The Fredholm Theory of Linear Equations in Banach spaces

by

T. LEŻAŃSKI (Warszawa).

In the first part of this paper[1]) we shall consider:

1. a fixed Banach space X;

2. a fixed closed linear subspace Ξ of the conjugate space $\bar{X}$ of all linear[2]) functionals on X;

3. a fixed closed linear subspace $\mathfrak{K}$ of the Banach space of all linear operations transforming Ξ into Ξ;

4. a fixed linear operation $T\epsilon\mathfrak{K}$;

5. a fixed linear functional F defined on K.

We shall use the following notations: If $x\epsilon X$ and $\varphi\epsilon\Xi$ then φx is the value of the functional φ at the point x. If K is a linear operation of Ξ into $\bar{X}$ (or: into Ξ), then $K\varphi$ is the functional associated with φ by K (i. e. the value of the mapping K at the point φ). Consequently $K\varphi x$ is the value of the functional $K\varphi$ at the point x. Obviously, the expression $K\varphi x$ can be interpreted as a bilinear functional on the Cartesian product $\Xi\times X$. Conversely, each bilinear functional $K\varphi x$ on $\Xi\times X$ can be interpreted as a linear operation K of Ξ into $\bar{X}$ (in particular, into Ξ). Subsequently we shall always speak about linear operations of Ξ into Ξ, although these operations will often be defined by expressions which are bilinear functionals on $\Xi\times X$.

The superposition of two operations $K_1,K_2\epsilon\mathfrak{K}$ will be denoted by K_1K_2. Of course, the expression $K_1K_2\varphi x$ should be read: $(K_1K_2)\varphi x$.

We shall suppose that the following conditions (**K**) and (**F**) are satisfied:

(**K**) The identical mapping I of Ξ into Ξ belongs to $\mathfrak{K}$. If $K_1,K_2\epsilon\mathfrak{K}$, then $K_1K_2\epsilon\mathfrak{K}$. If $K\epsilon\mathfrak{K}$, and if $x_0\epsilon X$ and $\varphi_0\epsilon\Xi$ are fixed, then the linear operation M,

$$M\psi y=K\psi x_0\cdot\varphi_0 y \quad \text{for} \quad y\epsilon X \quad \text{and} \quad \psi\epsilon\Xi,$$

also belongs to $\mathfrak{K}$, and

(**F**) $F(M)=KT\varphi_0x_0$.

The condition (**F**) yields that in some cases the values of the functional F are completely determined by T (see the example IV, p. 268). However, in general, F need not be uniquely determined by T (see the example **A** and the footnote[5]) on p. 262). Conversely, T always is completely determined by F.

Under hypotheses (**K**) and (**F**), we shall examine the linear equation

$$\varphi+KT\varphi=\psi_0, \tag{1}$$

where $K\epsilon\mathfrak{K}$, $\psi_0\epsilon\Xi$. The solution φ should also be in Ξ.

We shall develop the theory of the equation (1) in a way completely analogous to the Fredholm theory of integral equations. For instance, we shall define an expression D determined by K,T, and F only, such that the equation (1) has a (necessarily unique) solution for every $\psi_0\epsilon\Xi$ if and only if the number D is not equal to 0. If $D=0$, then the homogeneous equation

$$\varphi+KT\varphi=0 \tag{2}$$

has non-trivial solutions. We shall define a sequence of multilinear functionals

$$D_p\begin{pmatrix}\varphi_1,\ldots,\varphi_p\\ x_1,\ldots,x_p\end{pmatrix}$$

where $\varphi_1,\ldots,\varphi_p\epsilon\Xi$, $x_1,\ldots,x_p\epsilon X$, $p=1,2,\ldots$, such that the (necessarily finite) dimension of the space of all solutions of (2) is the smallest integer p such that D_p is not identically equal to 0. We shall also determine under what conditions relating to ψ_0 the equation (1) has a solution.

Clearly D should be called the *determinant* of the equation (1) by analogy with the finitely dimensional case. The functionals D_p are the analogue of the subdeterminants of the order p in the Fredholm theory of integral equations.

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[1]) Presented to the State Institute of Mathematics in Warsaw (Group of Functional Analysis) at May 1952.

[2]) The word "linear" always means "additive and continuous".

In the second part of this paper we shall specialize the spaces X and Ξ. The following cases will be under consideration:

$$X=\boldsymbol{L}^q,\ \boldsymbol{l}^q,\ \boldsymbol{L},\ \boldsymbol{M},\ \boldsymbol{V},\ \boldsymbol{l},\ \boldsymbol{m},$$

and

$$\Xi=\boldsymbol{L}^p,\ \boldsymbol{l}^p,\ \boldsymbol{M},\ \boldsymbol{L},\ \boldsymbol{C},\ \boldsymbol{m},\ \boldsymbol{l} \qquad (1/p+1/q=1)$$

respectively.

In the cases of $X=\boldsymbol{l}[\boldsymbol{m}]$ and $\Xi=\boldsymbol{m}[\boldsymbol{l}]$ the determinant D mentioned above is a generalization of Koch's determinant of an infinite system of linear equations with infinitely many variables (Koch's definitions [3]) are introduced under more restrictive conditions). We shall also show that, in the cases mentioned above, the determinant $D=D(I+T)$ of the equation

$$(I+T)\varphi=\psi_0 \tag{3}$$

is a multiplicative functional, i. e.

$$D\big((I+T)(I+T')\big)=D(I+T)\cdot D(I+T').$$

This equation was earlier obtained by Koch under more restrictive conditions.

The infinite determinant D has also many other properties of the usual finite determinants. In particular, a formula analogous to the Cramer formula will be found for the solution of (3).

The existence of the functional F satisfying the condition (F) plays the fundamental part in our theory of linear equations. The functional F may be conceived as a generalization of the trace of a square matrix. In fact, in the case of $X=\boldsymbol{l}[\boldsymbol{m}]$, $\Xi=\boldsymbol{m}[\boldsymbol{l}]$, linear operations K,T can be interpreted as certain infinite matrices, and $F(K)$ is then the trace of the matrix KT.

The restriction that we shall examine linear equations in conjugate spaces is not essential since each Banach space X can be interpreted as a closed subset of the space $\bar{\bar{X}}$ conjugate to $\bar{X}$.

A similar Fredholm theory of linear equations in Banach spaces was developed by A. F. Ruston [4]). The theory of Ruston makes use of the notion of cross-spaces, which does not appear in my paper.

[3]) H. v. Koch, *Sur la convergence des déterminants infinis*, Rendiconti del Circolo Mathematico di Palermo 28 (1909), p. 255-266.

[4]) Proc. London Math. Soc. 2 (1951), p. 109-124, and (3) 1 (1951), p. 327-384.

I. The general Fredholm theory.

In this part $X,\Xi,\mathfrak{K},T,F$ will have the meaning mentioned in the introduction. We suppose that conditions (K) and (F) are satisfied.

We adopt the following convenient notation. If

$$B(\varphi_1,\dots,\varphi_n,x_1,\dots,x_n)$$

is a functional ($\varphi_1,\dots,\varphi_n\in\Xi$, $x_1,\dots,x_n\in X$) such that, for some fixed $\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_n,x_1,\dots,x_{j-1},x_{j+1},\dots,x_n$, the linear operation M,

$$M\varphi_i x_j=B(\varphi_1,\dots,\varphi_n,x_1,\dots,x_n),$$

belongs to $\mathfrak{K}$, then $F_{\varphi_i,x_j}\{B(\varphi_1,\dots,\varphi_n,x_1,\dots,x_n)\}$ will denote the number $F(M)$. Obviously $F_{\varphi_i,x_j}\{B(\varphi_1,\dots,\varphi_n,x_1,\dots,x_n)\}$ is, in general, a function of variables

$$\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_n\in\Xi \qquad\text{and}\qquad x_1,\dots,x_{j-1},x_{j+1},\dots,x_n\in X;$$

however it does not depend on φ_i and x_j which are bound variables and can be replaced by other letters different from the remaining φ and x (compare e. g. the bound variable t in the expression $\int f(t)\,dt$).

Obviously

$$\begin{aligned}|F_{\varphi_i,x_j}\{B(\varphi_1,\dots,\varphi_n,x_1,\dots,x_n)\}|&=|F(M)|\leqslant\|F\|\cdot\|M\|\\&=\|F\|\cdot\sup_{\|\varphi_i\|\leqslant 1}\ \sup_{\|x_j\|\leqslant 1}|B(\varphi_1,\dots,\varphi_n,x_1,\dots,x_n)|.\end{aligned}\tag{i}$$

The condition (F) can now be formulated as follows:

$$F_{\psi,y}\{K\psi x\cdot\varphi y\}=KT\varphi x. \tag{F$'$}$$

It follows from (F′) that

(ii) *If $K_1,K_2\in\mathfrak{K}$, then for arbitrarily fixed $\varphi\in\Xi$, $x\in X$, the linear operation M,*

$$M\psi y=K_1\psi x\cdot K_2\varphi y \quad \textit{for} \quad \psi\in\Xi,\ y\in X,$$

belongs to K and the linear operation $L=F(M)$, i. e.

$$L\varphi x = F_{\psi,y}\{K_1\psi x\cdot K_2\varphi y\} \quad \textit{for} \quad \varphi\in\Xi,\ x\in X,$$

satisfies the equation $L=K_1TK_2$. *Consequently*, $L\in\mathfrak{K}$ *also*.

Let $\varphi_1=K_2\varphi$. We have

$$M\psi y = K_1\psi x\cdot K_2\varphi y = K_1\psi x\cdot\varphi_1 y.$$

Hence, for fixed x and φ, $M\in\mathfrak{K}$ by (**K**) and $F(M)=K_1T\varphi_1 x$ by (**F**), i. e.

$$L\varphi x = K_1T\varphi_1 x = K_1TK_2\varphi x.$$

We infer that $L=K_1TK_2\in\mathfrak{K}$ by (**K**).

(iii) *If* $K\in\mathfrak{K}$ *and* $\pi_1,\pi_2,\ldots,\pi_p$ *is a permutation of the sequence* $1,2,\ldots,p$, *then the expression* (*where* $q\leqslant p$)

$$B_q = F_{\psi_q,y_q}F_{\psi_{q-1},y_{q-1}}\ldots F_{\psi_1,y_1}\left\{\prod_{j=1}^{p}K\varphi_{\pi_j}y_j\right\}$$

is well defined and independent of the ordering of the sequence of operators $F_{\psi_1,y_1},\ldots,F_{\psi_q,y_q}$.

Let $K^{(m)}=\overbrace{(KT)(KT)\ldots(KT)}^{m\text{-times}}$, $K\in\mathfrak{K}$. We have $K^{(0)}=K$ and by (ii)

$$K^{(m)}\varphi x = F_{\psi,y}\{K^{(r)}\psi x\cdot K^{(s)}\varphi y\}$$

if $m=r+s+1$, $r,s\geqslant 0$.

