On the convergence of functionals representable as integrals over some classes of bounded functions

by

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1. We will denote by $M(\Delta)$ the Banach space composed of the bounded functions $x=x(t)$ in an interval Δ; this interval Δ may be closed or open, finite or not, addition and multiplication by numbers are defined in $M(\Delta)$ as usual, and the norm is

$$\|x\|=\sup_{\Delta}|x(t)|. \tag{1}$$

We shall be concerned with following linear sets in $M(\Delta)$:

(K_1) — the class of functions bounded in $\Delta=\langle a,b\rangle$, continuous everywhere in $\langle a,b\rangle$ except at t_0 (a fixed point in (a,b)), and vanishing at t_0;

(K_2) — the class of bounded and continuous functions in $\Delta=(-\infty,\infty)$;

(K_3) — the class of bounded functions in $\Delta=\langle a,b\rangle$, which vanish at t_0 (a fixed point in (a,b)), and are of finite variation in every interval $\langle a,t_0-\varepsilon\rangle$, $\langle t_0+\varepsilon,b\rangle$;

(K_4) — the class of bounded functions in $\Delta=(-\infty,\infty)$, which are of finite variation in every finite interval.

The above classes are subspaces of the space $M(\Delta)$.

By (K_1^*) will be denoted the (non-linear) class of bounded functions in $\langle a,b\rangle$, discontinuous at one point at most.

Let X be a linear space in which two norms $\|\ \|$ and $\|\ \|^*$ are defined, the first being of B- and the second of B- or F-type. By X_s we shall denote the metric space composed of those elements of X for which $\|x\|\leqslant 1$, the distance being defined as

$$d(x,y)=\|x-y\|^*.$$

[1]) The contents of this paper were presented to the Polish Mathematical Society, Warsaw Section, on the 9th May 1952.

If X_s is a complete space, it is called the *Saks space*. We shall consider the Saks spaces satisfying the following condition:

(Σ_1) If $x_0\in X_s$, $\varrho>0$, then there exists a $\delta>0$ such that $\|x\|^*<\delta$ implies $x=x_1-x_2$ where $x_i\in X_s, \|x_1-x_0\|^*<\varrho$, $\|x_2-x_0\|^*<\varrho$.

The functional $\xi(x)$ in X_s is called *linear* if it is continuous and distributive, i. e. if $\lambda_1x_1+\lambda_2x_2\in X_s$, where $x_1,x_2\in X_s$ and λ_1,λ_2 are rational, implies

$$\xi(\lambda_1x_1+\lambda_2x_2)=\lambda_1\xi(x_1)+\lambda_2\xi(x_2).$$

The following theorems are true in the Saks spaces satisfying the condition (Σ_1):

Theorem A. *If $\xi_n(x)$ is a sequence of linear functionals in X_s, bounded in a set of the second category, there exists a constant M such that $|\xi_n(x)|\leqslant M$ for $n=1,2,\ldots$, $x\in X_s$.*

Theorem B. *If the sequence $\xi_n(x)$ of linear functionals in X_s converges in a set of the second category, it converges in the whole of X_s, its limit is a linear functional, and $d(x_n,0)\to 0$, $x_n\in X_s$ implies $\xi_n(x_n)\to 0$* [2]).

Theorem A implies that the set of points of unboundedness for the sequence $\xi_n(x)$ is either residual or empty, and by Theorem B the set of points of divergence shares this property.

Choosing as X the class (K_1) or (K_2) respectively we can obtain Saks spaces, defining the norm $\|\ \|$ by the formula (1), and the norm $\|\ \|^*$ by

$$\|x\|^*=\sum_{n=1}^{\infty}\frac{1}{2^n}\|x\|_n;$$

here $\|\ \|_n$ is a sequence of pseudonorms defined as follows:

(a) In the case of the space (K_1), choosing t_n', t_n'' so that

$$a<t_n'<t_{n+1}'<t_0<t_{n+1}''<t_n''<b, \qquad t_n'\to t_0, \qquad t_n''\to t_0,$$

we set

$$\|x\|_n=\sup_{\langle a,t_n'\rangle}|x(t)|+\sup_{\langle t_n'',b\rangle}|x(t)|. \tag{2}$$

(b) In the case of the space (K_2) we choose two monotone sequences $t_n'\to-\infty$, $t_n''\to+\infty$ and set

$$\|x\|_n=\sup_{\langle t_n',t_n''\rangle}|x(t)|. \tag{3}$$

[2]) See W. Orlicz, *Linear operations in Saks spaces (II)*, to appear in Studia Mathematica 14.

