astérisques. Il y existe donc au moins une ligne qui contient $\geqslant k-2$ astérisques, car, en cas contraire, le nombre des astérisques serait

$$\leqslant (k-3)(2k-3) = 2k^2-9k+9,$$

ce qui est impossible.

La conclusion suit de la remarque 1 du N I.

Par raison de symétrie, on peut remplacer k par n et n par $\boldsymbol{k}$.

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Concerning the Homological Structure of the Functional Space S_m^X.

By

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1. Functional space. Let X be a compact space and Y a metric space. We denote the set of all continuous mappings of X in Y by Y^X and we define a metric in Y^X by setting

$$\varrho(\varphi,\psi) = \operatorname*{Sup}_{x\in X} \varrho(\varphi(x)),\psi(x)) \quad \text{for every} \quad \varphi,\psi \in Y^X.$$

The functional spaces Y^X offer scope for ample investigation especially when Y is the Euclidean m-dimensional sphere S_m. In particular the problem of what properties of X imply the connectedness of S_m^X is completely resolved for the case $\dim X \leqslant m$, by the celebrated theorem of H. Hopf[1]). The relations between the properties of X and the other properties of S_m^X have been less fully investigated. The purpose of this paper is to indicate a simple relation between some homological properties of X and of S_m^X.

2. ε-chains in a metric space. Let M be a metric space. By an *ε-simplex* of M we understand a finite subset of M with diameter $<\varepsilon$. In the known manner we introduce the notion of an *oriented ε-simplex, of an ε-chain* with arbitrarily given coefficients and especially of an *ε-cycle*. If the group of coefficients is the group of rational numbers $\mathfrak{R}$ then the chains will be said to be *rational*. The *boundary* of a chain $\varkappa$ will be denoted by $\partial\varkappa$. Let us point out

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[1]) H. Hopf, *Die Klassen der Abbildungen der n-dimensionalen Polyeder auf die n-dimensionale Sphäre*, Comm. Math. Helv. **5** (1933), p. 39-54 (for polytopes), and H. Freudenthal, *Bettische Gruppe* mod 1 *und Hopfsche Gruppe*, Compositio Math. **4** (1937), p. 235-238 (for compact spaces). See also W. Hurewicz and H. Wallmann, *Dimension Theory*, Princeton 1941, p. 147.

that by the boundary of a 0-dimensional simplex (a) we understand the number 1. Consequently a 0-dimensional chain is a cycle if and only if the sum of its coefficients vanishes.

Two ε-cycles γ and γ' of M (with coefficients belonging to an arbitrarily given abelian group $\mathfrak{A}$) are called *η-homologous in M* (notation: $\gamma \underset{\eta}{\sim} \gamma'$ in M) if there exists an η-chain $\varkappa$ of M (with the coefficients belonging to $\mathfrak{A}$) such that $\partial\varkappa = \gamma - \gamma'$.

An ε-simplex $(a_0, a_1, \ldots, a_n)$ is said to be *degenerated*, if not all vertices $a_0, a_1, \ldots, a_n$ are different. We assume that the degenerated simplexes can be added or cancelled in a chain without changing it. The boundary of a degenerate simplex vanishes.

3. Cartesian products. Let $\prec$ be an ordering relation defined in the set M, i. e. a relation defined for each pair of different points $x, y \in M$, asymmetric and transitive. Evidently each k-dimensional ε-chain $\varkappa$ of M can be uniquely represented by the sum

$$\varkappa = \sum_{\nu=1}^{n} a_\nu (a_{\nu,0}, a_{\nu,1}, \ldots, a_{\nu,k}), \tag{1}$$

where $a_\nu \neq 0$ and $a_{\nu,i} \prec a_{\nu,j}$ for every $\nu = 1, 2, \ldots, n$ and $i < j$. The representation (1) of $\varkappa$ will be said to be *consistent with the order* $\prec$.

Let M and M' be two metric spaces and let $M \times M'$ denote their Cartesian product, i. e. the metric space whose points are ordered pairs $x \times x'$ with $x \in M$ and $x' \in M'$ and such that the metric is defined by the formula

$$\varrho(x \times x', y \times y') = \sqrt{\varrho(x,y)^2 + \varrho(x',y')^2}.$$

Let $\prec$ denote an ordering relation in M and $\prec'$ an ordering relation in M'. Let us assign to each pair of non-degenerate simplexes $\Delta = (a_0, a_1, \ldots, a_k)$ of M and $\Delta' = (a'_0, a'_1, \ldots, a'_{k'})$ of M' in which the vertices are ordered consistently with the relations $\prec$ and $\prec'$, the chain $\Delta \times \Delta'$ (with integral coefficients) given by the formula:

$$\Delta \times \Delta' = (a_0, a_1, \ldots, a_k) \times (a'_0, a'_1, \ldots, a'_{k'}) =$$
$$= \sum \pm (a_{i_0} \times a'_{i'_0}, a_{i_1} \times a'_{i'_1}, \ldots, a_{i_{k+k'}} \times a'_{i'_{k+k'}}),$$

the sum being extended over all non-decreasing sequences of indices $i_0, i_1, \ldots, i_{k+k'}$ and $i'_0, i'_1, \ldots, i'_{k+k'}$ such that

$$0 \leqslant i_0 + i'_0 < i_1 + i'_1 < \ldots < i_{k+k'} + i'_{k+k'} \leqslant k + k' \text{ [2]}.$$

[2]) See H. Freudenthal, *Eine Simplizialzerlegung des Cartesischen Produktes zweier Simplexe*, Fund. Math. **29** (1937), p. 139.