First we shall prove by induction with respect to q $(q=0,\ldots,p)$ that B_q is well defined and

$$B_q = a_q\prod_{j=1}^{p-q}K^{(m_j)}\psi_{s_j}y_{q+j}, \tag{4}$$

where $s_1,\ldots,s_{p-q}$ is a permutation of the integers $q+1,\ldots,p$, a_q is a constant, and $m_1,\ldots,m_{p-q}$ is a sequence of non-negative integers.

The formula (4) holds, of course, if $q=0$. Suppose it is true for q $(0\leqslant q<p)$.

If $s_1=q+1$, the operation $F_{\psi_{q+1},y_{q+1}}\{B_q\}$ is feasible and

$$B_{q+1} = F_{\psi_{q+1},y_{q+1}}\{B_q\} = a_{q+1}\prod_{j=1}^{p-q}K^{(m_j)}\psi_{s_j}y_{q+j},$$

where $a_{q+1}=a_qF(K^{(m_1)})$. Hence B_{q+1} is also of the form (4).

If $q+1=s_r$, $r\neq 1$, then the operation $F_{\psi_{q+1},y_{q+1}}\{B_q\}$ is also feasible and

$$\begin{aligned} B_{q+1} &= F_{\psi_{q+1},y_{q+1}}\{B_q\} \\ &= a_{q+1}\cdot F_{\psi_{q+1},y_{q+1}}\{K^{(m_r)}\psi_{q+1}y_{q+r}\cdot K^{(m_1)}\psi_{s_1}y_{q+1}\}\prod_{\substack{j=2\\ j\neq r}}K^{(m_j)}\psi_{s_j}y_{q+j} \\ &= a_qK^{(m_r+m_1+1)}\psi_{s_1}y_{q+r}\prod_{\substack{j=2\\ j\neq r}}^{p-q}K^{(m_j)}\psi_{s_j}y_{q+j}. \end{aligned}$$

Thus B_{q+1} is also of the form (4).

Now we shall prove that

$$F_{\psi_{q+2},y_{q+2}}F_{\psi_{q+1},y_{q+1}}\{B_q\} = F_{\psi_{q+1},y_{q+1}}F_{\psi_{q+2},y_{q+2}}\{B_q\}. \tag{4'}$$

The following seven cases should be considered:

(a) $q+1=s_1$ and $q+2=s_2$. Then both sides of (4′) are equal to the number

$$a_qF(K^{(m_1)})\cdot F(K^{(m_2)})\prod_{j=3}^{p-q}K^{(m_j)}\psi_{s_j}y_{q+j}.$$

(b) $q+1=s_1$ and $q+2=s_r$, $r>2$. Then both sides of (4′) are equal to

$$a_qF(K^{(m_1)})\cdot K^{(m_r+m_2+1)}\psi_{s_2}y_{q+r}\cdot\prod_{\substack{j=3\\ j\neq r}}^{p-q}K^{(m_j)}\psi_{s_j}y_{q+j}.$$

(c) $q+1=s_2$ and $q+2=s_1$. Then both sides of (4′) are equal to

$$a_qF(K^{(m_1+m_2+2)})\cdot\prod_{j=3}^{p-q}K^{(m_j)}\psi_{s_j}y_{q+j}.$$

(d) $q+1=s_2$ and $q+2=s_r$, $r>2$. Then both sides of (4′) are equal to

$$a_qK^{(m_1+m_2+m_r+2)}\psi_{s_1}y_{q+r}\cdot\prod_{\substack{j=3\\ j\neq r}}^{p-q}K^{(m_j)}\psi_{s_j}y_{q+j}.$$

(e) $q+1=s_r$, $r>2$, and $q+2=s_1$. The proof is the same as that of (d).

(f) $q+1=s_r$, $r>2$, and $q+2=s_2$. The proof is the same as that of (b).

(g) $q+1=s_r$, $q+2=s_t$, $r,t>2$. Then both sides of (4′) are equal to

$$a_q K^{(m_1+m_r+1)}\psi_{s_1}y_{q+r}\cdot K^{(m_2+m_t+1)}\psi_{s_2}y_{q+t}\cdot\prod_{\substack{j=3\\ r\neq j\neq t}}^{p-q}K^{(m_j)}\psi_{s_j}y_{q+j}.$$

Notice moreover that

(iii′) $$F_{\zeta,z}\{F_{\psi,y}\{K\psi x\cdot K\zeta y\cdot\varphi z\}\}=F_{\psi,y}\{K\psi x\cdot F_{\zeta,z}\{K\zeta y\cdot\varphi z\}\},$$

(iii″) $$F_{\zeta,z}\{F_{\psi,y}\{K\psi x\cdot K\zeta x'\cdot K\varphi' y\cdot\varphi z\}\} = F_{\psi,y}\{K\psi x\cdot F_{\zeta,z}\{K\zeta x'\cdot K\varphi' y\cdot\varphi\zeta\}\}.$$

In fact both sides of (iii′) are equal to $KTKT\varphi x$ by (ii). Analogously both sides of (iii″) are equal to $KTK\varphi' x\cdot KT\varphi x'$.

We shall now examine the equation (1) which, of course, can be written in the form

(1′) $$\varphi+A\varphi=\psi_0 \quad\text{or}\quad (I+A)\varphi=\psi_0,$$

where $A=KT$, $K\in\mathfrak{K}$, $\psi_0\in\Xi$. The solution φ should also be in Ξ.

We introduce the following notations, analogous to those in the Fredholm theory of integral equations:

(5) $$K\begin{pmatrix}\varphi_1,\dots,\varphi_q\\ x_1,\dots,x_q\end{pmatrix}=\begin{vmatrix}K\varphi_1x_1 & \dots & K\varphi_1x_q\\ \cdot & \cdot & \cdot\\ K\varphi_qx_1 & \dots & K\varphi_qx_q\end{vmatrix},$$

(6) $$a_0=1,\qquad a_q=\frac{1}{q!}F_{\psi_1,y_1}F_{\psi_2,y_2}\dots F_{\psi_q,y_q}\left\{K\begin{pmatrix}\psi_1,\dots,\psi_q\\ y_1,\dots,y_q\end{pmatrix}\right\},$$

(7) $$A_0=A,\qquad A_q=a_qA-AA_{q-1},$$

(8) $$K_0=K,\qquad K_q\varphi x=a_qK\varphi x-F_{\psi,y}\{K\psi x\cdot K_{q-1}\varphi y\},$$

where $q=1,2,\dots$

Clearly $K_q\in\mathfrak{K}$. We shall prove that

(iv) $$K_q\varphi x=\frac{1}{q!}F_{\psi_1,y_1}\dots F_{\psi_q,y_q}\left\{K\begin{pmatrix}\varphi,\psi_1,\dots,\psi_q\\ x,y_1,\dots,y_q\end{pmatrix}\right\}.$$

The proof is by induction with respect to q. The case $q=0$ is obvious. Suppose (iv) is true for $q-1\geqslant 0$. We shall prove that (iv) is true for q.

We have

$$K\begin{pmatrix}\varphi,\psi_1,\dots,\psi_q\\ x,y_1,\dots,y_q\end{pmatrix}=K\varphi x\cdot K\begin{pmatrix}\psi_1,\dots,\psi_q\\ y_1,\dots,y_q\end{pmatrix}$$
$$+\sum_{i=1}^{q}(-1)^iK\psi_ix\cdot K\begin{pmatrix}\varphi,\ \psi_1,\dots,\psi_{i-1},\psi_{i+1},\dots,\psi_q\\ y_1,y_2,\dots,y_i,\quad y_{i+1},\dots,y_q\end{pmatrix}$$
$$=K\varphi x\cdot K\begin{pmatrix}\psi_1,\dots,\psi_q\\ y_1,\dots,y_q\end{pmatrix}-\sum_{i=1}^{q}K\psi_ix\cdot K\begin{pmatrix}\varphi,\psi_2,\dots,\psi_{i-1},\psi_{i+1},\dots,\psi_q\\ y_i,y_1,\dots,y_{i-1},y_{i+1},\dots,y_q\end{pmatrix}.$$

Hence, by the additivity, homogenity and commutativity (see (iii)) of the operators $F_{\psi,z}$ we obtain

$$F_{\psi_1,y_1}\dots F_{\psi_q,y_q}\left\{K\begin{pmatrix}\varphi,\psi_1,\dots,\psi_q\\ x,y_1,\dots,y_q\end{pmatrix}\right\}=K\varphi x\cdot F_{\psi_1,y_1}\dots F_{\psi_q,y_q}\cdot\left\{K\begin{pmatrix}\psi_1,\dots,\psi_q\\ y_1,\dots,y_q\end{pmatrix}\right\}$$
$$-\sum_{i=1}^{q}F_{\psi_i,y_i}\{K\psi_ix\cdot F_{\psi_1,y_1}\dots F_{\psi_{i-1},y_{i-1}}F_{\psi_{i+1},y_{i+1}}\dots F_{\psi_q,y_q}\}$$
$$\times\left\{K\begin{pmatrix}\varphi,\psi_1,\dots,\psi_{i-1},\psi_{i+1},\dots,\psi_q\\ y_i,y_1,\dots,y_{i-1},y_{i+1},\dots,y_q\end{pmatrix}\right\}.$$

Consequently, by the induction hypothesis,

$$F_{\psi_1,y_1}\dots F_{\psi_q,y_q}\left\{K\begin{pmatrix}\varphi,\psi_1,\dots,\psi_q\\ x,y_1,\dots,y_q\end{pmatrix}\right\}$$
$$=q!\,a_qK\varphi x-\sum_{i=1}^{q}F_{\psi_i,y_i}\{K\psi_ix\cdot(q-1)!\,K_{q-1}\varphi y_i\}$$
$$=q!\,a_qK\varphi x-(q-1)!\sum_{i=1}^{q}F_{\psi,y}\{K\psi x\cdot K_{q-1}\varphi y\}$$
$$=q!\,(a_qK\varphi x-F_{\psi,y}\{K\psi x\cdot K_{q-1}\varphi y\})=q!\,K_q\varphi x.$$

Now we prove

(v) $$\|K_q\|\leqslant\frac{(q+1)^{\frac{q+1}{2}}}{q!}\|F^q\|\cdot\|K\|^{q+1}.$$

In fact, by (iv) and (i),

$$\|K_q\| = \sup_{\|\varphi\|\leqslant 1} \sup_{\|x\|\leqslant 1} |K_q \varphi x| = \frac{1}{q!} \sup_{\|\varphi\|\leqslant 1} \sup_{\|x\|\leqslant 1} \left| F_{\psi_1 y_1} \dots F_{\psi_q y_q} \left\{ K \begin{pmatrix} \varphi, \psi_1, \dots, \psi_q \\ x, y_1, \dots, y_q \end{pmatrix} \right\} \right|$$

$$\leqslant \frac{\|F\|}{q!} \sup_{\|\varphi\|\leqslant 1} \sup_{\|x\|\leqslant 1} \sup_{\|\psi\|\leqslant 1} \sup_{\|y\|\leqslant 1} \left| F_{\psi_2 y_2} \dots F_{\psi_q y_q} \left\{ K \begin{pmatrix} \varphi, \psi_1, \dots, \psi_q \\ x, y_1, \dots, y_q \end{pmatrix} \right\} \right| \leqslant \dots$$

$$\leqslant \frac{\|F\|^q}{q!} \sup_{\|\varphi\|\leqslant 1} \sup_{\|x\|\leqslant 1} \sup_{\|\psi_1\|\leqslant 1} \sup_{\|y_1\|\leqslant 1} \dots \sup_{\|\psi_q\|\leqslant 1} \sup_{\|y_q\|\leqslant 1} \left| K \begin{pmatrix} \varphi, \psi_1, \dots, \psi_q \\ x, y_1, \dots, y_q \end{pmatrix} \right|,$$

and, by the Hadamard theorem,

$$\left| K \begin{pmatrix} \varphi, \psi_1, \dots, \psi_q \\ x, y_1, \dots, y_q \end{pmatrix} \right| \leqslant \|K\|^{q+1} (q+1)^{\frac{q+1}{2}}$$

if $\|\varphi\|\leqslant 1, \|x\|\leqslant 1, \|\psi_i\|\leqslant 1, \|x_i\|\leqslant 1$ $(i=1,2,\dots,q)$, since all terms of the determinant

$$K \begin{pmatrix} \varphi, \psi_1, \dots, \psi_q \\ x, y_1, \dots, y_q \end{pmatrix}$$

are $\leqslant \|K\|$.