The Saks space defined above we shall denote respectively by $C^0(a,b)$ and $C^0(-\infty,+\infty)$. The space $C^0(a,b)$ satisfies the condition (Σ_1); this is shown in an another paper[3]). A similar proof can be applied to the space $C^0(-\infty,+\infty)$.

Choosing as X the class (K_3) and (K_4) respectively we can obtain Saks spaces defining the norm $\| \; \|$ by the formula (1), and the norm $\| \; \|^*$ by

$$\|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|x\|_n}{1+\|x\|_n}.$$

The pseudonorms $\|x\|_n$ are defined in the case of the space (K_3) by

$$\|x\|_n = |x(a)| + |x(b)| + \underset{\langle a,t_n'\rangle}{\operatorname{var}}\, x(t) + \underset{\langle t_n'',b\rangle}{\operatorname{var}}\, x(t),$$

t_n' and t_n'' having the same meaning as in (a). In the case of the space (K_4) we define

$$\|x\|_n = \underset{\langle t_n',t_n''\rangle}{\operatorname{var}}\, x(t) + |x(0)|,$$

t_n' and t_n'' having the same meaning as in (b) moreover $t_n'<0, t_n''>0$. These Saks spaces will be denoted by $V^0(a,b)$ and $V^0(-\infty,+\infty)$ respectively; they satisfy the condition (Σ_1).

2. We shall be concerned with the following problem:

If the functions $f_n(t)$ are integrable in (a,b), what are the necessary and sufficient conditions for the convergence of the sequence

$$\int_a^b f_n(t)\,x(t)\,dt \tag{4}$$

for every $x \in (K_i)$, $i=1,2,3,4$?

The theorems obtained below for the integrals (4) can be proved also for Stieltjes integrals. We will return to this question in an another paper.

In the formula (4) a,b are finite when $x \in (K_1),(K_3)$, and infinite as $x \in (K_2),(K_4)$. The integrals (4) are linear functionals over the spaces $C^0(a,b)$, $V^0(a,b)$.

Theorem 1. *The sequence* (4) *is convergent for every* $x \in (K_1)$ *if and only if*

[3]) W. Orlicz, *Linear operations in Saks spaces (I)*, Studia Mathematica 11 (1950), p. 237-272.

(α) $\overline{\lim\limits_{n\to\infty}} \int_a^b |f_n(t)|\,dt < \infty$,

(β) *the functions*

$$\Phi_n(t) = \int_a^t |f_n(\tau)|\,d\tau$$

are equicontinuous at t_0,

(γ) *the functions*

$$F_n(t) = \int_a^t f_n(\tau)\,d\tau$$

converge asymptotically in $\langle a,b\rangle$,

(δ) *the sequence* $F_n(b)$ *converges.*

Proof. Necessity. Suppose the sequence (4) to be convergent in (K_1); then by Theorem A there exists a constant M such that

$$\left|\int_a^b f_n(t)\,x(t)\,dt\right| \leqslant M \tag{5}$$

for every $x(t) \in (K_1)_s$. Now, if $z(t)$ is continuous in $\langle a,b\rangle$ and $\|z\|\leqslant 1$, there exists a sequence $x_n \in (K_1)$ such that $x_n(t)\to z(t)$ for $t\neq t_0$ and $|x_n(t)|\leqslant|z(t)|$; hence (5) is satisfied for every continuous function, and this implies

$$\int_a^b |f_n(t)|\,dt \leqslant M.$$

Let $\tilde{t}_n'$ be an arbitrary increasing sequence tending to t_0, $\tilde{t}_n''$ a decreasing sequence tending to t_0, and $a<\tilde{t}_n'<\tilde{t}_n''<b$. It is easy to show that there exist continuous functions $x_n(t)$ belonging to $(K_1)_s$ such that

$$x_n(t)=0 \quad \text{for} \quad a\leqslant t\leqslant \tilde{t}_n' \quad \text{and} \quad \tilde{t}_n''\leqslant t\leqslant b, \tag{6'}$$

$$\left|\int_{\tilde{t}_n'}^{\tilde{t}_n''} f_n(t)\,x_n(t)\,dt\right| \geqslant \int_{\tilde{t}_n'}^{\tilde{t}_n''} |f_n(t)|\,dt - \frac{1}{n}. \tag{6''}$$