We see at once that if Δ is an ε-simplex and Δ' an ε'-simplex then $\Delta \times \Delta'$ is an $(\varepsilon + \varepsilon')$-chain.

Let $\varkappa$ be a k-dimensional ε-chain of M with coefficients belonging to an arbitrary abelian group $\mathfrak{A}$ and let $\varkappa'$ be a k'-dimensional ε'-chain of M' with integral coefficients. Let $\varkappa = \sum_{\nu=1}^{n} a_\nu \cdot \Delta_\nu$ and $\varkappa' = \sum_{\nu'=1}^{n'} a'_{\nu'} \cdot \Delta'_{\nu'}$ be the representations of $\varkappa$ and $\varkappa'$ respectively consistent with the orders $\prec$ and $\prec'$. Putting

$$\varkappa \times \varkappa' = \sum_{\nu=1}^{n} \sum_{\nu'=1}^{n'} a_\nu \cdot a'_{\nu'} \cdot \Delta_\nu \times \Delta'_{\nu'}$$

we obtain a $(k+k')$-dimensional $(\varepsilon+\varepsilon')$-chain of $M \times M'$ with the coefficients belonging to $\mathfrak{A}$, called the *product* of the chains $\varkappa$ and $\varkappa'$. It is known [2]) that

$$\partial(\varkappa \times \varkappa') = (-1)^k \cdot \varkappa \times \partial\varkappa' + \partial\varkappa \times \varkappa'. \tag{2}$$

It follows that:

(3) If γ and γ' are cycles, then $\gamma \times \gamma'$ is a cycle.

(4) If γ is an ε-cycle η-homologous to zero in M and γ' is an ε'-cycle in M', then $\gamma \times \gamma'$ is an $(\varepsilon+\varepsilon')$-cycle $(\eta+\varepsilon')$-homologous to zero in $M \times M'$.

4. True chains. A sequence of chains $\underline{\varkappa} = \{\varkappa_n\}$ is called a *true k-dimensional chain of M* if there exists a compact subset M_0 of M and a sequence $\{\varepsilon_n\}$ of positive numbers convergent to zero and such that $\varkappa_n$ is a k-dimensional ε_n-chain of M_0 (the coefficients of $\varkappa_n$ belong to an abelian group $\mathfrak{A}_n$, in general depending on n). If we multiply each of the chains $\varkappa_n$, constituting a true chain $\underline{\varkappa}$, by an integer m then we obtain a true chain, which we denote by $m \cdot \underline{\varkappa}$.

If $\underline{\varkappa} = \{\varkappa_n\}$ and $\underline{\lambda} = \{\lambda_n\}$ are two k-dimensional true chains of M and for every $n = 1, 2, \ldots$ the coefficients of $\varkappa_n$ and λ_n belong to the same abelian group $\mathfrak{A}_n$, then the sequence $\{\varkappa_n + \lambda_n\}$ is a k-dimensional true chain of M called the *sum* $\underline{\varkappa} + \underline{\lambda}$ of the chains $\underline{\varkappa}$ and $\underline{\lambda}$.

The true chain $\underline{\gamma} = \{\gamma_n\}$ of the space M such that every one of the chains γ_n is a k-dimensional cycle is called a *k-dimensional true cycle of M*. If $\underline{\varkappa} = \{\varkappa_n\}$ is a true $(k+1)$-dimensional chain of M, then the sequence $\{\partial\varkappa_n\}$ is a true k-dimensional cycle of M, called the *boundary* $\partial\underline{\varkappa}$ of the true chain $\underline{\varkappa}$. A true k-dimensional cycle $\underline{\gamma}$ of M is said to be *homologous to zero in* M (notation: $\underline{\gamma}\sim 0$ in M) if there exists a true $(k+1)$-dimensional chain $\underline{\varkappa}$ of M such that $\partial\underline{\varkappa}=\underline{\gamma}$. We say that a true cycle $\underline{\gamma}$ is *weakly homologous to zero in* M, if there exists an integer $n_0\neq 0$ such that the true cycle $n_0\cdot\underline{\gamma}$ is homologous to zero in M.

A true cycle $\underline{\gamma}=\{\gamma_n\}$ of M such that all coefficients of the cycles γ_n belong to an arbitrarily given abelian group $\mathfrak{A}$ is said to be *convergent in* M if the true cycle $\{\gamma_{n+1}-\gamma_n\}$ is homologous to zero in M.

