L'inégalité (20), appliquée à la fonction $f_n(x)$, donne pour $|h|\leqslant l$,

$$|f_n(x+h)-f_n(x)|\leqslant M_n\omega(|h|),$$

où

$$M_n=K\alpha_n\beta_n\gamma(2l\beta_n^{-1})<2Ksh_n^{-1}n\omega_1(2h_nls^{-1})\gamma(2h_nls^{-1})\to 0.$$

La norme $\|f_n\|$ tend vers 0, car on a de plus

$$2Kl\alpha_n<M_n\omega(2l\beta_n^{-1})\to 0.$$

Mais, d'autre part,

$$|U_n(f_n;x)|=\frac{|f_n(x+h_n)-f_n(x)|}{\omega_1(2h_n)}=n|\varphi(\beta_nx+s)-\varphi(\beta_nx)|,$$

et on démontre, comme au cas du théorème 1, à l'aide d'un théorème cité [8]), que la suite

$$\{U_n(f_n;x)\}$$

ne tend pas asymptotiquement vers 0 sur A, contrairement à la propriété 2°. L'ensemble A est par conséquent de mesure 0, donc la propriété 3° et le lemme 1 impliquent la thèse du théorème.

La fonction $\omega_1(h)$ étant donnée d'avance, on peut construire une fonction $\omega(h)$ telle que la condition (6) soit satisfaite. Toutes les fonctions de l'espace $\boldsymbol{L}^\omega$ correspondantes à la fonction $\omega(h)$, à l'exception d'un ensemble de première catégorie, jouissent de la propriété (b). Mais l'espace $\boldsymbol{L}^\omega$, considéré comme sous-ensemble de l'espace $\boldsymbol{C}$ de fonctions continues de période l, est de première catégorie dans $\boldsymbol{C}$. Or, l'argument appliqué pour démontrer le théorème 7 donne le

Théorème 8. *La fonction $\omega_1(h)$ étant fixée, toutes les fonctions de l'espace $\boldsymbol{C}$, à l'exception d'un ensemble de première catégorie, satisfont à la condition* (b).

[8]) l. c. [5]).

(Reçu par la Rédaction le 12. 6. 1951)

On Parseval equation for almost periodic vectors

by

K. MAURIN (Warszawa).

1. Introduction. The Parseval equation is the kernel of H. Weyl's theory of almost periodic vectors[1]). In this paper I give a proof which is much simpler than Weyl's original proof of this basic relation. The proof is based on some quite elementary properties of hermitian operators in Hilbert space.

2. Definitions and notations. We recall some fundamental notions of Weyl's theory[2]). Let $\mathfrak{H}$ be a (non-complete) Hilbert space, i. e. a complex linear space with the scalar product (f,g) defined for all $f, g, h\in\mathfrak{H}$ and such that

$$(f,g)=\overline{(g,f)},\qquad (f+g,h)=(f,h)+(g,h),$$

$$(af,g)=a(f,g)\ \text{for each complex number}\ a,$$

$$(f,f)\neq 0\quad\text{for}\quad f\neq 0,\quad (f,f)\geqslant 0.$$

Following Weyl we suppose that, besides the usual Hilbert norm $\|f\|=\sqrt{(f,f)}$, there is defined another norm $|f|$, called *length* of f, which satisfies the usual conditions:

$$|f+g|\leqslant|f|+|g|;\qquad |af|=|a|\,|f|.$$

The two norms $\|\ \|$ and $|\ |$ are related as follows:

$$\|f\|\leqslant|f|.$$

[1]) H. Weyl, *Almost periodic invariant vector sets in metric vector space*, American Journal of Mathematics 71 (1949), p. 178-205.

[2]) Ibidem, p. 178-180.

Furthermore, let Σ be an abstract group which operates in $\mathfrak{H}$, i. e. with every $\sigma\in\Sigma$ there is associated a linear transformation $f\to f'=\sigma f$ of $\mathfrak{H}$ into itself, such that $(\sigma f,\sigma g)=(f,g)$ and $|\sigma f|=|f|$.

Weyl calls a vector f *almost periodic* if the group Σ is compact in the topology defined by the metric $\varrho_f(\sigma,\tau)=|\sigma f-\tau f|$.

In the sequel the letter f will always denote a fixed almost periodic vector in $\mathfrak{H}$. The group Σ will be considered as the topological group with the metric $\varrho_f(\sigma,\tau)$.

All complex continuous functions ξ on Σ form the (non-complete) Hilbert space Ξ with the usual scalar product

$$(\xi_1,\xi_2)=\int_\Sigma \xi_1(\sigma)\,\overline{\xi_2(\sigma)}\,d\sigma.$$

Weyl introduces the two linear operations F and F^* ($F(\Xi)\subset\mathfrak{H}$, $F^*(\mathfrak{H})\subset\Xi$), defined as follows[3]):

(2.1) $$g=F\xi=\int_\Sigma \xi(\sigma)\,\sigma f\,d\sigma \quad \text{for} \quad \xi\in\Xi,$$

and

(2.2) $$\xi(\sigma)=F^*h=(\sigma f,h) \quad \text{for} \quad h\in\mathfrak{H}.$$

These operations are continuous and conjugated, i. e.