Since $\|x_n(t)\|^*\to 0$, Theorem B implies

$$\int_{\tilde{t}_n'}^{\tilde{t}_n''} |f_n(t)|\,dt = \Phi_n(\tilde{t}_n'') - \Phi_n(\tilde{t}_n') \to 0,$$

and the condition (β) is satisfied. The condition (δ) is satisfied too, for the sequence (4) converges if

$$x(t)=\begin{cases}1 & \text{for} \quad t\neq t_0,\\ 0 & \text{for} \quad t=t_0.\end{cases}$$

Let

$$l(\tau)=\begin{cases}0 & \text{for} \quad a\leqslant\tau\leqslant t,\\ \tau-t & \text{for} \quad t\leqslant\tau\leqslant b.\end{cases}$$

Since

$$\int_a^b l(\tau)f_n(\tau)\,d\tau=(b-t)F_n(b)-\int_t^b F_n(\tau)\,d\tau, \tag{7}$$

and since the sequence on the left hand side of (7) converges by hypothesis, (δ) implies that the sequence

$$\int_t^b F_n(\tau)\,d\tau$$

converges in $\langle a,b\rangle$ to a function $h(t)$. The sequence $F_n(t)$ is bounded by (α) and the variations of $F_n(t)$ are bounded, hence we can extract a sequence $F_{n_i}(t)$ convergent to a function $G(t)$ of bounded variation. Similarly, we can extract from every sequence $F_{p_i}(t)$ a subsequence $F_{p_i'}(t)$ such that $F_{p_i'}(t)$ converges to a function $\overline{G}(t)$ of bounded variation. Since for every t

$$\int_t^b F_{n_i}(\tau)\,d\tau\to\int_t^b G(\tau)\,d\tau,$$

$$\int_t^b F_{p_i'}(\tau)\,d\tau\to\int_t^b \overline{G}(\tau)\,d\tau,$$

$G(t)$ must be equal to $\overline{G}(t)$ at every point of continuity of both functions, i. e. everywhere except a denumerable set. Thus we have proved that every subsequence $F_{p_i}(t)$ contains a sequence convergent almost everywhere to $G(t)$, hence $F_n(t)$ converges asymptotically to $G(t)$ [4]).

[4]) Compare G. Fichtenholz, *Sur les opérations linéaires dans l'espace des fonctions continues,* Bulletin de l'Académie Royale de Belgique 22 (1936), p. 26-33.

Sufficiency. By (α), (γ), (δ) and (7) the sequence (4) converges for every polygonal function. Since these functions compose a dense set in the space of continuous functions, the sequence converges for every continuous function by (α). Now, if $x_0(t)$ is an arbitrary function of (K_1) such that $\|x_0\|\leqslant 1$, we can choose in virtue of (β) a $\delta>0$ such that

$$\int_{t_0-\delta}^{t_0+\delta} |f_n(\tau)|\,d\tau<\varepsilon \quad \text{for} \quad n=1,2,\ldots \tag{8}$$

The function $z(t)$ equal to 0 for $t_0-\delta/2\leqslant t\leqslant t_0+\delta/2$, equal to $x_0(t)$ for $a\leqslant t\leqslant t_0-\delta$ and $t_0+\delta\leqslant t\leqslant b$, and linear in the intervals $\langle t_0-\delta, t_0-\delta/2\rangle, \langle t_0+\delta/2, t_0+\delta\rangle$ is continuous, and the function $y(t)=x_0(t)-z(t)$ satisfies by (8) the inequality

$$\left|\int_a^b y(t)f_n(t)\,dt\right|\leqslant 2\varepsilon \quad \text{for} \quad n=1,2,\ldots$$

Since the sequence (4) converges for $x(t)=z(t)$, then, for sufficiently large m and n,

$$\left|\int_a^b x_0(t)\,[f_n(t)-f_m(t)]\,dt\right|$$

$$\leqslant\left|\int_a^b y(t)\,[f_n(t)-f_m(t)]\,dt\right|+\left|\int_a^b z(t)\,[f_n(t)-f_m(t)]\,dt\right|\leqslant 4\varepsilon+\varepsilon=5\varepsilon;$$

hence the sequence (4) converges for every $x_0(t)\,\epsilon\,(K_1)$.

