delta, Ax) \neq 0. \tag{17}$$

This proof can be repeated word by word in the system s owing to the circumstance that (15), (16), and recursion-equations for the functions F, Q, Sb, Imp, Neg, and Z are available in s. Hence if we denote by w the Gödel number of the sentence $N(s)$ and observe that the Gödel number of the sentence $(x)F(x,\delta,Ax)\neq 0$ is δ, we obtain as an arithmetical counterpart of (17) in s the equation

$$F(h, w\ \mathrm{Imp}\ \delta, Ax)=0, \tag{18}$$

where h is the Gödel number of the proof of (17) in s.

It follows from (18) that if $N(s)$ were provable in s (i. e. if w were *provable* in s), then δ would be *provable* in s and hence by (17) s would be inconsistent.

Lemma 8 is thus proved.

Theorem XIX results from lemma 8 by the same argument which was used in the proof of theorem X.

The following result can also be obtained from XVIII. Following Tarski and Szmielew[28]) we shall call a system s *interpretable* in S if there exists a model of the first kind of s in S. We have then

XX [29]). *For every finitely axiomatizable sub-system s_1 of the arithmetic of integers S there is a finitely axiomatizable sub-system s of S such that s is not interpretable in s_1.*

Proof. Define s as a system obtained from s_1 by the adjunction of the sentence $N(s_1)$ to the axioms of s_1. According to XVIII s is a finitely axiomatizable subsystem of S. If s were interpretable in s_1, then the implication $N(s_1) \supset N(s)$ would be provable in s_1 and hence the sentence $N(s)$ would be provable in s. But this is impossible since $N(s)$ is provable in s only if s is inconsistent (cf. lemma 8, p. 155).

To complete our discussion we remark that the results established in section 7 hold not only when S is the system of the arithmetic of positive integers but more generally for all systems S which contain arithmetic and have the property that inductive definitions of the form (7) and (8) are expressible by single matrices of the system S.

Bibliography.

[1] Alonzo Church, *Introduction to Mathematical Logic. Part I*, Annals of Mathematics Studies **13**. Princeton 1944.

[2] Kurt Gödel, *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, Monatshefte für Mathematik und Physik **37** (1930), pp. 349-360.

[3] — *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, Ibidem **38** (1931), pp. 173-198.

[4] David Hilbert, *Grundlagen der Geometrie*, 7-th edition, Leipzig-Berlin 1930.

[5] David Hilbert and Paul Bernays, *Grundlagen der Mathematik*, vol. **2**. Berlin 1939.

[6] Jerzy Łoś, *O matrycach logicznych*, Prace Wrocławskiego Towarzystwa Naukowego, seria B, Nr **19**. 1949.

[7] Andrzej Mostowski, *Logika matematyczna*, Monografie Matematyczne, vol. **18**. Warszawa-Wrocław 1948.

[8] Ilse L. Novak, *A Construction for Models of Consistent Systems*, Fundamenta Mathematicae **37** (1950), pp. 87-110.

[9] J. Barkley Rosser, *Review of* [15], The Journal of Symbolic Logic **16** (1951), pp. 143-144.

[28]) Tarski-Szmielew [14].

[29]) Theorem XX and its proof were communicated to me by A. Tarski.

[10] Czesław Ryll-Nardzewski, *The Role of the Axiom of Induction in the Elementary Arithmetic,* Fundamenta Mathematicae **39** (1953).

[11] Alfred Tarski, *Der Wahrheitsbegriff in den formalisierten Sprachen,* Studia Philosophica **1** (1936), pp. 261-405.

[12] — *Über den Begriff der logischen Folgerung,* Actes du Congrès International de Philosophie scientifique, VII Logique. Actualités Scientifiques et Industrielles **394**, Paris 1936, pp. 1-11.

[13] — *Einige Betrachtungen über die Begriffe der ω-Widerspruchsfreiheit und der ω-Vollständigkeit,* Monatshefte für Mathematik und Physik **40** (1933), pp. 97-112.

[14] — and Wanda Szmielew, *Mutual Interpretability of Some Essentially Undecidable Theories,* Proceedings of the International Congress of Mathematics, vol. 1, Cambridge 1950, p. 734.

[15] Hao Wang, *The Non-finitizability of Impredicative Principles,* Proceedings of the National Academy of Sciences of the USA **36** (1950), pp. 479-484.

[16] — *Arithmetic Translations of Axiom Systems,* Transactions of the American Mathematical Society **71** (1951), pp. 283-293.

[17] — *The Irreducibility of Impredicative Principles,* Mathematische Annalen **125** (1952), pp. 56-66.

On Labil and Stabil Points.

By

K. Borsuk and J. W. Jaworowski (Warszawa).

1. The concept of the (homotopically) labil point is due to H. Hopf and E. Pannwitz[1]). Its definition can be formulated as follows:

Definition 1. A point a of a space M is *homotopically labil* whenever for every neighbourhood U of a there exists a continuous mapping $f(x,t)$ which is defined in the Cartesian product $M\times I$ of M and of the interval $I: 0\leqslant t\leqslant 1$ and which satisfies the following conditions:

(1) $f(x,t)\in M$ for every $(x,t)\in M\times I$,

(2) $f(x,0)=x$ for every $x\in M$,

(3) $f(x,t)=x$ for every $(x,t)\in (M-U)\times I$,

(4) $f(x,t)\in U$ for every $(x,t)\in U\times I$,

(5) $f(x,1)\neq a$ for every $x\in M$.

A point $a\in M$ will be called *homotopically stabil*[2]) if it is not homotopically labil.

Remark. If a is a homotopically labil point of a space M and b a point of another space N and if there exists a homeomorphic mapping h of a neighbourhood U_0 of a in M onto a neighbourhood V_0 of b in N such that $h(a)=b$, then b is a homotopically

[1]) H. Hopf and E. Pannwitz, *Über stetige Deformationen von Komplexen in sich,* Math. Ann. **108** (1933), pp. 433-465. See also P. Alexandroff and H. Hopf, *Topologie I,* Berlin 1935, p. 523. In the present paper we slightly modify the terminology. Namely we shall refer to the points called by H. Hopf and E. Pannwitz *labil, as homotopically labil.* The term "labil" will be used here in the other sense.

[2]) H. Hopf and E. Pannwitz use the term "locally stabil point".