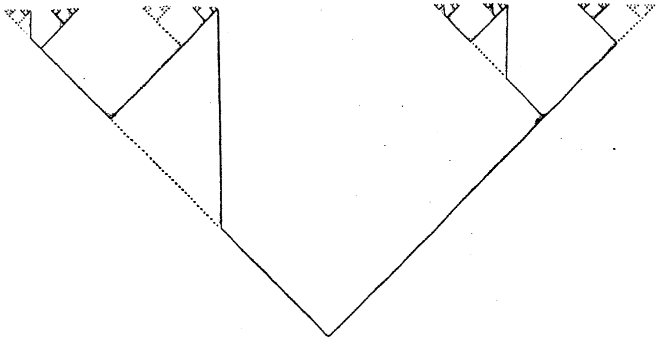

Enfin, tous les points de Γ sont d'ordre 1 dans Ψ. En effet, considérons, pour un point quelconque $x \in \Gamma$, l'ensemble $Y_{m_1 m_2 \dots m_i}$ contenant x et ayant le diamètre aussi petit que l'on veut. Le sommet $s_{m_1 m_2 \dots m_i}$, qui est la frontière de cet ensemble dans Δ et — comme on l'a vu — dans $\Delta - Y$, est d'ordonnée inférieure à celles de tous les $R(g)$ ajoutés à lui; ce sommet est donc disjoint d'eux. Leur diamètre n'augmentant pas, par leur définition, celui de $Y_{m_1 m_2 \dots m_i}$, leur somme avec cet ensemble, soit Σ, est encore de diamètre aussi petit que l'on veut et sa frontière dans Ψ se réduit encore au même sommet (puisque tous les autres $R(g)$ ajoutés à $\Delta - Y$ sont, dans leur ensemble, à la distance positive de Σ, l'ensemble $C_{m_1 m_2 \dots m_i}$ étant fermé-ouvert dans $\mathcal{C}$). Le point x est donc contenu dans un $\Sigma \cdot \Psi$ de diamètre arbitrairement petit et de frontière composée d'un point. L'ordre 1 de tout point $x \in \Gamma$ est ainsi établi.

Wrocław, le 14 mars 1953.

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Correction to "On Real-Valued Functions in Topological Spaces"

(Fund. Math. **38** (1951), p. 85—91)

By

M. Katětov (Praha)

There are simple counterexamples showing that the lemma in [1], p. 86 is false. I am indebted to C. H. Dowker for calling my attention to this fact, as well as for pointing out that the extension to normal spaces of Hahn's insertion theorem (cf. Theorem 1 of [1]) was announced by Hing Tong in 1948 (Bulletin Amer. Math. Soc. **54**, p. 65).

To state the correct propositions, some additional definitions are necessary.

If ϱ is a binary relation in R, then $\bar{\varrho}$ is defined as follows: $x\bar{\varrho}y$ if and only if $y\varrho v$ implies $x\varrho v$, and $u\varrho x$ implies $u\varrho y$ for any $u \in R$, $v \in R$. Clearly, $\bar{\varrho}$ is always transitive.

If $M \subset R$, $N \subset R$, then $M\varrho N$ means that $x\varrho y$ for any $x \in M$, $y \in N$.

We shall say that ϱ possesses *property* (I) if the following assertion holds: if $M \subset R$, $N \subset R$ are countable, and there exist $a \in R$, $b \in R$ such that $M\bar{\varrho}a$, $a\varrho N$, $M\varrho b$, $b\bar{\varrho}N$, then there exists $c \in R$ such that $M\varrho c$, $c\varrho N$.

We shall say that ϱ possesses *property* (L) if, for any finite $A \subset R$, there exist elements $a \in R$, $b \in R$ such that 1^0 $A\bar{\varrho}a$, and $a\varrho x$ whenever $A\varrho x$; 2^0 $b\bar{\varrho}A$, and $x\varrho b$ whenever $x\varrho A$.