All k-dimensional true cycles with rational coefficients convergent in M constitute a group $C^k(M)$ and the k-dimensional true cycles with rational coefficients convergent and homologous to zero in M constitute its subgroup $H^k(M)$. The rank of the factor group $C^k(M)/H^k(M)$ is said to be the *k-th Betti number* of M. We denote it by $p^k(M)$.

Let M and M' be two metric spaces and let $\prec$ denote an ordering relation in M and $\overset{\prime}{\prec}$ an ordering relation in M'. If $\underline{\varkappa}=\{\varkappa_n\}$ is an arbitrary k-dimensional true chain in M and $\underline{\varkappa}'=\{\varkappa_n'\}$ a k'-dimensional true chain in M' with integral coefficients then putting

$$\underline{\varkappa}\times\underline{\varkappa}'=\{\varkappa_n\times\varkappa_n'\}$$

we obtain a $(k+k')$-dimensional true chain in $M\times M'$. With regard to (2) we have

$$\partial(\underline{\varkappa}\times\underline{\varkappa}')=(-1)^k\cdot\underline{\varkappa}\times\partial\underline{\varkappa}'+\partial\underline{\varkappa}\times\underline{\varkappa}'. \tag{5}$$

It follows that if $\underline{\gamma}$ and $\underline{\gamma}'$ are true cycles then $\underline{\gamma}\times\underline{\gamma}'$ is also a true cycle and if $\underline{\gamma}$ (or $\underline{\gamma}'$) is homologous (or weakly homologous) to zero in M then $\underline{\gamma}\times\underline{\gamma}'$ is homologous (or weakly homologous) to zero in $M\times M'$. And if the true cycles $\underline{\gamma}$ and $\underline{\gamma}'$ are convergent then also the true cycle $\underline{\gamma}\times\underline{\gamma}'$ is convergent.

5. Mappings of true chains. Let f be a mapping of a metric space M in a metric space M'. If we assign to each simplex $\Delta=(a_0,a_1,\dots,a_k)$ of M the simplex $\Delta_f=(f(a_0),f(a_1),\dots,f(a_k))$ of M' then we obtain a transformation mapping each k-dimensional chain $\varkappa$ of M into a k-dimensional chain $\varkappa_f$ of M'. Evidently this chain-mapping commutes with addition and with the operation of boundary ∂.

If the mapping f is continuous then we see at once that the corresponding chain mapping assigns to every k-dimensional true chain $\underline{\varkappa}=\{\varkappa_n\}$ of M a k-dimensional true chain $\underline{\varkappa}_f=\{\varkappa_{n,f}\}$ of M' and to every k-dimensional true cycle $\underline{\gamma}=\{\gamma_n\}$ of M a k-dimensional true cycle $\underline{\gamma}_f=\{\gamma_{n,f}\}$ of M' and it commutes with addition and with the operation of boundary ∂. Moreover if the true cycle $\underline{\gamma}=\{\gamma_n\}$ is convergent in M, then the corresponding true cycle $\underline{\gamma}_f=\{\gamma_{n,f}\}$ is convergent in M'.

If f is a homeomorphism mapping M on M' then we obtain in this manner a (1—1)-correspondence between the true cycles of the spaces M and M' conserving the convergence and the relation of homology.

6. Cycles on S_k. The k-dimensional sphere S_k is homeomorphic with the boundary B_k of a $(k+1)$-dimensional simplex Δ_{k+1}. Consequently instead of the cycles on S_k we shall consider the cycles on B_k. Let P_k denote the complex made up of all k-dimensional faces of Δ_{k+1}. Choosing a positive orientation in P_k, let us denote by $\pi_{k,n}$ the k-dimensional cycle defined as the sum of all k-dimensional positively oriented simplexes of the n-th barycentric subdivision of P_k. It is known that the sequence $\underline{\pi}_k=\{\pi_{k,n}\}$ is a convergent k-dimensional true cycle of the polytope B_k and that, for each k-dimensional true cycle $\underline{\gamma}=\{\gamma_n\}$ of B_k with the coefficients of γ_n belonging to a group $\mathfrak{A}_n$, there exists a sequence $\{a_n\}$ such that $a_n\in\mathfrak{A}_n$ and that $\underline{\gamma}$ is homologous in B_k to the true cycle $\{a_n\cdot\pi_{k,n}\}$. Evidently the last cycle is homologous to zero in B_k if and only if almost all coefficients a_n vanish.

Since S_k and B_k are homeomorphic, we infer that:

There exists in S_k a convergent k-dimensional true cycle $\underline{\sigma}=\{\sigma_{k,n}\}$ with integral coefficients such that for every k-dimensional true cycle $\underline{\gamma}=\{\gamma_n\}$ of S_k, with the coefficients of γ_n belonging to an abelian group $\mathfrak{A}_n$, there exists a sequence $\{a_n\}$ with $a_n\in\mathfrak{A}_n$ such that $\underline{\gamma}$ is homologous in S_k to the true cycle $\{a_n\cdot\sigma_{k,n}\}$. The last cycle is homologous to zero in S_k if and only if almost all coefficients a_n vanish.