By an analogous argumentation

(vi) $$|a_q| \leqslant \frac{q^{2/q}}{q!} \|F\|^q \cdot \|K\|^q.$$

We prove by an easy induction with respect to q that (see (ii))

(vii) $$A_q = K_q T.$$

Hence, by (v),

(viii) $$\|A_q\| \leqslant \frac{(q+1)^{\frac{q+1}{2}}}{q!} \|F\|^q \cdot \|K\|^{q+1} \cdot \|T\|.$$

It follows from (vi) and (viii) that

(ix) *The series*

$$D_{0,\lambda} = \sum_{q=0}^{\infty} a_q \lambda^q$$

and

$$\sum_{q=0}^{\infty} \|A_q\| \lambda_q$$

are uniformly convergent in each interval $-r \leqslant \lambda \leqslant r$.

Now let

$$V_q \begin{pmatrix} \varphi_1, \dots, \varphi_p \\ x_1, \dots, x_p \end{pmatrix} = F_{\psi_1 y_1} \dots F_{\psi_q y_q} \left\{ K \begin{pmatrix} \varphi_1, \dots, \varphi_p, \psi_1, \dots, \psi_q \\ x_1, \dots, x_p, y_1, \dots, y_q \end{pmatrix} \right\}.$$

V_q is a functional linear with respect to each variable $\varphi_1, \dots, \varphi_p$; $x_1, \dots, x_p$ running through Ξ and X respectively. The norm of V_q is

$$\|V_q\| = \sup_{\substack{\|\varphi_i\|\leqslant 1, \|x_i\|\leqslant 1 \\ i=1,2,\dots,p}} \left| V_q \begin{pmatrix} \varphi_1, \dots, \varphi_p \\ x_1, \dots, x_p \end{pmatrix} \right| \leqslant (p+q)^{\frac{p+q}{2}} \cdot \|F\|^q \cdot \|K\|^{p+q}$$

by an argumentation analogous to that in the proof of (v).

Hence

(x) $$\left| V_q \begin{pmatrix} \varphi_1, \dots, \varphi_p \\ x_1, \dots, x_p \end{pmatrix} \right|$$
$$\leqslant (p+q)^{\frac{p+q}{2}} \cdot \|F\|^q \cdot \|K\|^{p+q} \|\varphi_1\| \cdot \dots \cdot \|\varphi_p\| \cdot \|x_1\| \cdot \dots \cdot \|x_p\|.$$

This implies that

(xi) *The series*

$$D_{p,\lambda} \begin{pmatrix} \varphi_1, \dots, \varphi_p \\ x_1, \dots, x_p \end{pmatrix} = (-1)^p \sum_{q=0}^{\infty} \frac{\lambda^q}{q!} V_q \begin{pmatrix} \varphi_1, \dots, \varphi_p \\ x_1, \dots, x_p \end{pmatrix}$$

is an integral function of λ *and a functional linear with respect to each variable* $\varphi_1, \dots, \varphi_p \in \Xi$, $x_1, \dots, x_p \in X$ $(p>0)$.

(xii) *For every* λ, *there is an integer* $p \geqslant 0$ *such that* $D_{p,\lambda} \neq 0$ (*i. e.* $D_{p,\lambda}$ *is not identically equal to* 0).

Suppose the converse, i. e.

$$D_{p,\lambda_0} = 0 \quad (p=0,1,2,\dots)$$

for a number λ_0. Since

$$\frac{d^p}{d\lambda^p} D_{0,\lambda} = F_{\varphi_1 x_1} \dots F_{\varphi_p x_p} \left\{ \sum_{q=0}^{\infty} \frac{\lambda^q}{q!} V_q \begin{pmatrix} \varphi_1, \dots, \varphi_p \\ x_1, \dots, x_p \end{pmatrix} \right\},$$

we obtain that

$$\left(\frac{d^p D_{0,\lambda}}{d\lambda^p} \right)_{\lambda=\lambda_0} = 0,$$

i. e. $D_{0,\lambda} = 0$ identically. This is impossible since $D_{0,0} = a_0 = 1$.

We shall write, for simplicity, D_p instead of $D_{p,1}$, and D instead of $D_0=D_{0,1}$. Hence

$$D=D_0=\sum_{q=0}^{\infty} a_q\,,$$

$$D_p\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix}=(-1)^p\sum_{q=0}^{\infty}\frac{1}{q!}V_q\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix}\qquad\text{for}\quad p=1,2,\dots$$

We have

$$\begin{aligned}D_{p+1}\begin{pmatrix}\varphi,\varphi_1,\dots,\varphi_p\\ x,x_1,\dots,x_p\end{pmatrix}&=-K\varphi x\cdot D_p\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix}\\ &-\sum_{i=1}^{p}(-1)^i K\varphi x_i\cdot D_p\begin{pmatrix}\varphi_1,\ \dots\dots\dots\dots\ ,\varphi_p\\ x,x_1,\dots,x_{i-1},x_{i+1},\dots,x_p\end{pmatrix}\\ &-F_{\psi,y}\left\{D_{p+1}\begin{pmatrix}\psi,\varphi_1,\dots,\varphi_p\\ x,x_1,\dots,x_p\end{pmatrix}\cdot K\varphi y\right\},\end{aligned}\tag{xiii}$$

Hence

$$\begin{aligned}D_{p+1}\begin{pmatrix}\varphi,\varphi_1,\dots,\varphi_p\\ x,x_1,\dots,x_p\end{pmatrix}&=-K\varphi x\cdot D_p\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix}\\ &-\sum_{i=1}^{p}(-1)^i K\varphi_i x\cdot D_p\begin{pmatrix}\varphi,\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_p\\ x_1,\ \dots\dots\dots\dots\ ,x_p\end{pmatrix}\\ &-F_{\psi,y}\left\{K\psi x\cdot D_{p+1}\begin{pmatrix}\varphi,\varphi_1,\dots,\varphi_p\\ y,x_1,\dots,x_p\end{pmatrix}\right\}.\end{aligned}\tag{xiv}$$

Expand the determinant

$$K\begin{pmatrix}\varphi,\varphi_1,\dots,\varphi_p,\psi_1,\dots,\psi_q\\ x,x_1,\dots,x_p,y_1,\dots,y_q\end{pmatrix}$$

on the first row:

$$\begin{aligned}K&\begin{pmatrix}\varphi,\varphi_1,\dots,\varphi_p,\psi_1,\dots,\psi_q\\ x,x_1,\dots,x_p,y_1,\dots,y_q\end{pmatrix}\\ &=K\varphi x\cdot K\begin{pmatrix}\varphi_1,\dots,\varphi_p,\psi_1,\dots,\psi_q\\ x_1,\dots,x_p,y_1,\dots,y_q\end{pmatrix}\\ &\quad+\sum_{i=1}^{p}(-1)^i K\varphi x_i\cdot K\begin{pmatrix}\varphi_1,\ \dots\dots\dots\dots\ ,\varphi_p,\psi_1,\dots,\psi_q\\ x,x_1,\dots,x_{i-1},x_{i+1},\dots,x_p,y_1,\dots,y_q\end{pmatrix}\\ &\quad+\sum_{i=1}^{q}(-1)^{p+i}K\varphi y_i\cdot K\begin{pmatrix}\varphi_1,\varphi_2,\dots,\varphi_p,\quad \psi_1,\psi_2,\ \dots\dots\dots\ ,\psi_q\\ x,x_1,\dots,x_{p-1},x_p,y_1,\dots,y_{i-1},y_{i+1},\dots,y_q\end{pmatrix}\end{aligned}$$

$$\begin{aligned}&=K\varphi x\cdot K\begin{pmatrix}\varphi_1,\dots,\varphi_p,\psi_1,\dots,\psi_q\\ x_1,\dots,x_p,y_1,\dots,y_q\end{pmatrix}\\ &\quad+\sum_{i=1}^{p}(-1)^i K\varphi x_i\cdot K\begin{pmatrix}\varphi_1,\ \dots\dots\dots\dots\ ,\varphi_p,\psi_1,\dots,\psi_q\\ x,x_1,\dots,x_{i-1},x_{i+1},\dots,x_p,y_1,\dots,y_q\end{pmatrix}\\ &\quad-\sum_{i=1}^{q}K\varphi y_i\cdot K\begin{pmatrix}\psi_i,\varphi_1,\dots,\varphi_p,\psi_1,\dots,\psi_{i-1},\psi_{i+1},\dots,\psi_q\\ x,x_1,\dots,x_p,y_1,\dots,y_{i-1},y_{i+1},\dots,y_q\end{pmatrix}.\end{aligned}$$

Apply the operator

$$F_{\psi_1,y_1}\dots F_{\psi_q,y_q}$$

to both sides of the above equation. By the commutativity of $F_{\psi,y}$ (see (iii)) we obtain

$$\begin{aligned}V_q\begin{pmatrix}\varphi,\varphi_1,\dots,\varphi_p\\ x,x_1,\dots,x_p\end{pmatrix}&=K\varphi x\cdot V_q\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix}\\ &+\sum_{i=1}^{p}(-1)^i K\varphi x_i\cdot V_q\begin{pmatrix}\varphi_1,\ \dots\dots\dots\dots\ ,\varphi_p\\ x,x_1,\dots,x_{i-1},x_{i+1},\dots,x_p\end{pmatrix}\\ &-\sum_{i=1}^{q}F_{\psi_i,y_i}\left\{K\varphi y_i\cdot V_{q-1}\begin{pmatrix}\psi_i,\varphi_1,\dots,\varphi_p\\ x,x_1,\dots,x_p\end{pmatrix}\right\}.\end{aligned}$$

Replace ψ_i,y_i by ψ,y respectively. Consequently

$$\begin{aligned}V_q\begin{pmatrix}\varphi,\varphi_1,\dots,\varphi_p\\ x,x_1,\dots,x_p\end{pmatrix}&=K\varphi x\cdot V_q\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix}\\ &+\sum_{i=1}^{p}(-1)^p K\varphi x_i\cdot V_q\begin{pmatrix}\varphi_1,\ \dots\dots\dots\dots\ ,\varphi_p\\ x,x_1,\dots,x_{i-1},x_{i+1},\dots,x_p\end{pmatrix}\\ &-qF_{\psi,y}\left\{K\varphi y\cdot V_{q-1}\begin{pmatrix}\psi,\varphi_1,\dots,\varphi_p\\ x,x_1,\dots,x_p\end{pmatrix}\right\}.\end{aligned}$$

This implies immediately the equation (xiii). The proof of (xiv) is analogous. We should now expand the determinant

$$K\begin{pmatrix}\varphi,\varphi_1,\dots,\varphi_p,\psi_1,\dots,\psi_q\\ x,x_1,\dots,x_p,y_1,\dots,y_q\end{pmatrix}$$

on the first column.