In a similar way we can prove

Theorem 2. *Let $a=-\infty$, $b=+\infty$. The sequence (4) converges for every $x(t)\,\epsilon\,(K_2)$ if and only if*

(α′) $\quad \overline{\lim_{n\to\infty}} \int_{-\infty}^{+\infty} |f_n(\tau)|\,d\tau<\infty,$

(β′) *for every $\varepsilon>0$ there is an $r>0$ such that*

$$\int_r^{\infty} |f_n(t)|\,dt<\varepsilon,$$

and

$$\int_{-\infty}^{-r} |f_n(t)|\,dt<\varepsilon \quad \textit{for} \quad n=1,2,\ldots,$$

(γ') *the functions $F_n(t)$ converge asymptotically in every finite interval,*

(δ') *the sequence $F_n(t)$ converges at the point $t=\infty$.*

Theorem 3. *The sequence* (4) *converges for every $x(t)\in(K_1^*)$ if and only if the condition* (α) *and the following conditions are satisfied:*

(β'') *the functions $\Phi_n(t)$ are equicontinuous in $\langle a,b\rangle$,*

(γ'') *the sequence $F_n(t)$ converges uniformly in $\langle a,b\rangle$.*

Proof. The equicontinuity of $\Phi_n(t)$ in (a,b) is implied by Theorem 1. Since bounded functions, continuous except at a or b, belong to the class (K_1^*) one can prove in the same way as in Theorem 1 that the functions $\Phi_n(t)$ are also equicontinuous at the boundary points of the interval. The inequality

$$|F_n(t+h)-F_n(t)|\leqslant\int_t^{t+h}|f_n(\tau)|\,d\tau=|\Phi_n(t+h)-\Phi_n(t)|$$

implies equicontinuity in $\langle a,b\rangle$ of the functions $F_n(t)$. Hence, as the functions $F_n(t)$ converge asymptotically to a function $G(t)$, we infer that the functions must converge uniformly in $\langle a,b\rangle$ to a continuous function $\overline{G}(t)$ equivalent to $G(t)$, and this implies (γ'').

To prove that the conditions are sufficient we apply Theorem 1, and in the case of functions discontinuous at a boundary point of $\langle a,b\rangle$, we prove the convergence of the sequence (4) in the same way as in the proof of Theorem 1.

Theorem 4. *The sequence* (4) *converges for every $x(t)\in(K_3)$ if and only if the condition* (β) *and the following are satisfied:*

(λ) $\overline{\lim\limits_{n\to\infty}}\max\limits_{\langle a,b\rangle}|F_n(t)|<\infty$,

(μ) *the functions $F_n(t)$ converge in the interval $\langle a,b\rangle$.*

Proof. Necessity. If the sequence (4) is convergent for every $x(t)\in(K_3)$, it converges for every function $x(t)$ of finite variation in $\langle a,b\rangle$; then we have (λ) and (μ) by a well known theorem of Lebesgue[5]). In the same way as in the proof of Theorem 1 we can choose such functions $x_n(t)\in(K_3)_s$ that the conditions ($6'$) and ($6''$) are satisfied and arguing as in that case we obtain (β).

[5]) H. Lebesgue, *Sur les intégrales singulières*, Annales de Toulouse 1 (1909), p. 25-117.

Sufficiency. Given $x_0(t)\in(K_3)$, $\|x_0\|\leqslant 1$, choose $\delta>0$ so that (8) is satisfied and consider the function $z(t)$ defined as in the proof of Theorem 1. Plainly $z(t)\in(K_3)_s$, thus by (λ), (μ) and the theorem of Lebesgue we see that the sequence (4) converges for $x(t)=z(t)$. We finish the proof arguing in the same way as in the proof of Theorem 1.