Lemma 1. *Let p be a set and let R denote the set of all subsets of p. Let ϱ be a binary relation in R such that 1^0 ϱ has the Interpolation Property; 2^0 $x \subset y \subset p$ implies $x\bar{\varrho}y$; 3^0 $x\varrho y$ implies $x \subset y$. Then ϱ possesses properties (I) and (L).*

Lemma 2. *Let a binary relation ϱ in R possess properties (I) and (L). Let T be countable and let τ be a transitive irreflexive relation in T. If $g \in R^T$, $h \in R^T$, $h\varrho^\tau g$, $h\bar{\varrho}^\tau h$, $g\bar{\varrho}^\tau g$, then there exists $f \in R^T$ such that $h\,\varrho^\tau f$, $f\bar{\varrho}^\tau f$, $f\varrho^\tau g$.*

The relations denoted by ϱ in the proofs of Theorems 1 and 3 in [1] satisfy, evidently, conditions (1)-(3) of Lemma 1 and possess, therefore, properties (I) and (L). Since, clearly, we have $G\bar{\varrho}^\tau G$, $H\bar{\varrho}^\tau H$, for the transformations G, H occurring in the proofs, we may apply Lemma 2 instead of the incorrect lemma of [1], p. 86 without making any other changes in the proofs in question.

Proof of Lemma 1. As for property (L), given a finite set $A=\{a_1,\ldots,a_n\}\subset R$, we have only to set $a=\sum_{i=1}^{n}a_i$, $b=\prod_{i=1}^{n}a_i$. Then, if $x\in R$, $A\varrho x$, there exists, by the Interpolation Property, $y\in R$ such that $A\varrho y$, $y\varrho x$; we have $a_i\subset y$ $(i=1,\ldots,n)$, $a\subset y$, $a\bar{\varrho}y$, $a\varrho x$. Similarly, $x\varrho A$ implies $x\varrho b$.

To show that ϱ has property (I), suppose that $a_i\in R$, $b_j\in R$, $a\in R$, $b\in R$, $a_i\bar{\varrho}a$, $a\varrho b_j$, $b\bar{\varrho}b_j$, $a_i\varrho b$ $(i,j=1,2,\ldots)$. We have to find $c\in R$ such that $a_i\varrho c$, $c\varrho b_j$ $(i,j=1,2,\ldots)$.

Suppose that, for some $n=1,2,\ldots$, we have defined $c_k\in R$, $d_k\in R$ $(k=1,\ldots,n-1)$ such that

$$(\mathrm{Q}_n)\qquad a_i\varrho c_i,\quad d_j\varrho b_j,\quad a\varrho d_j,\quad c_i\varrho b,\quad c_i\varrho d_j,$$

where $i,j=1,2,\ldots,n-1$.

We have $a_n\bar{\varrho}a$ and $a\varrho d_j$, hence $a_n\varrho d_j$ $(j=1,2,\ldots,n-1)$. Therefore, by the Interpolation Property, there exists $c_n\in R$ such that $a_n\varrho c_n$, $c_n\varrho b$, $c_n\varrho d_j$ $(j=1,2,\ldots,n-1)$. Since $a\varrho b_n$, $c_i\varrho b_n$ $(i=1,2,\ldots,n-1)$, $c_n\varrho b_n$ (this follows from $c_n\varrho b$, $b\bar{\varrho}b_n$), there exists, by the Interpolation Property, $d_n\in R$ such that $a\varrho d_n$, $c_i\varrho d_n$ $(i=1,2,\ldots,n)$, $d_n\varrho b_n$. Thus we have defined c_n, d_n in such a way that (Q_{n+1}) holds true. Therefore, by induction, there exist $c_k\in R$, $d_k\in R$ $(k=1,2,\ldots)$ such that (Q_n) holds for $n=1,2,\ldots$

Now set $c=\sum_{k=1}^{\infty}c_k$. Then $c_k\subset c$, $c_k\bar{\varrho}c$, hence $a_k\varrho c$; for any $j=1,2,\ldots$, we have $c_k\varrho d_j$ $(k=1,2,\ldots)$, hence $c_k\subset d_j$, $c\subset d_j$, $c\bar{\varrho}d_j$, and therefore $c\varrho b_j$. This completes the proof.