Replace now φ by ζ in the equations (xiii) and (xiv), multiply both sides of these equations by φz and apply the operator $F_{\zeta,z}$. We obtain by (iii′) and (iii″)

$$\text{(xv)}\quad F_{\zeta,z}\left\{D_{p+1}\begin{pmatrix}\zeta,\varphi_1,\dots,\varphi_p\\ x,x_1,\dots,x_p\end{pmatrix}\cdot\varphi z\right\}=-F_{\zeta,z}\{K\zeta x\cdot\varphi z\}\cdot D_p\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix}$$
$$-F_{\psi,y}\left\{D_{p+1}\begin{pmatrix}\psi,\varphi_1,\dots,\varphi_p\\ x,x_1,\dots,x_p\end{pmatrix}\cdot F_{\zeta,z}\{K\zeta y\cdot\varphi z\}\right\}$$
$$-\sum_{i=1}^{p}(-1)^i F_{\zeta,z}\{K\zeta x_i\cdot\varphi z\}D_p\begin{pmatrix}\varphi_1, & \dots\dots\dots\dots & ,\varphi_p\\ x,x_1,\dots,x_{i-1},x_{i+1},\dots,x_p\end{pmatrix}.$$

$$\text{(xvi)}\quad F_{\zeta,z}\left\{D_{p+1}\begin{pmatrix}\zeta,\varphi_1,\dots,\varphi_p\\ x,x_1,\dots,x_p\end{pmatrix}\cdot\varphi z\right\}=-F_{\zeta,z}\{K\zeta x\cdot\varphi z\}\cdot D_p\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix}$$
$$-F_{\psi,y}\left\{K\psi x\cdot F_{\zeta,z}\left\{D_{p+1}\begin{pmatrix}\zeta,\varphi_1,\dots,\varphi_p\\ y,x_1,\dots,x_p\end{pmatrix}\cdot\varphi z\right\}\right\}$$
$$-\sum_{i=1}^{p}(-1)^i K\varphi_i x\cdot F_{\zeta,z}\left\{D_p\begin{pmatrix}\zeta,\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_p\\ x_1, & \dots\dots\dots\dots & ,x_p\end{pmatrix}\cdot\varphi z\right\}.$$

We introduce now the following notations, where $p,\ \varphi_1,\dots,\varphi_p\,\epsilon\,\Xi$ and $x_1,\dots,x_p\,\epsilon\,X$ are fixed:

$$U_p\varphi x=F_{\zeta,z}\left\{D_{p+1}\begin{pmatrix}\zeta,\varphi_1,\dots,\varphi_p\\ x,x_1,\dots,x_p\end{pmatrix}\cdot\varphi\zeta\right\},$$

$$\Psi_i\varphi=F_{\zeta,z}\left\{D_p\begin{pmatrix}\zeta,\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_p\\ x_1, & \dots\dots\dots\dots & ,x_p\end{pmatrix}\cdot\varphi z\right\},$$

$$g_i x=(-1)^i D_p\begin{pmatrix}\varphi_1, & \dots\dots\dots\dots & ,\varphi_p\\ x,x_1,\dots,x_{i-1},x_{i+1},\dots,x_p\end{pmatrix},$$

$$\bar\varphi_i x=(-1)^i K\varphi_i x,$$

$$\Phi_i\varphi=A\varphi x_i,$$

$$\delta_p=D_p\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix},$$

where $\varphi\,\epsilon\,\Xi$, $x\,\epsilon\,X$, and $i=1,\dots,p$. If $p=0$, the expressions Ψ_i, g_i, $\bar\varphi_i$, Φ_i are not defined, and

$$U_0\varphi x=F_{\zeta,z}\left\{D_1\begin{pmatrix}\zeta\\ x\end{pmatrix}\cdot\varphi z\right\}.$$

Clearly, $g_i,\bar\varphi_i\,\epsilon\,\Xi$, U_p is a linear operation of Ξ into Ξ, and Ψ_i, Φ_i are linear functionals on Ξ.

The equations (xv) and (xvi) can be written as follows (see (F′)):

$$\text{(xvii)}\qquad U_p\varphi=-\delta_pA\varphi-U_pA\varphi-\sum_{i=1}^{p}\Phi_i\varphi\cdot g_i,$$

$$\text{(xviii)}\qquad U_p\varphi=-\delta_pA\varphi-AU_p\varphi-\sum_{i=1}^{p}\Psi_i\varphi\cdot\bar\varphi_i.$$

The operation $I+A$ is said to be of an *order* p if $D_p\neq0$ and $D_r=0$ for all $r<p$ $(r\geqslant0)$.

(xix) *Suppose the order of* $I+A$ *is* $p>0$. *Then* $\Phi_i g_j=-\delta_{ij}\delta_p$ *and* $\Psi_i\bar\varphi_j=-\delta_{ij}\delta_p$, *where* δ_{ij} *is the Kronecker symbol.*

In fact, from (xiv) where p is replaced by $p-1$ we obtain

$$\text{(9)}\qquad D_p\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix}=-F_{\psi,y}\left\{K\psi x_1\cdot D_p\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ y,x_2,\dots,x_p\end{pmatrix}\right\}.$$

Replace now x_i by x_1 and conversely. We obtain

$$\text{(10)}\qquad (-1)^iD_p\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix}$$
$$=-F_{\psi,y}\left\{K\psi x_i\cdot D_p\begin{pmatrix}\varphi_1, & \dots\dots\dots\dots & ,\varphi_p\\ y,x_1,\dots,x_{i-1},x_{i+1},\dots,x_p\end{pmatrix}\right\},$$

i. e., by the definition of Φ_i and g_i (see (F′)),

$$-\delta_p=\Phi_i g_i.$$

Replace x_i by x_j $(i\neq j)$ in the formula (10). The left side is equal to 0, and the other is equal to $\Phi_j(g_i)$. This proves the first equation in (xix). The proof of the second is analogous (see (xiv)).

Subsequently we shall suppose that $I+A$ is of the order $p\geqslant0$ and (in the case of $p>0$) that

$$\varphi_1,\dots,\varphi_p\,\epsilon\,\Xi,\qquad x_1,\dots,x_p\,\epsilon\,X$$

are so chosen that

$$\delta_p=D_p\begin{pmatrix}\varphi_1,\dots,\varphi_p\\ x_1,\dots,x_p\end{pmatrix}\neq0.$$

Under these hypotheses, it follows immediately from (xix) that

(xx) *If* $I+A$ *is of order* $p>0$, *then the functionals* $g_1,\ldots,g_p$ *are linearly independent. Analogously, the functionals* $\Psi_1,\ldots,\Psi_p$ *are linearly independent.*

Theorem 1. *If the operation* $I+A$ *has the order* $p>0$, *then the sequence* $g_1,\ldots,g_p$ *is the basis of the linear space of all* φ *which are solutions of the homogeneous equation*

$$\varphi+A\varphi=0 \tag{11}$$

(*i. e.* φ *satisfies* (11) *if and only if* $\varphi=\sum_{i=1}^{p}a_i g_i$).

If we replace x_1 by x in (9) we obtain

$$D_p\begin{pmatrix}\varphi_1,\varphi_2,\ldots,\varphi_p\\ x,x_2,\ldots,x_p\end{pmatrix}+F_{\psi,y}\left\{K\psi x\cdot D_p\begin{pmatrix}\varphi_1,\varphi_2,\ldots,\varphi_p\\ y,x_2,\ldots,\varphi_p\end{pmatrix}\right\}=0,$$

i. e.

$$g_1x+F_{\psi,y}\{K\psi x\cdot g_1 y\}=0.$$

By (F′) this equation can be written in the form

$$g_1+KTg_1=0.$$

Thus g_1 is a solution of (11). The proof that $g_2,\ldots,g_p$ are also solutions of (11) is analogous.

Suppose now that φ is a solution of (11). It follows from (xvii) and (11) that

$$U_p\varphi=\delta\varphi+U_p\varphi-\sum_{i=1}^{p}\Phi_i\varphi\cdot g_i.$$

Hence

$$\varphi=\frac{1}{\delta_p}\cdot\sum_{i=1}^{p}(\Phi_i\varphi)\cdot g_i,$$

which completes the proof of Theorem 1.

(xxi) *If the order of* $I+A$ *is* $p>0$, *and if the equation* (1) *has a solution* φ, *then the functional*

$$\varphi_0=\psi_0+\frac{1}{\delta_p}U_p\psi_0$$

is also a solution (1), *i. e.*

$$\varphi_0+A\varphi_0=\psi_0.$$

Suppose $\varphi+A\varphi=\psi_0$, i. e. $A\varphi=\psi_0-\varphi$. Replace $A\varphi$ by $\psi_0-\varphi$ in the equation (xvii). We obtain

$$U_p\varphi=-\delta_p(\psi_0-\varphi)-U_p\psi_0+U_p\varphi-\sum_{i=1}^{p}\Phi_i\varphi\cdot g_i,$$

i. e.

$$\delta_p\psi_0+U_p\psi_0=\delta_p\varphi-\sum_{i=1}^{p}\Phi_i\varphi\cdot g_i.$$

Hence

$$\varphi_0=\varphi-\sum_{i=1}^{p}a_i\cdot g_i \quad \text{where} \quad a_i=\frac{1}{\delta_p}\Phi_i\varphi.$$

Since φ is a solution of (1) and g_i are solutions of the homogeneous equation (11), φ_0 is a solution of (1).

Theorem 2. *Suppose the order of* $I+A$ *is* $p>0$. *The equation* (1) *has solutions if and only if* $\Psi_i\psi_0=0$ *for* $i=1,\ldots,p$.

By lemma (xxi), if (1) has solutions, then

$$\varphi_0=\psi_0+\frac{1}{\delta_p}U_p\psi_0$$

is also a solution of (1). Replacing φ by φ_0 in (1) we obtain

$$U_p\psi_0+\delta_pA\psi_0+AU_p\psi_0=0. \tag{12}$$

By (xviii) the left side of the above equation is equal to

$$-\sum_{i=1}^{p}\Psi_i\psi_0\cdot g_i.$$

Hence

$$\sum_{i=1}^{p}\Psi_i\psi_0\cdot g_i=0. \tag{13}$$

By (xx) the functionals g_i are linearly independent. Hence $\Psi_i\psi_0=0$ for $i=1,\ldots,p$.