7. Spherical cycles and spherically essential cycles. A k-dimensional true cycle $\underline{\gamma}$ of a metric space M will be said to be *spherical* if there exists a k-dimensional true cycle $\underline{\gamma}'$ in S_k and a continuous mapping $f\in M^{S_k}$ such that

$$\underline{\gamma}\sim\underline{\gamma}'_f \quad \text{in } M.$$

A k-dimensional true cycle $\underline{\gamma}$ of M will be called *spherically essential* in M if there exists a continuous mapping f of M in S_k such that $\underline{\gamma}_f$ is not homologous to zero in S_k. Evidently if $\underline{\gamma}$ is homologous to zero in M then $\underline{\gamma}$ is not spherically essential in M.

Theorem[3]. *Let M be a metric space of dimension $\leqslant k$. Each k-dimensional true cycle $\underline{\gamma}$ not homologous to zero in M is spherically essential in M.*

First we establish a lemma constituting a slight extension of the known lemma by Lebesgue[4].

Lemma. *Let $A_1, A_2, \ldots, A_r$ be compact subsets of a metric space E. For every $\eta>0$ there exists a $\delta>0$ such that if B is a subset of E with diameter $\leqslant\delta$ and $i_1, i_2, \ldots, i_\alpha$ is a system of indices $\leqslant r$ such that $B\cdot A_{i_\nu}\neq 0$ for $\nu=1,2,\ldots,\alpha$, then there exists a point $a\in A_{i_1}\cdot A_{i_2}\cdot\ldots\cdot A_{i_\alpha}$ such that $\varrho(a,B)<\eta$.*

Proof. Suppose that for an $\eta>0$ such a $\delta>0$ does not exist. Then for a system of indices $i_1, i_2, \ldots, i_\alpha$ there exists for every natural n a set B_n with the diameter $<1/n$ such that $B_n\cdot A_{i_\nu}\neq 0$ for $\nu=1,2,\ldots,\alpha$ and that

(6) $$\varrho(a,B_n)\geqslant\eta \quad \text{for every} \quad a\in A_{i_1}\cdot A_{i_2}\cdot\ldots\cdot A_{i_\alpha}.$$

Let $a_{\nu,n}\in A_{i_\nu}\cdot B_n$. Since A_{i_1} is compact we can assume that the points $a_{1,n}$ converge to a point $a_0\in A_{i_1}$. Then also $a_{\nu,n}\to a_0$ for every $\nu=1,2,\ldots,\alpha$ and we infer that $a_0\in A_{i_1}\cdot A_{i_2}\cdot\ldots\cdot A_{i_\alpha}$. But

$$\varrho(a_0,B_n)\leqslant\varrho(a_0,a_{1,n})\to 0,$$

hence $\varrho(a_0,B_n)<\eta$ for almost all n, contrary to (6).

Proof of the theorem. We first show, by induction, that the theorem is true for compact spaces.

If $k=0$ then the cycles γ_n constituting the 0-dimensional true cycle $\underline{\gamma}=\{\gamma_n\}$ are of the form

$$\gamma_n=\alpha_{n,1}(a_{n,1})+\alpha_{n,2}(a_{n,2})+\ldots+\alpha_{n,l_n}(a_{n,l_n})$$

where

$$\alpha_{n,1}+\alpha_{n,2}+\ldots+\alpha_{n,l_n}=0.$$

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[3]) Compare P. Alexandroff, *Dimensionstheorie. Ein Beitrag zur Geometrie der abgeschlossenen Mengen*, Math. Ann. **106** (1932), p. 223, where an analogous theorem is proved by another method under a slightly more restrictive hypothesis.

[4]) See, for instance, P. Alexandroff and H. Hopf, *Topologie I*, Berlin 1935, p. 101.

Since $\underline{\gamma}$ is not homologous to zero in M there exists an $\varepsilon>0$ and an increasing sequence of indices $\{n_\nu\}$ such that no one of the cycles γ_{n_ν} is ε-homologous to zero in M. It follows that the compact space M is the sum of two closed sets X_ν and Y_ν such that:

(7) If $x\in X_\nu$ and $y\in Y_\nu$ then $\varrho(x,y)\geqslant\varepsilon$.

(8) The sum β_ν of the coefficients $\alpha_{n_\nu,j}$ such that $a_{n_\nu,j}\in X_\nu$ does not vanish.