Similarly one can prove

Theorem 5. *Let $a=-\infty$, $b=+\infty$. The sequence* (4) *converges for every $x(t)\in(K_4)$ if and only if the conditions* (β') *and the following are satisfied:*

(λ') $\overline{\lim\limits_{n\to\infty}}\sup\limits_{(-\infty,\infty)}|F_n(t)|<\infty$,

(μ') *the functions $F_n(t)$ converge everywhere.*

3. Now, we shall present some applications of the preceding theorems.

Theorem 6. *Let the functions $K_n(s,t)$ be integrable in $\langle a,b\rangle$ for every $s\in\langle a,b\rangle$ and let us write*

$$S_n(x,s)=\int_a^b K_n(s,t)x(t)\,dt, \tag{9}$$

where $x(t)\in M(\Delta)$ and is measurable. Assume that $x(t)\in M(\Delta)$ ($x(t)$ being measurable) implies

$$S_n(x,s)\overset{\text{as}}{\to}x(s). \tag{10}$$

Then there exists a set $A\subset\langle a,b\rangle$ of mesure $b-a$ such that:

(a) *for every $s_0\in A$ there exists a bounded function $x(t)$, continuous except at most at s_0, for which the sequence $S_n(x,s_0)$ diverges;*

(b) *for every sequence $s_i\in A$ there exists a bounded function $x(t)$, continuous except at most at s_i, and such that the sequence $S_n(x,s_i)$ diverges for $i=1,2,\ldots$;*

(c$'$) *the functions $x(t)$ considered in* (a) *compose a residual set in the Saks space $C^0(a,b)$ (if we choose $t_0=s_0$);*

(c$''$) *the functions considered in* (b) *compose a residual set in the Banach space $M^*(\Delta)$ consisting of the functions bounded in $\Delta=\langle a,b\rangle$, continuous except at the points $s_i\in A$ (s_i being fixed), with norm defined by* (1).

Proof. By (10) we infer that for every measurable $x(t)\in M(\Delta)$ there is a sequence of indices n_i such that

$$S_{n_i}(x,s)\to x(s)$$

almost everywhere. Then there exists a set $A\subset\langle a,b\rangle$ of measure $b-a$ and such a sequence m_i of indices that, for $s_0\in A$ and for every rational c,d such that $c<s_0<d$, we have

$$\int_c^d K_{m_i}(s_0,t)\,dt\to 1.$$

Hence the condition (β) of Theorem 1 is not satisfied for $s_0\in A$ and this implies (a), and implies in turn (c′) in virtue of the remark made with regard to Theorem A and B.

The sequence $S_n(x,s)$ may be considered as a sequence of linear functionals in the Banach space $M^*(\Delta)$, divergent in virtue of (a) in a non-empty set of elements in $M^*(\Delta)$ if $s=s_i\in A$. Hence the set of divergence of this sequence is residual if $s=s_i$. Hence we get (c″) and this implies (b).

Remark. Replacing in Theorem 6 the hypothesis (10) by the assumption that $S_n(x,s)\to x(s)$ in $\langle a,b\rangle$ for every function continuous in $\langle a,b\rangle$, we can state that $A=\langle a,b\rangle$.

Theorem 6 and the above remark show that the problem of local divergence of classical singular integrals of several types becomes trivial if the functions considered are allowed to be discontinuous even at one point.

Theorem 7. *Let $\Phi\{\varphi_i(t)\}$ be an orthonormal system in $\langle a,b\rangle$, complete in L^2, and let denote by $S_n(x,s)$ the n-th sum of the corresponding development of the function $x(t)$. Then there exists a set $A\subset\langle a,b\rangle$ of measure $b-a$ such that the statements* (a)-(c″) *of Theorem 6 are satisfied.*

An analogous statement holds if we replace the sums $S_n(x,s)$ by their transforms corresponding to a row-finite Toeplitz method of summation.

Proof. We observe that if we write

$$K_n(s,t)=\sum_{i=1}^{n}\varphi_i(s)\,\varphi_i(t),$$

$S_n(x,s)$ is represented by the formula (9), the completeness of the system Φ in L^2 implies that

$$\|S_n(x,s)-x(s)\|_2\to 0$$

for every measurable function $x(t)\in M(\Delta)$, and this implies already (10).

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