Conversely, if $\Psi_i\psi_0=0$ for $i=1,\ldots,p$, then (13) holds. Consequently (12) holds, i. e.

$$\varphi_0=\Psi_0+\frac{1}{\delta_p}U_p\psi_0$$

is a solution of (1).

Theorem 3. *The equation* (1) *is solvable for every* $\psi_0\epsilon\,\Xi$ *if and only if* $I+A$ *has the order* 0, *i. e. if* $D\neq 0$.

In fact, if $D\neq 0$, then the operation (see (7))

$$I-\frac{1}{D}\sum_{q=0}^{\infty}A_q$$

is converse to the operation $I+A$ since, by (7),

$$\begin{aligned}(I+A)\Big(I-\frac{1}{D}\sum_{q=0}^{\infty}A_q\Big)&=I+A-\frac{1}{D}\Big(\sum_{q=0}^{\infty}A_q(I+A)\Big)\\&=I+A-\frac{1}{D}\Big(\sum_{q=0}^{\infty}A_q+\sum_{q=0}^{\infty}(a_{q+1}A-A_{q+1})\Big)\\&=I+A-\frac{1}{D}\cdot DA=I.\end{aligned}$$

On the other hand, if $I+A$ has the order $p>0$, then, by Theorem 2, the equation $\varphi+A\varphi=\bar{\varphi}_1$ has no solution since $\Psi_1\bar{\varphi}_1\neq 0$ by (xix).

Let $\bar{\Xi}$ be the Banach space of all linear functionals on Ξ, and let $\bar{A}$ be the operation (of $\bar{\Xi}$ into $\bar{\Xi}$) conjugate to A, i. e. $\Psi=\bar{A}\Phi$ is equivalent to: $\Psi(\varphi)=\Phi\big(A(\varphi)\big)$ for each $\varphi\in\Xi$.

Theorem 4. *Suppose the order of $I+A$ is $p>0$. The sequence $\Psi_1,\dots,\Psi_p$ is a basis for the linear space of all solutions of the homogeneous equation*

$$\Psi+\bar{A}\Psi=0 \qquad (\Psi\in\bar{\Xi}). \tag{14}$$

By the definition of Ψ_i and by the equation (see (F′))

$$A\Psi y=F_{\zeta,z}\{K\zeta y\cdot\varphi z\}$$

we obtain

$$\Psi_i(\varphi)+\Psi_i\big(A(\varphi)\big)=F_{\zeta,z}\Big\{D_p\begin{pmatrix}\zeta,\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_p\\ x_1,\ \dots\dots\dots\dots,x_p\end{pmatrix}\varphi z$$
$$+F_{\psi y}\Big\{D_p\begin{pmatrix}\zeta,\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_p\\ x_1,\ \dots\dots\dots\dots,x_p\end{pmatrix}\cdot K\zeta y\cdot\varphi z\Big\}\Big\}.$$

The expression in the outer brackets $\{\ \ \}$ is equal to 0 by (xiii) (where p is replaced by $p-1$) since $D_{p-1}=0$. Consequently

$\Psi_i(\varphi)+\Psi_i\big(A(\varphi)\big)=0$ for every $\varphi\in\Xi$, i. e. $\Psi_i+\bar{A}\Psi_i=0$ for $i=1,\dots,p$. Thus the linearly independent (see (xx)) functionals $\Psi_1,\dots,\Psi_p$ are solutions of (14).

Suppose now that (14) holds, i. e. $\Psi+\bar{A}\Psi=0$. By (xviii)

$$\Psi U_p\varphi=-\delta_p\Psi(A\varphi)-\Psi(AU_p\varphi)-\sum_{i=1}^{p}\Psi_i\varphi\cdot(\bar{\varphi}_i).$$

Replacing $\Psi(A\varphi)$ by $-\Psi(\varphi)$ we obtain

$$\Psi U_p\varphi=\delta_p\Psi_\varphi+\Psi(U_p\varphi)-\sum_{i=1}^{p}\Psi_i\varphi\cdot\Psi\bar{\varphi}_i$$

for every $\varphi\in\Xi$, i. e.

$$\Psi=\frac{1}{\delta_p}\sum_{i=1}^{\infty}\Psi\bar{\varphi}_i\cdot\Psi_i, \qquad \text{q. e. d.}$$

For applications it is important to know that D depends continuously on F and K. Let $\mathfrak{F}$ be the set of all functionals F on $\mathfrak{K}$ such that there is a linear operation $T\in\mathfrak{K}$ such that (F) holds. Clearly $\mathfrak{F}$ is a linear subspace of the normed space $\bar{\mathfrak{K}}$ conjugate to $\mathfrak{K}$. The expression

$$a_q(F^{(1)},F^{(2)},\dots,F^{(q)},K^{(1)},K^{(2)},\dots,K^{(q)})$$
$$=\frac{1}{q!}F^{(1)}_{\psi_1,y_1}\dots F^{(q)}_{\psi_q,y_q}\left\{\begin{vmatrix}K^{(1)}\psi_1y_1 & K^{(1)}\psi_1y_2 & \dots & K^{(1)}\psi_1y_q\\ K^{(2)}\psi_2y_1 & K^{(2)}\psi_2y_2 & \dots & K^{(2)}\psi_2y_q\\ \dots & \dots & \dots & \dots\\ K^{(q)}\psi_qy_1 & K^{(q)}\psi_qy_2 & \dots & K^{(q)}\psi_qy_q\end{vmatrix}\right\}$$

is a $2k$-linear operation on

$$\underbrace{\mathfrak{F}\times\mathfrak{F}\times\dots\times\mathfrak{F}}_{q\text{-times}}\times\underbrace{\mathfrak{K}\times\mathfrak{K}\times\dots\times\mathfrak{K}}_{q\text{-times}}$$

since

$$|a_q(F^{(1)},\dots,F^{(q)},K^{(1)},\dots,K^{(q)})|\leqslant\frac{q^{q/2}}{q!}\|F^{(1)}\|\cdot\ldots\cdot\|F^{(q)}\|\cdot\|K^{(1)}\|\cdot\ldots\cdot\|K^{(q)}\|$$

by an argumentation similar to that in the proof of (v) and (vi). Hence

$$a_q(F,K)=a_q(F,\dots,F;K,\dots,K)=\frac{1}{q!}F_{\psi_1,y_1}\dots F_{\psi_q,y_q}K\begin{pmatrix}\psi_1,\dots,\psi_q\\ y_1,\dots,y_q\end{pmatrix}$$

is a continuous polynomial on $\mathfrak{F}\times\mathfrak{K}$ and

$$|a_q(F,K)|\leqslant\frac{q^{q/2}}{q!}\,\|F\|^q\cdot\|K\|^q.$$

Therefore the series

$$D(F,K)=\sum_{q=0}^{\infty}a_q(F,K)\qquad (a_0(F,K)=1)$$

is uniformly convergent on each bounded subset of $\mathfrak{F}\times\mathfrak{K}$, consequently it is a continuous function on $\mathfrak{F}\times\mathfrak{K}$, i. e.:

(xxii) *If* $\|F^{(n)}-F\|\to 0$ *and* $\|K^{(n)}-K\|\to 0$, *then*

$$D(F^{(n)},K^{(n)})\to D(F,K).$$

II. Applications.

I. Consider first the case where X is the space L^q of all measurable functions $x(t)$ on $\langle 0,1\rangle$ with

$$\|x\|=\left(\int_0^1|x(t)|^q\,dt\right)^{1/q}<\infty.$$

Ξ is the conjugates pace L^q $(1/p+1/q=1,\ 1<p,\ q<\infty)$. Clearly, each function $\varphi\in\Xi$ determines uniquely a functional on X denoted by the same letter φ:

(15) $$\varphi x=\int_0^1\varphi(t)x(t)\,dt.$$

Each measurable function $K(t,s)$ defined on the square $0\leqslant t,\ s\leqslant 1$, such that

(16) $$\int_0^1\int_0^1|K(t,s)x(t)\varphi(s)|\,dt\,ds<\infty\qquad\text{for}\quad x\in X,\ \varphi\in\Xi,$$

determines uniquely a linear operation of Ξ into Ξ denoted by the same letter K:

(17) $$K\varphi(t)=\int_0^1K(t,s),\varphi(s)\,ds,\quad\text{i. e.}\quad K\varphi x=\int_0^1\int_0^1K(t,s)x(t)\varphi(s)\,ds\,dt.$$

Let $\mathfrak{K}_0$ be the class of all operations K of the form (17) where $K(t,s)$ satisfies (16), and let $\mathfrak{K}$ be the least closed linear set of linear operations of Ξ into Ξ such that $I\in\mathfrak{K}$ and $\mathfrak{K}_0\subset\mathfrak{K}$. The linear space $\mathfrak{K}$ satisfies the condition (**K**).

In fact, if $K_1,K_2\in\mathfrak{K}_0$, then $K=K_1K_2\in\mathfrak{K}_0$ since K is determined by the function

$$K(t,s)=\int_0^1K_1(t,r)K_2(r,s)\,dr$$

which satisfies the condition (16). Consequently, if $K_1,K_2\in\mathfrak{K}$, then $K_1K_2\in\mathfrak{K}$.

If $K\in\mathfrak{K}_0$, and if $x_0\in X$, $\varphi_0\in\Xi$, then

(18) $$M\psi y=K\psi x_0\cdot\varphi_0 y=\int_0^1\int_0^1M(t,s)\psi(s)y(t)\,ds\,dt$$

where

$$M(t,s)=\int_0^1K(r,s)\varphi_0(t)x_0(r)\,dr,$$

which proves that $M\in\mathfrak{K}_0$. If $K=I$, we have

(19) $$M\psi y=I\psi x_0\cdot\varphi_0 y=\psi x_0\cdot\varphi_0 y=\int_0^1\int_0^1M(t,s)\psi(s)y(t)\,ds\,dt$$

where $M(t,s)=\varphi_0(t)\cdot x_0(s)$ which proves that $M\in\mathfrak{K}_0$. By continuity, $M\in\mathfrak{K}$ for any $K\in\mathfrak{K}$, q. e. d

Now let $R,S\in\mathfrak{K}_0$ be two operations such that

(20) $$\left(\int_0^1\left(\int_0^1|R(t,r)|^p\,dt\right)^{u/p}dr\right)^{1/u}<\infty,$$

(21) $$\left(\int_0^1\left(\int_0^1|S(r,s)|^q\,ds\right)^{v/q}dr\right)^{1/v}<\infty,$$

where $1/u+1/v=1$, $1<u,\ v<\infty$ [5]).

Let $T=RS\in\mathfrak{K}_0$, i. e.

$$T(t,s)=\int_0^1R(t,r)S(r,s)\,dr.$$

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[5]) The cases $u=1$, $v=\infty$, and $u=\infty$, $v=1$ are also admissible. Evidently the norm $(\int|\cdot|^\infty dr)^{1/\infty}$ should be replaced by $\sup_r|\cdot|$.