Proof of Lemma 2. Let all $t\in T$ be arranged in a sequence $\{t_n\}$, $t_m\neq t_n$ for $m\neq n$. Let T_n denote the set of all t_k, $k<n$, let U_n denote the set of all $t\in T$, $t\tau t_n$, and let V_n denote the set of all $t\varepsilon T$, $t_n\tau t$. Suppose (which is, clearly, true for $n=2$) that, for some $n=2,3,\ldots$ there exist $c_k\in R$, $1\leqslant k<n$, such that if we set $f(t_k)=c_k$, then

$$(\mathrm{Z}_n)\qquad \begin{array}{c} t\tau t_k,\ k<n \text{ implies } h(t)\varrho f(t_k);\\ t_k\tau t,\ k<n \text{ implies } f(t_k)\varrho g(t);\\ t_i\tau t_j,\ i<n,\ j<n \text{ implies } f(t_i)\varrho f(t_j).\end{array}$$

Set $M=f(T_nU_n)+h(U_n)$, $N=f(T_nV_n)+g(V_n)$, $A=f(T_nU_n)+\big(h(t_n)\big)$, $B=f(T_nV_n)+\big(g(t_n)\big)$. Then (Z_n) implies, τ being transitive, $M\varrho B$, $A\varrho N$. Property (L) of ϱ implies the existence of $a\in R$, $b\in R$ such that 1° $A\bar{\varrho}a$, and $a\varrho x$ whenever $A\varrho x$; 2° $b\bar{\varrho}B$, and $x\varrho b$ whenever $x\varrho B$. Clearly, $M\varrho b$, $a\varrho N$. Since $h(U_n)\bar{\varrho}h(t_n)$ (this follows from $h\bar{\varrho}^{\tau}h$), $h(t_n)\varepsilon A$, $A\bar{\varrho}a$, $M\subset A+h(U_n)$, we have $M\bar{\varrho}a$; since $g(t_n)\bar{\varrho}g(V_n)$, $g(t_n)\in B$, $b\bar{\varrho}B$, we have $b\bar{\varrho}N$. Hence, by property (I), there exists $c_n\in R$ such that $M\varrho c_n$, $c_n\varrho N$. Setting $f(t_n)=c_n$, we see at once that (Z_{n+1}) holds. Therefore, by induction, there exist c_k $(k=1,2,\ldots)$ such that, with $f(t_k)=c_k$, (Z_n) holds for $n=2,3,\ldots$ This proves the lemma.

Remarks. 1° If we do not want to formulate explicitly (as has been done in Lemmata 1 and 2) the underlying common abstract kernel of Theorems 1 and 3 of [1], then it is, of course, possible to render their proofs, each separately, more perspicuous by applying the argument of Lemmata 1 and 2 to definite, for each case, relations ϱ.

2° It is to be noted that, in a recent paper [2] by Hing Tong, he proves his result announced in 1948. The proof is different from that given by the present author; it rests on a result concerning lattices which is simpler than Lemmata 1 and 2 of the present article.

3° In [1], p. 86, line 13 from bottom, read $(z,x)\in U$ instead of $(z,x)\in\mathfrak{U}$; p. 87, line 2 from bottom read $y\in\operatorname{Int}F(t_2)-\overline{F(t_1)}$ instead of $y\in\operatorname{Int}\overline{F(t_2)}-F(t_1)$; p. 91, l. 22-23, instead of "therefore, $|F(x)-F(y)|\leqslant\varepsilon$ whenever $(x,y)\in U$", read: "therefore, $F(x)-F(y)<\varepsilon$ whenever $(x,y)\in U$; since we may suppose that $(x,y)\in U$ implies $(y,x)\in U$, we have $|F(x)-F(y)|<\varepsilon$ whenever $(x,y)\in U$".

References

[1] M. Katětov, *On Real-Valued Functions in Topological Spaces*. Fundamenta Mathematicae **38** (1951), p. 85—91.

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