Since M is compact there exists a subsequence $\{X_{\nu_i}\}$ of the sequence $\{X_\nu\}$ convergent to a set $X_0\subset M$. From (7) it follows that:

(9) $$X_{\nu_i}=X_0 \quad \text{for almost all indices } i.$$

Putting

$$Y_0=M-X_0$$

we have

$$Y_{\nu_i}=Y_0 \quad \text{for almost all indices } i.$$

Hence

(10) $$M=X_0+Y_0 \text{ and if } x\in X_0 \text{ and } y\in Y_0 \text{ then } \varrho(x,y)\geqslant\varepsilon.$$

Let β_n^0 denote the sum of all coefficients $\alpha_{n,j}$ such that $a_{n,j}\in X_0$. By (8) and (9)

(11) $$\beta^0_{n_{\nu_i}}=\beta_{\nu_i}\neq 0 \quad \text{for almost all indices } i.$$

The sphere S_0 contains only two points p and q. Putting

$$f(x)=p \quad \text{for} \quad x\in X_0,$$
$$f(x)=q \quad \text{for} \quad x\in Y_0,$$

we obtain a continuous mapping of M in S_0. The function f maps the true cycle $\underline{\gamma}=\{\gamma_n\}$ onto the true cycle $\{\beta_n\cdot(p)-\beta_n\cdot(q)\}$ which, by (11), is not homologous to zero in S_0.

Assume now that $k>0$ and that the theorem is true (for compact spaces) for dimensions $<k$. Let $\underline{\gamma}=\{\gamma_n\}$ be a k-dimensional true cycle not homologous to zero in a compact space M of dimension $\leqslant k$. Then there exists a positive number ε and an increasing sequence of indices $\{n_\nu\}$ such that γ_{n_ν} is not ε-homologous to 0 in M. Since M is compact and $\dim M\leqslant k$, there exist open sets $V_1, V_2, \ldots, V_r$ covering M such that for every $i=1,2,\ldots,r$ the diameter of V_i is $<\varepsilon$ and that

$$\dim(\overline{V}_i-V)\leqslant k-1.$$

Putting

$$M_0=\sum_{i=1}^{r}(\bar V_i - V_i); \qquad G_i = V_i - \sum_{j<i} \bar V_j$$

we see at once that M_0 is a closed subset of M such that

$$\dim M_0 < k$$

and $G_1, G_2, \dots, G_r$ are disjoint open subsets of M with diameters $<\varepsilon$ and such that

$$M - M_0 = \sum_{i=1}^{r} G_i \quad \text{and} \quad M = \sum_{i=1}^{r} \bar G_i .$$

Let $\delta_n/2$ denote the maximum diameter of simplexes of γ_n. Then $\delta_n \to 0$. By the lemma there exists for almost every n a positive number η_n such that

$$\lim_{n=\infty} \eta_n = 0$$

and that for every subset B of M with diameter $\leqslant \delta_n$ if $B\cdot \bar G_{i_\nu} = 0$ for $\nu = 1,2,\dots,a$, then there exists a point $a \in \bar G_{i_1}\cdot \bar G_{i_2}\cdot \ldots \cdot \bar G_{i_a}$ such that $\varrho(a,B) < \eta_n$.

Now we associate with each vertex p of γ_n a point $\omega_n(p) \in M$ in the following manner:

Denote by $B(p)$ the set composed by all points $q \in M$ such that there exists in γ_n a simplex Δ containing p and q among its vertices. Evidently $p \in B(p)$ and the diameter of $B(p)$ is $\leqslant \delta_n$.

Let $i_1, i_2, \dots, i_a$ be the maximal system of indices $\leqslant r$ such that $B(p)\cdot \bar G_{i_\nu} \neq 0$ for every $\nu = 1,2,\dots,a$. By the lemma there exists a point $a \in \bar G_{i_1}\cdot \bar G_{i_2}\cdot \ldots \cdot \bar G_{i_a}$ such that $\varrho(a,B_n) \leqslant \eta_n$. We put $\omega_n(p) = a$.

Evidently the cycle $\omega_n(\gamma_n)$ is $(\eta_n + \delta_n)$-homologous in M to the cycle γ_n. It follows that the cycles $\omega_n(\gamma_n)$ constitute a true k-dimensional cycle in M and that for almost all ν the cycle $\omega_{n_\nu}(\gamma_{n_\nu})$ is not ε-homologous to 0 in M.

Moreover let us observe that for almost all n each simplex $\omega_n(\Delta)$ of $\omega_n(\gamma_n)$ having a vertex belonging to G_i lies in $\bar G_i$. In fact, by the construction of the function ω_n, if a vertex $\omega_n(p)$ of $\omega_n(\Delta)$ belongs to G_i then also $p \in G_i$ and for every vertex q of Δ it is $\omega_n(q) \in \bar G_i$.