We define the functional F on $\mathfrak{K}_0$ by the formula

$$F(K)=\int_0^1\int_0^1 T(t,s)K(s,t)\,dt\,ds. \tag{22}$$

F is linear on $\mathfrak{K}_0$ since

$$F|(K)|\leqslant \int_0^1\left|\int_0^1\int_0^1 K(s,t)R(t,r)S(r,s)\,ds\,dt\right|dr$$

$$\leqslant \int_0^1\left(\|K\|\left(\int_0^1|R(t,r)|^p\,dt\right)^{1/p}\cdot\left(\int_0^1|S(r,s)|^q\,ds\right)^{1/q}\right)dr \tag{23}$$

$$\leqslant\|K\|\cdot\left(\int_0^1\left(\int_0^1|R(t,r)|^p\,dt\right)^{u/p}dr\right)^{1/u}\cdot\left(\int_0^1\left(\int_0^1|S(r,s)|^q\,ds\right)^{v/q}dr\right)^{1/v}.$$

Now we extend F to a linear functional on $\mathfrak{K}$ in an arbitrary way[6]). The extended functional F satisfies the condition (**F**). It suffices to verify the case of $K\epsilon\mathfrak{K}_0$ and $K=I$.

If M is defined by (18), then

$$F(M)=\int_0^1\int_0^1 T(s,t)\,M(t,s)\,ds\,dt$$

$$=\int_0^1\int_0^1\left(\int_0^1 T(s,t)K(r,s)\,ds\right)\varphi_0(t)\,x_0(r)\,dr\,dt=KT\varphi_0x_0.$$

If M is defined by (19), then

$$F(M)=\int_0^1\int_0^1 T(s,t)\,M(t,s)\,ds\,dt$$

$$=\int_0^1\int_0^1 T(s,t)\varphi_0(t)x_0(s)\,ds\,dt=T\varphi_0x_0=TI\varphi_0x_0, \qquad \text{q. e. d.}$$

Consequently, the theory developed in the first part can be applied to the equation (1) where $K\epsilon\mathfrak{K}$ and $\varphi,\psi_0\epsilon\Xi=\boldsymbol{L}^p$. If $A=KT\epsilon\mathfrak{K}_0$, then (1) is the integral equation

$$\varphi(t)+\int_0^1 A(t,s)\varphi(s)\,ds=\psi_0(t). \tag{24}$$

[6]) Since I does not belong to the closure of $\mathfrak{K}_0$, the number $F(I)$ can be arbitrary.

Consider, in particular (the case $K=I$), the equation

$$\varphi+T\varphi=\psi_0, \tag{25}$$

i. e. the integral equation

$$\varphi(t)+\int_0^1 T(t,s)\varphi(s)\,ds=\psi_0(t) \qquad (\psi_0,\varphi\epsilon\boldsymbol{L}^p). \tag{26}$$

The determinant D of the equation (25) coincides with the Fredholm determinant of the integral equation (26). In fact, we have (see the definition (7))

$$A_0=T, \qquad A_n=a_nT-TA_{n-1},$$

i. e.

$$A_0(t,s)=T(t,s),$$
$$A_n(t,s)=a_nT(t,s)-\int_0^1 T(t,r)A_{n-1}(r,s)\,dr \qquad (n=1,2,\dots), \tag{27}$$

where $A_n(t,s)$ denotes clearly the function determining the operation $A_n\epsilon\mathfrak{K}_0$. By (iv) and (vii),

$$a_0=1,$$

$$a_{n+1}=\frac{1}{n+1}F(K_n)=\frac{1}{n+1}\int_0^1 T(s,t)\,K_n(t,s)\,dt\,ds \tag{28}$$

$$=\frac{1}{n+1}\int_0^1 A_n(s,s)\,ds \qquad (n=1,2,\dots).$$

We may suppose that

$$F(I)=\int_0^1 T(s,s)\,ds$$

(if not, we can modify the function $T(s,t)$ on the diagonal of the unit square). Therefore

$$a_1=F(K_0)=F(I)=\int_0^1 A_0(s,s)\,ds. \tag{28'}$$

The formulae (27), (28), (28′) show that

$$D=\sum_{n=0}^{\infty}a_n$$

is the Fredholm determinant of (26), and

$$\frac{1}{D}\sum_{n=0}^{\infty} A_n(t,s)$$

is Fredholm's resolvent kernel of equation (26).

The restriction that the kernel T of (26) is a superposition of two other kernels is not essential since the examination of (26) can be reduced to the examination of the equation with the iterated kernel TT.

Analogous results can be obtained in the case where $X=l^q$ [7]) and $\Xi=l^p$ $(1<p,\ q<\infty,\ 1/p+1/q=1)$. The necessary modifications in the text are obvious.

II. Now let X be the space $\boldsymbol{M}$ of all bounded measurable functions $x(t)$ on the interval $\langle 0,1\rangle$ with the norm [8])

$$\|x\|=\sup_t \operatorname{ess}|x(t)|$$

and let Ξ be the space $\boldsymbol{L}$ of all integrable functions $\varphi(s)$ on $\langle 0,1\rangle$ with the norm

$$\|\varphi\|=\int_0^1|\varphi(s)|\,ds.$$

Clearly each $\varphi\in\Xi$ determines a functional on X denoted by the same letter φ (see the formula (15)).

Each measurable function $K(t,s)$ defined on the square $0\leqslant t,$ $s\leqslant 1$, such that

$$\int_0^1\int_0^1|K(t,s)\varphi(s)x(t)|\,ds\,dt<\infty \quad \text{for each} \quad \varphi\in\Xi,\ x\in X, \tag{29}$$

determines a linear operation of Ξ into Ξ denoted by the same letter K and defined by the equation

$$K\varphi x=\int_0^1\int_0^1 K(t,s)\varphi(s)x(t)\,ds\,dt. \tag{30}$$

The norm of K is

$$\|K\|=\sup_s \operatorname{ess}\int_0^1|K(t,s)|\,dt. \tag{31}$$

Let $\mathfrak{K}_0$ be the set of all linear operations K of the form (30), and let $\mathfrak{K}$ be the least closed linear set of linear operations of Ξ into Ξ such that $I\in\mathfrak{K}$ and $\mathfrak{K}_0\subset\mathfrak{K}$. $\mathfrak{K}$ satisfies the condition (**K**). The proof is the same as in **I**.

Let $T(t,s)$ be a fixed measurable function defined on the square $0\leqslant t,$ $s\leqslant 1$, such that

$$\int_0^1\sup_s \operatorname{ess}|T(t,s)|\,dt<\infty.$$

Clearly $T\in\mathfrak{K}_0$. Let

$$F(K)=\int_0^1\int_0^1 T(t,s)K(s,t)\,ds\,dt \quad \text{for} \quad K\in\mathfrak{K}_0.$$

F is a linear operation on $\mathfrak{K}_0$. In fact

$$|F(K)|\leqslant\int_0^1\int_0^1|T(t,s)K(s,t)|\,ds\,dt$$

$$\leqslant\int_0^1\Big(\sup_s \operatorname{ess}|T(t,s)|\cdot\int_0^1|K(s,t)|\,ds\Big)\,dt$$

$$\leqslant\sup_t \operatorname{ess}\int_0^1|K(s,t)|\,ds\cdot\int_0^1\sup_s \operatorname{ess}|T(t,s)|\,dt=\|K\|\cdot\int_0^1\sup_s \operatorname{ess}|T(t,s)|\,dt.$$

Now we extend the functional F to a linear functional over $\mathfrak{K}$. The functional F on $\mathfrak{K}$ satisfies condition (**F**). The proof is the same as in **I**.

Consequently, the Fredholm theory from the first part of this paper can be applied to the equation

$$\varphi+KT\varphi=\psi_0,$$

where $K\in\mathfrak{K}$. In particular, if $A=KT\in\mathfrak{K}_0$, this equation is the integral equation

$$\varphi(t)+\int_0^1 A(t,s)\varphi(s)\,ds=\psi_0(t).$$

---

[7]) l^q is the space of all sequences $x=(x_n)$ such that

$$\|x\|=\Big(\sum_{i=1}^{\infty}|x_n|^q\Big)^{1/q}<\infty.$$

[8]) $\sup_t \operatorname{ess} f(t)$ is, by definition, the number $\inf_Z \sup_{t\,\text{non}\in Z} f(t)$ where Z runs over all sets $\subset\langle 0,1\rangle$ of measure zero.

It is easy to see that the case $X=\boldsymbol{L}$, $\Xi=\boldsymbol{M}$ can be discussed in a completely analogous way to the case examined above: $X=\boldsymbol{M}$, $\Xi=\boldsymbol{L}$.

III. Now let X be the space $\boldsymbol{V}$ of all function $x(t)$ on $\langle 0,1\rangle$ with finite total variation $\operatorname*{var}_t x(t)$. The norm in $\boldsymbol{V}$ is

$$\|x\|=\operatorname*{var}_t x(t).$$

Let Ξ be the space $\boldsymbol{C}$ of all continuous functions φ on $\langle 0,1\rangle$ with the norm $\|\varphi\|=\sup_s|\varphi(s)|$. Clearly each $\varphi\epsilon\boldsymbol{C}$ determines a linear functional on $X=\boldsymbol{V}$, which will be denoted by the same letter φ:

$$\varphi x=\int_0^1\varphi(s)\,d_t x(t).$$

All integrals in this section are of the Riemann-Stieltjes type.

Let $K(T,s)$ be a function on the square $0\leqslant t,\ s\leqslant 1$ such that

(a) for each fixed t, the function $K(t,s)$ of one variable s is of bounded variation, and

$$\sup_t\operatorname*{var}_s K(t,s)<\infty,$$

(b) for each fixed s, the function $K(t,s)$ of one variable t is continuous.

The function $K(t,s)$ determines linear operation of Ξ into Ξ denoted by the same letter K:

$$K\varphi x=\int_0^1\Big(\int_0^1\varphi(s)\,d_sK(t,s)\Big)d_tx(t)=\int_0^1\varphi(s)\,d_s\Big(\int_0^1K(t,s)\,d_tx(t)\Big).\tag{32}$$

The norm of K is

$$\|K\|=\sup_t\operatorname*{var}_s K(t,s).\tag{33}$$

Let $\mathfrak{K}_0$ be the class of all functions $K(t,s)$ satisfying (a) and (b), and let $\mathfrak{K}$ be the least closed linear set such that $I\epsilon\mathfrak{K}$ and $\mathfrak{K}_0\subset\mathfrak{K}$. $\mathfrak{K}$ satisfies the condition (**K**).

The superposition $K=K_1K_2$ ($K_1,K_2\epsilon\mathfrak{K}_0$) is determined by the function

$$K(t,s)=\int_0^1K_1(r,s)\,d_rK_2(t,r).$$

If $K\epsilon\mathfrak{K}_0$ and $\varphi_0\epsilon\Xi$, $x_0\epsilon X$ are fixed, then

$$M\psi y=K\psi x_0\cdot\varphi_0 y=\int_0^1\Big(\int_0^1\psi(s)\,d_sM(t,s)\Big)d_ty(t),\tag{34}$$

where

$$M(t,s)=\varphi_0(t)\int_0^1K(r,s)\,d_rx_0(r).$$

If $K=I$, then

$$M\psi y=\psi x_0\cdot\varphi_0 y=\int_0^1\int_0^1\psi(s)\,d_sM(t,s)\,d_ty(t),$$

where $M(t,s)=\varphi_0(t)\cdot x_0(s)$.