(2.3) $$(Fg,\xi)=(g,F^*\xi) \quad \text{for every} \quad g\in\mathfrak{H},\ \xi\in\Xi.$$

Furthermore let

(2.4) $$\mathfrak{F}=FF^*.$$

Clearly $\mathfrak{F}(\mathfrak{H})\subset\mathfrak{H}$ and $\mathfrak{F}$ is symmetric (=hermitian), i. e.

(2.5) $$(\mathfrak{F}g,h)=(g,\mathfrak{F}h).$$

$\mathfrak{F}$ commutes with σ:

(2.6) $$\mathfrak{F}(\sigma g)=\sigma(\mathfrak{F}g).$$

3. Proof of the Parseval equation. Now we shall prove the following theorem of Weyl:

Theorem. *Let $f\in\mathfrak{H}$ be almost periodic. There is a finite or denumerable sequence of mutually orthogonal finitely dimensional subspaces $\mathfrak{H}_\varkappa\subset\mathfrak{H}$ spanned by mutually orthogonal unit vectors $g_{n_\varkappa+1},\dots,g_{n_{\varkappa+1}}$ and such that*

(i) *$\mathfrak{H}_n$ is invariant, i. e. $\sigma h\in\mathfrak{H}_\varkappa$ for every $h\in\mathfrak{H}_\varkappa$ ($\varkappa=1,2,\dots$),*

(ii) *$f=\sum_\nu (f,g)g_\nu$.*

[3]) Ibidem, p. 185.

The summation index ν runs always through the set of all positive integers ν such that g_ν is defined. In the case where the sequence $\{g_\nu\}$ is infinite, each series $\sum_\nu$ of vectors is always meant to be convergent after the norm $\|\ \|$.

Clearly (ii) implies the Parseval equation

$$\|f^2\|=\sum_\nu |(f,g)|^2.$$

Lemma. *$\mathfrak{F}=FF^*$ is completely continuous*[4]).

It suffices to prove the complete continuity of F^*. Let $\|g\|<L$. If $\xi=F^*g=(\sigma f,g)$ we have

$$|\xi(\sigma)-\xi(\tau)|=|(\sigma f,g)-(\tau f,g)|=|(\sigma f-\tau f,g)|$$

(3.1) $$\leqslant\|g\|\,\|\sigma f-\tau f\|<L|\sigma f-\tau f|=L\varrho_f(\sigma,\tau),$$

(3.2) $$|\xi(\sigma)|=|(\sigma f,g)|\leqslant\|\sigma f\|\,\|g\|<L\|f\|=K,$$

and (3.1) and (3.2) show that the set of all functions F^*g, where $\|g\|<L$, $g\in\mathfrak{H}$, is a set of equicontinuous functions with common bound K. By Arzela's theorem, this set is compact.

We may apply to the operation $\mathfrak{F}$ theorems on completely continuous symmetric operations in (non-complete) Hilbert spaces[5]).

Let $\{\gamma_\varkappa\}$ be the sequence of all (real) eigenvalues of $\mathfrak{F}$, and let $\{\mathfrak{H}_\varkappa\}$ be the sequence of the corresponding eigenspaces, i. e. $\mathfrak{F}g=\gamma_\varkappa g$ for $g\in\mathfrak{H}_\varkappa$. The set of eigenvalues is non empty. The eigenspaces $\mathfrak{H}_\varkappa$ are mutually orthogonal and finitely dimensional.

[4]) An operation U of a normal space X into another (non-complete) normed space Y is said to be *completely continuous* if for every bounded set $A\subset X$, the set $U(A)$ is compact in Y, i. e. every sequence $y_n\in U(A)$ contains a subsequence convergent to an element $y\in Y$.

[5]) F. Rellich, *Spektraltheorie in nichtseparablen Räumen*, Math. Ann. 110 (1935), p. 342. H. J. Ahiezer and J. M. Glazman, *Teoria liniejnych operatorov v gilbertovom prostranstvie*, Moskva-Leningrad 1950, p. 174-188.

We suppose that $\mathfrak{H}_1$ is spanned by mutually orthogonal unit vectors $g_1,\dots,g_{n_2}\in\mathfrak{H}$, $\mathfrak{H}_2$ is spanned by mutually orthogonal unit vectors $g_{n_2+1},\dots,g_{n_3}\in\mathfrak{H}$ etc.