Let us denote by $\varkappa_{n,i}$ the chain made up of all simplexes of $\omega_n(\gamma_n)$ having a vertex belonging to G_i with the same coefficients as in $\omega_n(\gamma_n)$. Then

$$\omega_n(\gamma_n) = \varkappa_{n,0} + \varkappa_{n,1} + \ldots + \varkappa_{n,r}$$

where $\varkappa_{n,0}$ is a chain lying in M_0 and for $i = 1,2,\dots,r$, $\varkappa_{n,i}$ is a chain lying in $\bar G_i$. Then $\partial \varkappa_{n,i}$ is, for $i=1,2,\dots,r$, a $(k-1)$-dimensional cycle lying in the boundary $D_i \subset M_0$ of the set G_i and $\{\partial\varkappa_{n,i}\}$ is a true $(k-1)$-dimensional cycle of M_0. If for every $i = 1,2,\dots,r$, the true cycle $\{\varkappa_{n,i}\}$ were homologous to zero in D_i, then there would exist a true k-dimensional chain $\{\lambda_{n,i}\}$ of D_i such that $\partial\lambda_{n,i} = \partial\varkappa_{n,i}$ for every $n = 1,2,\dots$, $i = 1,2,\dots,r$. Since the diameter of D_i is $<\varepsilon$ the cycle $\lambda_{n,i} - \varkappa_{n,i}$ is ε-homologous to zero in M. It follows that the chain

$$\varkappa_{n_\nu,0} + \lambda_{n_\nu,1} + \ldots + \lambda_{n_\nu,r} = \omega_{n_\nu}(\gamma_{n_\nu}) - (\varkappa_{n_\nu,1} - \lambda_{n_\nu,1}) - \ldots - (\varkappa_{n_\nu,r} - \lambda_{n_\nu,r})$$

would be a k-dimensional cycle lying in M_0 and ε-homologous in M to $\omega_{n_\nu}(\gamma_{n_\nu})$. But $\omega_{n_\nu}(\gamma_{n_\nu})$ is not ε-homologous to zero in M. Hence the cycle $\varkappa_{n_\nu,0} + \lambda_{n_\nu,1} + \ldots + \lambda_{n_\nu,r}$ would not be ε-homologous to zero in M_0 and the true k-dimensional cycle $\{\varkappa_{n,0} + \lambda_{n,1} + \ldots + \lambda_{n,r}\}$ of M_0 would not be homologous to zero in M_0. But this contradicts the assumption $\dim M_0 \leqslant k-1$.

It follows that there exists an index i_0 such that the true $(k-1)$-dimensional cycle $\underline{\delta} = \{\partial\varkappa_{n,i_0}\}$ is not homologous to zero in D_{i_0}. By the hypothesis of the induction there exists a continuous function φ mapping D_{i_0} in S_{k-1} and carrying the true cycle $\underline{\delta}$ into a true cycle homologous on S_{k-1} to a true cycle of the form $\{a_n \cdot \sigma_{k-1,n}\}$, where $a_n \neq 0$ for an infinite collection of the indices n. We may assume that S_{k-1} is the „equator" of the sphere S_k dividing it into two halfspheres H_1 and H_2. Evidently there exists a continuous extension f of φ over M such that $f(G_{i_0}) \subset H_1$ and $f(M - G_{i_0}) \subset H_2$. One readily sees that f maps the true cycle $\{\omega_n(\gamma_n)\}$, hence also the true cycle $\underline{\gamma}$ into the cycle homologous in S_k to the true cycle of the form $\{a_n \cdot \sigma_{k,n}\}$. Hence $\underline{\gamma}$ is spherically essential in M and the proof of the theorem for compact spaces is complete.

Passing to the case in which M is an arbitrary metric space we find a compact subset N of M containing $\underline{\gamma}$ and a continuous function g mapping N in S_k in such a manner that $\underline{\gamma}_g$ is a true cycle not homologous to zero in S_k. Since $\dim M \leqslant k$ there exists [5]) a continuous extension f of g over M with the values lying on S_k. Then f carries $\underline{\gamma}$ into the true cycle $\underline{\gamma}_f = \underline{\gamma}_g$ not homologous to zero in S_k. This proves the theorem.

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[5]) See, for instance, W. Hurewicz and H. Wallman, l. c., p. 83.

8. Homology and extension of mappings. We now prove the following

Lemma. *Let f be a continuous mapping of a compact space X in a metric space Y and let $\underline{\gamma}=\{\gamma_n\}$ be a true cycle lying in X. The true cycle $\underline{\gamma}_f=\{\gamma_{nf}\}$ is homologous to zero in Y if and only if there exists a continuous extension f' of f, with the values belonging to Y, over a compact space $X'\supset X$ such that $\underline{\gamma}\sim 0$ in X'.*

Proof. The condition is sufficient because the relation $\underline{\gamma}\sim 0$ in X' implies the relation $\underline{\gamma}_{f'}=\underline{\gamma}_f\sim 0$ in Y. Thus it remains to prove its necessity.

First let us observe that if h is a homeomorphic mapping of X onto any compact space X_0 and $\underline{\gamma}_f\sim 0$ in Y then it suffices to show that there exists a continuous extension φ of the mapping $fh^{-1}\in Y^{X_0}$ with the values belonging to Y, over a compact space $X_0'\supset X_0$ such that the true cycle $\underline{\gamma}_h$ is homologous to zero in X_0'.