Consequently, $M\epsilon\mathfrak{K}$ for every $K\epsilon\mathfrak{K}$, q. e. d.

Let $T(t,s)$ be a fixed continuous function on the square $0\leqslant s,\ t\leqslant 1$. This function determines an operation $T\epsilon\mathfrak{K}$ defined as follows:

$$T\varphi x=\int_0^1\Big(\int_0^1T(t,s)\varphi(s)\,ds\Big)d_tx(t).\tag{35}$$

For each $K\epsilon\mathfrak{K}$ take

$$F(K)=\int_0^1dt\int_0^1T(s,t)\,d_sK(t,s).$$

F is a linear functional on $\mathfrak{K}$ since

$$|F(K)|\leqslant\int_0^1dt\Big|\int_0^1T(s,t)\,d_sK(t,s)\Big|$$

$$\leqslant\int_0^1\sup_{t,s}|T(s,t)|\cdot\operatorname*{var}_s K(t,s)\,ds=\|K\|\cdot\sup_{t,s}|T(s,t)|.$$

The functional F has the property (F). In fact, if M is defined by (34), then

$$F(M)=\int_0^1dt\int_0^1T(s,t)\,d_sM(t,s)$$

$$=\int_0^1dt\int_0^1T(s,t)\,d_s\Big(\int_0^1\varphi_0(t)K(r,s)\,d_rx_0(r)\Big)=KT\varphi_0x_0.$$

The case $K=I$ can be verified in an analogous way. Therefore we may apply the theory from the first part of this paper to the equation

$$(1') \qquad \varphi+A\varphi=\psi_0,$$

which can be written in the form

$$\varphi(t)+\int_0^1 \varphi(s)\,d_sA(t,s)=\psi_0(t).$$

IV. We shall now apply the results of the first part of this paper to the case where $X=\boldsymbol{l}$ or $\boldsymbol{m}$ and $\Xi=\boldsymbol{m}$ or $\boldsymbol{l}$[9]) respectively.

We shall write explicitly all formulae and equations for the case where $X=\boldsymbol{l}$. In square brackets [ ] we shall write analogous expressions for the case where $X=\boldsymbol{m}$.

Let $X=\boldsymbol{l}$ $[X=\boldsymbol{m}]$, and let $\Xi=\boldsymbol{m}$ $[\Xi=\boldsymbol{l}]$. Elements of X or Ξ are sequences of real numbers. The n-th term of a sequence $x\epsilon X$ or $\varphi\epsilon\Xi$ will always be denoted by x_n or φ_n respectively. We shall also write $x=(x_n)$ or $\varphi=(\varphi_n)$ respectively.

Infinite square matrices will be denoted by letters $K,T,S,\ldots$, their elements will always be denoted by the same Greek letters with two indices: $K=(\varkappa_{ki})$, $T=(\tau_{ki})$, $S=(\sigma_{ki})$, etc. However, the unit matrix (δ_{ki}) (where δ_{ki} is the Kronecker symbol) will be denoted by I.

We adopt the usual notations: $S+T$ is the matrix $(\sigma_{ki}+\tau_{ki})$, ST is the matrix

$$\left(\sum_{r=1}^{\infty}\sigma_{kr}\tau_{ri}\right),$$

whenever all series defining the terms of ST are absolutely convergent.

Let $\mathfrak{K}$ be the set of all matrices $K=(\varkappa_{ki})$ such that

$$(36) \qquad \sum_{k,i=1}^{\infty}|\varkappa_{ki}x_k\varphi_i|<\infty \quad \text{for each} \quad \varphi\epsilon\Xi \text{ and } x\epsilon X.$$

Each matrix $K\epsilon\mathfrak{K}$ determines uniquely a linear operation of Ξ into Ξ denoted by the same letter K:

[9]) $\boldsymbol{m}$ is the space of all bounded sequences $x=(x_n)$ with the norm $\|x\|=\sup\limits_n|x_n|$. $\boldsymbol{l}$ is the space of all sequences $x=(x_n)$ such that $\|x\|=\sum\limits_{n=1}^{\infty}|x_n|<\infty$.

$$K\varphi x=\sum_{k,i=1}^{\infty}\varkappa_{ki}x_k\varphi_i.$$

The superposition KT of two linear operations K and T is determined by the product of the matrices K,T. The unit matrix I determines the identical mapping of Ξ onto Ξ.

We shall interprete $\mathfrak{K}$ as the Banach space of linear operations of Ξ into Ξ. The norm of an operation $K\epsilon\mathfrak{K}$ is

$$(37) \qquad \|K\|=\sup_k\left(\sum_{i=1}^{\infty}|\varkappa_{ki}|\right) \qquad \left[\|K\|=\sup_i\left(\sum_{k=1}^{\infty}|\varkappa_{ki}|\right)\right].$$

$\mathfrak{K}$ satisfies the condition (**K**).

Let $\mathfrak{T}$ denote the class of all matrices $T=(\tau_{ki})$ such that

$$(38) \qquad \|T\|^*=\sum_{i=1}^{\infty}\sup_k|\tau_{ki}|<\infty \qquad \left[\|T\|^*=\sum_{k=1}^{\infty}\sup_i|\tau_{ki}|<\infty\right].$$

With respect to the norm $\|\ \|^*$, $\mathfrak{T}$ is a Banach algebra, since $T,S\epsilon\mathfrak{T}$ implies $TS\epsilon\mathfrak{T}$ and

$$\|TS\|^*\leqslant\|T\|^*\cdot\|S\|^*.$$

Clearly $\mathfrak{T}\subset\mathfrak{K}$. Consequently, each T can be interpreted as a linear operation of Ξ onto Ξ. Notice that

$$\|T\|\leqslant\|T\|^*.$$

Each $T\epsilon\mathfrak{T}$ determines also a linear functional F defined on $\mathfrak{K}$ by the formula

$$F(K)=\sum_{k,i=1}^{\infty}\tau_{ik}\varkappa_{ki} \quad \text{for} \quad K=(\varkappa_{ki})\epsilon\mathfrak{K}.$$

This double series is absolutely convergent.

In the same way as in **II** we prove that $|F(K)|\leqslant\|K\|\cdot\|T\|^*$. Hence

$$(39) \qquad \|F\|\leqslant\|T\|^*.$$

Now take a fixed $T\epsilon\mathfrak{T}$ and consider the functional F determined by T. The functional F fulfils the condition (**F**). The proof is the same as in case **II**.

Consequently we can apply the results from the first part of this paper to the equation

$$(1') \qquad \varphi+A\varphi=\psi \quad \text{or} \quad (I+A)\varphi=\psi,$$

where $\varphi,\psi\epsilon\Xi$, $A=KT$, $K=(\varkappa_{ki})\epsilon\mathfrak{K}$.

The equation (1′) is, in fact, a system of infinitely many linear equations

$$\varphi_k+\sum_{i=1}^{\infty}a_{ki}\varphi_i=\psi_k \qquad (k=1,2,\ldots), \tag{40}$$

where $\varphi=(\varphi_k)$, $\psi=(\psi_k)$ and $A=(a_{ki})=KT\,\epsilon\,\mathfrak{R}$.

Since we shall examine in this section the equation (1′) for different operations A, we shall write $D(I+A)$ instead of D. By definition,

$$D(I+A)=\sum_{k=0}^{\infty}\frac{1}{k!}F_{\varphi_1,y_1}\ldots F_{\varphi_k,y_k}\left\{K\begin{pmatrix}\varphi_1,\ldots,\varphi_k\\ y_1,\ldots,y_k\end{pmatrix}\right\}.$$

Now

$$F_{\varphi_1,y_1}\ldots F_{\varphi_k,y_k}\left\{K\begin{pmatrix}\varphi_1,\ldots,\varphi_k\\ y_1,\ldots,y_k\end{pmatrix}\right\}$$

$$=\sum_{i_1,j_1=1}^{\infty}\tau_{j_1i_1}\sum_{i_2,j_2=1}^{\infty}\tau_{j_2i_2}\ldots\sum_{i_k,j_k=1}^{\infty}\tau_{j_ki_k}\begin{vmatrix}\varkappa_{i_1j_1}\ldots\varkappa_{i_1j_k}\\ \cdots\cdots\\ \varkappa_{i_kj_1}\ldots\varkappa_{i_kj_k}\end{vmatrix}$$

$$=\sum_{(j_1,\ldots,j_k)}\begin{vmatrix}a_{j_1j_1}\ldots a_{j_1j_k}\\ \cdots\cdots\\ a_{j_kj_1}\ldots a_{j_kj_k}\end{vmatrix},$$

where the last sign Σ is extended over all sequences of different positive integers. The determinants under the last sign Σ are the principal subdeterminants[10]) of the matrix $A=(a_{ki})$ of the order k. Consequently[11]),

$$D(I+A)=\sum_{k=0}^{\infty}d_k, \tag{41}$$

where d_k is the sum of all principal subdeterminants of the order k of the matrix A if $k>0$, and $d_0=1$. By (ix), the series (41) is absolutely convergent.

Subsequently we shall examine the equations (1) only in the case where $K=I$, i. e. the equation

$$\varphi+T\varphi=\psi \qquad\text{or}\qquad (I+T)\varphi=\psi, \tag{42}$$

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[10]) *Principal subdeterminants* (of an order k) of a matrix (a_{ji}) are the determinants of finite matrices $(a_{r_jr_i})$ $(j,i=1,\ldots,k)$ where $r_1,\ldots,r_k$ is a sequence of different positive integers.

[11]) An analogous result holds in the case $\Xi=l^p$, $X=l^q$, see p. 244.

where $T\,\epsilon\,\mathfrak{T}$. In other words, we shall examine the following infinite system of linear equations:

$$\varphi_k+\sum_{i=1}^{\infty}\tau_{ki}\varphi_i=\psi_k \qquad\text{or}\qquad \sum_{i=1}^{\infty}(\delta_{ki}+\tau_{ki})\varphi_i=\psi_k, \tag{43}$$

where $k=1,2,\ldots$, $T=(\tau_{ki})\,\epsilon\,\mathfrak{T}$.