Each vector of the form $\mathfrak{F}g$ has an expansion in Fourier series

$$\mathfrak{F}g=\sum_\nu(\mathfrak{F}g,g)g_\nu.$$

In particular

$$\mathfrak{F}f=\sum_\nu(\mathfrak{F}f,g_\nu)g_\nu=\sum_\nu(f,\mathfrak{F}g_\nu)g_\nu=\sum_\nu(f,g_\nu)\gamma_\nu g_\nu=\sum_\nu(f,g_\nu)\mathfrak{F}g_\nu.$$

Now we extend the space $\mathfrak{H}$ to a complete Hilbert space $\overline{\mathfrak{H}}$. We may assume that F^* and, consequently, $\mathfrak{F}$ are defined on $\overline{\mathfrak{H}}$ (see (2.2) and (2.4)). We have in $\overline{\mathfrak{H}}$

$$\mathfrak{F}f=\sum_\nu(f,g_\nu)\mathfrak{F}g_\nu=\mathfrak{F}\big(\sum_\nu(f,g_\nu)g_\nu\big).$$

Consequently

(3.3) $$f=\sum_\nu(f,g_\nu)g_\nu+h,$$

where $h\in\overline{\mathfrak{H}}$ and $\mathfrak{F}h=0$. Since $0=(\mathfrak{F}h,h)=(FF^*h,h)=(F^*h,F^*h)$, we obtain $F^*h=0$ and, by (2.2),

(3.4) $$(f,h)=0.$$

Hence, by (3.4) and (3.3),

$$(h,h)=(f,h)-\sum_\nu(f,g_\nu)(g_\nu,h)=0$$

since, by (3.3), $(f,g_\nu)=(f,g_\nu)+(h,g_\nu)$, i. e. $(h,g_\nu)=0$.

We infer that $h=0$, i. e., by (3.3),

$$f=\sum_\nu(f,g_\nu)g_\nu$$

which completes the proof of Parseval's equation.

The invariance of $\mathfrak{H}_\varkappa$ follows immediately from (2.6). In fact, if $g\in\mathfrak{H}_\varkappa$, i. e. $\mathfrak{F}g=\gamma_\varkappa g$, then, by (2.6), $\mathfrak{F}\sigma g=\sigma\mathfrak{F}g=\sigma\gamma_\varkappa g=\gamma_\varkappa\sigma g$. Therefore $\sigma g\in\mathfrak{H}_\varkappa$*.

* I am gratefully indebted to R. Sikorski for his valuable suggestions.

(Reçu par la Rédaction le 18. 4. 1952)

Zur Gitterpunktverteilung bei Verschiebungen von Mengen

von

S. HARTMAN (Wrocław).

1. $x=(x_i)$ bedeute einen Punkt oder einen Vektor des Euklidischen Raumes R^n mit den Koordinaten $x_1,\dots,x_n$. Der Würfel $0\leqslant x_i\leqslant 1$ heiße W. $|E|$ bedeute das Lebesguesche Maß der Menge E. Ist $\{a_k\}$ irgendeine Folge, P irgendeine Eigenschaft, N' die Anzahl der a_k mit dieser Eigenschaft und mit $k\leqslant N$, so werde der gegebenenfalls vorhandene Grenzwert $\lim_{N\to\infty}N'/N$ mit $\operatorname{fr}_k(P(a_k))$ bezeichnet; in Worten heißt $\operatorname{fr}_k(P(a_k))$ die *Häufigkeit* derjenigen a_k, welchen die Eigenschaft P zukommt. Wir setzen noch $\overline{\lim}_{N\to\infty}N'/N=\overline{\operatorname{fr}}_k(P(a_k))$.

Es sei nun E eine beliebige bis auf Veschiebungen bestimmte L-meßbare Menge von endlichem Maße in R^n. Kommt bei einer Verschiebung ein in E festgehaltener Punkt A in den Raumpunkt x zu liegen, so deuten wir x als die *Lage* von E. Dann soll

$$g(x)=g(x_1,\dots,x_n)$$

die (vielleicht unendliche) Anzahl der Gitterpunkte in E bedeuten.

2. Man denke sich E in punktfremde L-meßbare Teile I_j $(j=1,2,\dots)$ vom Durchmesser <1 zerlegt. Die Menge T_j der Lagen x von E, bei welchen sich in I_j ein (und dann genau ein) Gitterpunkt befindet, ist offenbar eine Summe von auseinander durch Verschiebung um Vektoren (r_i) $(r_1=0,\dots,r_p=\pm1,\dots,r_n=0;\ p=1,2,\dots$ oder $n)$ entstehenden, zu I_j kongruenten Mengen. Weiter besteht für $s=0,1,2,\dots,\infty$ die Menge $M_s=\underset{x\in W}{E}[g(x)=s]$ aus allen $x\in W$, die für genau s verschiedene j-Werte zu T_j gehören.