Consequently we may assume that X is a subset of the Hilbert cube Q_ω [6]). By our hypothesis there exists a compact set $Y_0\subset Y$ such that $f(X)\subset Y_0$ and the true cycle $\underline{\gamma}_f$ is homologous to zero in Y_0. Let X_0 be the subset of the compact space $Q_\omega\times Y_0$ composed of all points of the form $x\times f(x)$ with $x\in X$. Putting

$$h(x)=x\times f(x) \quad \text{for every } x\in X$$

we obtain a homeomorphic mapping of X onto X_0. From the remark just made, we infer that it suffices to show that there exists a continuous extension g of fh^{-1} over a compact space $X_0'\supset X_0$ such that the values of g belong to Y and that $\underline{\gamma}_h\sim 0$ in X_0'.

Let us put

$$X_0'=\underset{p}{E}[p=tx\times f(x),\ x\in X \text{ and } 0\leqslant t\leqslant 1]+(0)\times Y_0.$$

Evidently X_0' is a compact space and $X_0\subset X_0'$. Putting

$$g(t\cdot x\times f(x))=f(x) \quad \text{for } x\in X \text{ and } 0\leqslant t\leqslant 1,$$
$$g(0\times y)=y \quad \text{for } y\in Y_0$$

[6]) By the *Hilbert cube* Q_ω we understand the set of points $\{x_n\}$ in Hilbert space whose n-th coordinate x_n satisfies the inequality $0\leqslant x_n\leqslant 1/n$. If $x=\{x_n\}\in Q_\omega$ and $0\leqslant t\leqslant 1$ then $t\cdot x$ denotes the points $\{t\cdot x_n\}\in Q_\omega$. In particular we denote by 0 the point $\{x_n\}$ such that $x_n=0$ for every $n=1,2,\dots$ Consequently $0\cdot x=0$ for every $x\in Q_\omega$. By the known theorem of P. Urysohn each metric separable space is homeomorphic to a subset of Q_ω.

we obtain a continuous function g mapping X_0' into Y. The value of g at the point $h(x)=x\times f(x)$ is equal to $f(x)$. Hence g is an extension of fh^{-1}.

It remains only to show that the true cycle $\underline{\gamma}_h$ is homologous to zero in X_0'. To do it let us observe that putting

$$\varphi_\lambda(x\times f(x))=\lambda\cdot x\times f(x) \quad \text{for } x\in X \text{ and } 0\leqslant\lambda\leqslant 1$$

we obtain a continuous family of the functions $\{\varphi_\lambda\}$ deforming homotopically the set X_0 in the space X_0'. By this homotopic deformation the true cycle $\underline{\gamma}_h=\underline{\gamma}_{h\varphi_1}$ is carried into the true cycle $\underline{\gamma}_{h\varphi_0}$ being the image of the true cycle $\underline{\gamma}_f$ by the homeomorphic mapping $y\to 0\times y$ of the set Y_0 on the set $(0)\times Y_0$. But the true cycle $\underline{\gamma}_f$ is homologous to zero in Y_0, hence $\underline{\gamma}_{h\varphi_0}$ is homologous to zero in $(0)\times Y_0\subset X_0'$. Thus we have

$$\underline{\gamma}_h\sim\underline{\gamma}_{h\varphi_0} \quad \text{in } X_0' \quad \text{and} \quad \underline{\gamma}_{h\varphi_0}\sim 0 \quad \text{in } X_0'$$

and finally $\underline{\gamma}_h\sim 0$ in X_0'. This proves our lemma.

9. Cycles in the functional space. We now come to the main result of this paper.

Theorem. *If a compact space X contains a k-dimensional spherically essential true cycle $\underline{\gamma}=\{\gamma_n\}$, with arbitrary coefficients, then the functional space S_m^X $(m\geqslant k)$ contains a convergent $(m-k)$-dimensional true spherical cycle $\underline{\gamma}^*$ with integral coefficients not homologous to zero in S_m^X.*

In the case for which almost all of the groups $\mathfrak{A}_n$, to which the coefficients of γ_n belong, contain no elements of finite order, the true cycle $\underline{\gamma}^*$ is not weakly homologous to zero in S_m^X.

Proof. By Nrs **6** and **7** there exists a mapping $\varphi\in S_k^X$ such that the true cycle $\underline{\gamma}_\varphi$ is homologous in S_k to a true cycle of the form $\{\alpha_n\cdot\sigma_{k,n}\}$ where $\alpha_n\in\mathfrak{A}_n$ and $\alpha_n\neq 0$ for an infinite set of indices n. Putting

$$f(x\times y)=\varphi(x)\times y \quad \text{for } x\times y\in X\times S_{m-k},$$

we obtain a continuous mapping f of the product $X\times S_{m-k}$ onto the m-dimensional orientable manifold $S_k\times S_{m-k}$ such that the true m-dimensional cycle $\{\gamma_n\times\sigma_{m-k,n}\}$ is carried by f into an m-dimensional true cycle homologous in $S_k\times S_{m-k}$ to the true cycle $\{\alpha_n\cdot\sigma_{k,n}\times\sigma_{m-k,n}\}$.