For each $T\,\epsilon\,\mathfrak{T}$ let $T_n=(\tau_{ki}^n)$ denote the matrix defined as follows:

$$\tau_{ki}^n=\begin{cases}\tau_{ki} & \text{if}\quad i\leqslant n\quad [k\leqslant n],\\ 0 & \text{if}\quad i>n\quad [k>n].\end{cases}$$

It is clear that $T_n\,\epsilon\,\mathfrak{T}$ and $\|T-T_n\|^*\to 0$. The letters $F^{(n)}$ and F will denote respectively the linear functionals determined by T_n and T. By (39), $\|F^{(n)}-F\|\to 0$. Hence, by lemma (xxii),

$$D(I+T_n)\to D(I+T).$$

We have

$$D(I+T_n)=\sum_{k=0}^{\infty}d_k^{(n)},$$

where $d_k^{(n)}$ is the sum of all principal subdeterminants of the matrix T_n. Now each principal subdeterminant of T_n which is not equal to 0 is the principal subdeterminant of the finite matrix (τ_{ki}) $(k,i=1,\ldots,n)$. Hence, by the known properties of determinants, $D(I+T_n)$ is equal to the determinant of the finite matrix $(\delta_{ki}+\tau_{ki})$, where $k,i=1,\ldots,n$. Consequently:

Theorem 5. *If $T\,\epsilon\,\mathfrak{T}$, then the sum d_k of all principal subdeterminants of the matrix T exists for each k, the series $d_0+d_1+d_2+\ldots$ is absolutely convergent, and*

$$D(I+T)=\sum_{k=0}^{\infty}d_k=\lim_{n}\begin{vmatrix}1+\tau_{11} & \tau_{12} & \ldots & \tau_{1n}\\ \tau_{21} & 1+\tau_{22} & \ldots & \tau_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ \tau_{n1} & \tau_{n2} & \ldots & 1+\tau_{nn}\end{vmatrix}. \tag{44}$$

The number $D(I+T)$ is called the *determinant of the infinite matrix* $C=I+T$. The determinant $D(C)$ of an infinite matrix C is thus well defined if $C-I\,\epsilon\,T$.

Now let $S\,\epsilon\,T$ be another matrix, and let $S_n=(\sigma_{ki}^n)$ be the matrix defined analogously to T_n, i. e.

$$\sigma_{ki}^n=\begin{cases}\sigma_{ki} & \text{if}\quad i\leqslant n\quad [k\leqslant n],\\ 0 & \text{if}\quad i>n\quad [k>n].\end{cases}$$

We have $\|S_n - S\|^* \to 0$. Since $\mathfrak{T}$ is a Banach algebra, we obtain $\|T_nS_n - T_nS_n\|^* \to 0$, and consequently

$$\|(T_n+S_n+T_nS_n)-(T+S+TS)\|^* \to 0.$$

Hence, by (39) and (xxii),

$$D\big((I+T)(I+S)\big)=D(I+T+S+TS)=\lim_n D(I+T_n+S_n+T_nS_n).$$

Now, if $i>n$, the i-th column [line] of the matrices T_n, S_n, $T_n+S_n+T_nS_n$ contains exclusively the numbers 0. Consequently (see the proof of Theorem 5), $D(I+T_n)$ is the determinant of the finite matrix $(\delta_{ki}+\tau_{ki})$, where $k,i=1,2,\ldots,n$, and $D(I+S_n)$ is the determinant of the matrix $(\delta_{ki}+\sigma_{ki})$, where $i,k=1,\ldots,n$. Analogously, $D(I+T_n+S_n+T_nS_n)$ is equal to the determinant of the finite matrix $(\delta_{ki}+\tau_{ki})(\delta_{ki}+\sigma_{ki})$, where $k,i=1,\ldots,n$. Hence

$$D(I+T_n+S_n+T_nS_n)=D(I+T_n)\cdot D(I+S_n)$$

and, consequently,

Theorem 6. *$D\big((I+T)(I+S)\big)=D(I+T)\cdot D(I+S)$ for arbitrary $T,S\,\epsilon\, T$.*

In other words, if $C_1-I\,\epsilon\,\mathfrak{T}$ and $C_2-T\,\epsilon\,\mathfrak{T}$, then

$$D(C_1C_2)=D(C_1)\cdot D(C_2).$$

During the print of this paper R. Sikorski and the author have proved that the Theorem 6 holds in arbitary Banach space. Since the determinant of the equation $(I+T)\varphi=\psi$ is uniquely determined by F, but not by T, the extension of Theorem 6 to the case of an arbitrary Banach space must be otherwise formulated.

The above theorem shows that the infinite determinant $D(C)$ $(C-I\,\epsilon\,\mathfrak{T})$ has the multiplication property of usual determinants of a finite order. Obviously it has also other properties of finite determinants. For instance:

(xxiii) *Let $C-I\,\epsilon\,\mathfrak{T}$, let $a_i\,\epsilon\,\boldsymbol{m}$ $[a_i\,\epsilon\,\boldsymbol{l}]$ for $i=1,2,3$, and let C_i be the matrix obtained from C by replacing the r-th column of C by the sequence a_i. Since $C_i-I\,\epsilon\,\mathfrak{T}$, the determinant $D(C_i)$ exists $(i=1,2,3)$. If $a_3=a_1+a_2$, then $D(C_3)=D(C_1)+D(C_2)$.*

This additive property follows immediately from Theorem 5. Analogously:

(xxiv) *Let $C-I\,\epsilon\,\mathfrak{T}$, let $a_i\,\epsilon\,\boldsymbol{l}$ $[a_i\,\epsilon\,\boldsymbol{m}]$ for $i=1,2,3$, and let C_i be the matrix obtained from C by replacing the r-th line of C by the sequence a_i. Since $C_i-T\,\epsilon\,\mathfrak{T}$, the determinant $D(C_i)$ exists $(i=1,2,3)$. If $a_3=a_1+a_2$, then $D(C_3)=D(C_1)+D(C_2)$.*

(xxv) *Let C_n, C_n^0, C' denote respectively the matrices obtained from the matrix C $(C-I\,\epsilon\,\mathfrak{T})$ by replacing the r-th column by the sequence $(a_1,a_2,\ldots,a_n,0,0,0,\ldots)$ or $(0,0,\ldots,0,\overset{(n)}{1},0,0,\ldots)$, or by $a=(a_1,a_2,a_3,\ldots)\,\epsilon\,\boldsymbol{c}_0$[12]) $[\epsilon\,\boldsymbol{l}]$. Then the determinants $D(C_n)$, $D(C_n^0)$, $D(C')$ exist, and*

$$D(C_n)=\sum_{i=1}^{n} a_i\cdot D(C_i^0), \tag{45}$$

$$D(C')=\lim_n D(C_n)=\sum_{i=1}^{\infty} a_i D(C_i^0). \tag{46}$$

The formula (46) is the expansion of C' on the r-th column. The equation (45) follows from (xxiii). (47) follows from (45), (39) and (xxii) since $\|(C_n-I)-(C'-I)\|^*\to 0$.

The analogous statement is true for the expansion on the t-th row. The sequence a should belong to $\boldsymbol{l}$ $[\boldsymbol{c}_0]$.

Let $C=(\gamma_{ki})$ be such that

$$\lim_k \gamma_{ki}=0 \quad \text{for } i=1,2,\ldots \quad \text{and} \quad C-I\,\epsilon\,\mathfrak{T} \quad [C-I\,\epsilon\,T]. \tag{47}$$

Let μ_{ki} denote the determinant of the matrix obtained from the matrix C by replacing the term γ_{ki} by the number 1, and all other terms in the k-th line and in the i-th column by 0. Let $M=(\mu_{ik})$. We shall prove that

$$M\cdot C=D(C)\cdot I. \tag{48}$$

Set $a=(\gamma_{1s},\gamma_{2s},\gamma_{3s},\ldots)$ in lemma (xxv). We obtain

$$C'=\sum_{i=1}^{\infty}\mu_{ri}\gamma_{is}.$$

On the other hand, $C'=C$ if $s=r$; hence

$$\sum_{i=1}^{\infty}\mu_{ri}\gamma_{is}=D(C) \quad \text{if} \quad s=r.$$

[12]) $\boldsymbol{c}_0$ is the space of all sequences convergent to 0. It is not known whether $\boldsymbol{c}_0$ can be replaced here by $\boldsymbol{m}$. Consequently, it is not known whether the condition $\lim_k \gamma_{ki}=0$ in (47) is necessary.

If $s \neq r$, the matrix C' has two identical columns: the s-th and the r-th one. Therefore $D(C')=0$ which follows easily from Theorem 5. Consequently,

$$\sum_{i=1}^{\infty} \mu_{ri} \gamma_{is} = D(C) \cdot \delta_{rs}, \qquad \text{q. e. d.}$$

Theorem 7. *Suppose T fulfils the condition* (48) *and $\psi \in \boldsymbol{m}$ $[\psi \in \boldsymbol{l}]$. If $D(I+T) \neq 0$ and if φ is a solution of* (43) *then*

$$\varphi_i = \frac{1}{D(I+T)} \sum_{i=1}^{\infty} \mu_{ik} \psi_k. \tag{49}$$

This follows immediately from (48). Clearly, (49) defines the operation inverse to $I+T$ (see Theorem 3).

If $\psi \in \boldsymbol{c}_0$[13]) $[\psi \in \boldsymbol{l}]$, then the formula (49) is analogous to the Cramer formula for a finite system of linear equations since

$$\sum_{i=1}^{\infty} \mu_{ik} \psi_k$$

is the determinant of the matrix obtained from $C=I+T$ by replacing the i-th column by the sequence $\psi = (\psi_k) \in \boldsymbol{c}_0 [\in \boldsymbol{l}]$.

I am indebted to Prof. S. Mazur for suggesting this problem to me; I also wish to thank Prof. R. Sikorski for essential advices and correction of several proofs.

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[13]) It is not known whether $\boldsymbol{c}_0$ can be replaced here by $\boldsymbol{m}$. See footnote [12]).

Sur un type de conditions mixtes pour les équations aux dérivées partielles

par

J. G.-MIKUSIŃSKI (Wrocław).

§ 1. Introduction.

Considérons l'équation différentielle

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} \frac{\partial^{\mu+\nu}}{\partial t^{\mu} \partial \lambda^{\nu}} x(t,\lambda) = \varphi(t,\lambda) \tag{1}$$

dont les coefficients $a_{\mu\nu}$ sont constants (réels ou complexes). Dans un travail antérieur [1], j'ai discuté la méthode opérationnelle de résolution de cette équation et le problème d'unicité, lorsque les

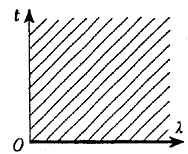

Fig. 1.

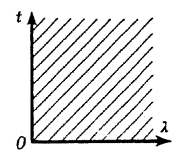

Fig. 2.

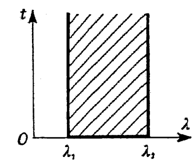

Fig. 3.

conditions initiales sont données sur une seule axe, à savoir $O\lambda$ (Fig. 1), ou sur les deux axes $O\lambda$ et Ot (Fig. 2). Or, dans les applications physiques et techniques, un autre type de conditions est d'une grande importance (Fig. 3): ce sont des conditions sur la frontière d'une demi-bande

$$D:\quad 0 \leqslant t < \infty, \qquad \lambda_1 \leqslant \lambda \leqslant \lambda_2.$$

Ce type de conditions intervient, par exemple, dans le problème de la propagation de la chaleur, lorsque la température est connue sur les deux extremités d'une barre, dans certains problèmes de la ligne électrique et dans beaucoup d'autres problèmes,