Now let us consider a continuous mapping ψ of the m-dimensional orientable manifold $S_k \times S_{m-k}$ onto the sphere S_m carrying the true m-dimensional convergent cycle $\{\sigma_{k,n} \times \sigma_{m-k,n}\}$ with integral coefficients into a convergent true cycle homologous in S_m to the cycle $\{\sigma_{m,n}\}$. We readilly see that ψf maps the product $X \times S_{m-k}$ into S_m in such a manner that it carries the true cycle $\{\gamma_n \times \sigma_{m-k,n}\}$ into a true cycle homologous in S_m to the true cycle $\{\alpha_n \cdot \sigma_{m,n}\}$. If we assign to each point $y \in S_{m-k}$ the mapping $g_y \in S_m^X$ defined by the formula

$$g_y(x) = \psi f(x \times y)$$

then we obtain a continuous mapping g of the sphere S_{m-k} into the space S_m^X. The mapping g carries the convergent true $(m-k)$-dimensional cycle $\{\sigma_{m-k,n}\}$ into a convergent true $(m-k)$-dimensional spherical cycle $\underline{\gamma}^*$ lying in S_m^X and having integral coefficients.

Were $\underline{\gamma}^*$ homologous to zero in S_m^X then, by the lemma of Nr **8**, there would exist a compact space $Q \supset S_{m-k}$ such that the true cycle $\{\sigma_{m-k,n}\}$ is homologous to zero in Q, and a continuous extension g' of g over Q, with the values belonging to S_m^X. The mapping g', being an extension of g, carries the true cycle $\{\sigma_{m-k,n}\}$ into $\underline{\gamma}^*$ and it assigns to each point $y \in Q$ a mapping $g'(y) = \varphi_y \in S_m^X$. Putting

$$\vartheta(x \times y) = \varphi_y(x) \quad \text{for} \quad x \times y \in X \times Q$$

we obtain a continuous function ϑ mapping $X \times Q$ into S_m. The mapping ϑ coincides in the set $X \times S_{m-k} \subset X \times Q$ with the mapping ψf and consequently it maps the m-dimensional true cycle $\{\gamma_n \times \sigma_{m-k,n}\}$ onto a true cycle homologous to $\{\alpha_n \cdot \sigma_{m,n}\}$ in S_m.

But the relation

$$\{\sigma_{m-k,n}\} \sim 0 \quad \text{in} \quad Q$$

implies

$$\{\gamma_n \times \sigma_{m-k,n}\} \sim 0 \quad \text{in} \quad X \times Q.$$

It follows that ϑ maps $\{\gamma_n \times \sigma_{m-k,n}\}$ into a true cycle homologous to zero in S_m. Hence

$$\{\alpha_n \cdot \sigma_{m,n}\} \sim 0 \text{ in } S_m$$

and finally $\alpha_n = 0$ for almost all indices n, contrary to our assumption.

Let us assume now that almost all of the groups $\mathfrak{A}_n$ contain no elements of finite order. If $\underline{\gamma}^*$ were weakly homologous to zero in S_m^X then there would exist an integer $n_0 \neq 0$ such that

$$n_0 \cdot \underline{\gamma}^* \sim 0 \quad \text{in} \quad S_m^X.$$

Repeating the reasoning just applied we infer that there would exist a compact set $Q \supset S_{m-k}$ such that

$$\{n_0 \cdot \sigma_{m-k,n}\} \sim 0 \quad \text{in} \quad Q$$

and a continuous function ϑ mapping $X \times Q$ into S_m in such a manner that it carries the m-dimensional true cycle $\{\gamma_n \times n_0 \cdot \sigma_{m-k,n}\}$ into a true cycle homologous to $\{n_0 \cdot \alpha_n \cdot \sigma_{m,n}\}$ in S_m. It follows that

$$\{n_0 \cdot \alpha_n \cdot \sigma_{m,n}\} \sim 0 \text{ in } S_m$$

and consequently $n_0 \cdot \alpha_n = 0$, hence also $\alpha_n = 0$ for almost all indices n. But this contradicts our assumption.

Combining the last theorem with the theorem of Nr **7** we obtain the following

Corollary. *If X is a compact space such that $\dim X \leq k$ and $p^k(X) > 0$ then $p^{m-k}(S_m^X) > 0$ for $m = k, k+1, \ldots$*

Problem 1. *Let X be a compact space of dimension $\leq k$ and such that every true k-dimensional cycle of X is homologous to zero in X. Is it true that $p^{m-k}(S_m^X) = 0$ for $m = k, k+1, \ldots$?*

Problem 2. *Let X be a compactum of dimension $\leq k$. Do the homological properties of the space S_m^X (for $m = k, k+1, \ldots$) depend only on the homological properties of X?*

Problem 3. *Is the space $S_2^{S_1}$ unicoherent?*

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