A new proof of Titchmarsh's theorem on convolution

by

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1. E. TITCHMARSH proved the following theorem:

(I) *If the functions f and g are integrable over* $[0,T]$ *and*

$$\int_0^t f(t-\tau)g(\tau)\,d\tau=0$$

a. e.[1]) *in* $[0,T]$, *then* $f=0$ *a. e. in* $[0,t_1]$ *and* $g=0$ *a. e. in* $[0,t_2]$, *where* $t_1+t_2\geqslant T$.

There exist several proofs ([1], [2], [3] and [5]) of this theorem; they are all based on the theory of analytic or harmonic functions. In the sequel, we are going to give a simple proof based only on methods of analysis of functions of a real variable. Precisely, we shall apply the following *Theorem on bounded moments* [4]:

If $\left|\int_0^T e^{a_n t}f(t)\,dt\right|\leqslant M$ $(n=1,2,\ldots)$, *where* $a_1>0$, $a_{n+1}-a_n>\varepsilon>0$ *and* $\sum_{n=1}^{\infty}1/a^n=\infty$, *then* $f=0$ *a. e. in* $[0,T]$.

2. The theorem (I) can be written in the following equivalent form:

(II) *If the functions f and g satisfy the assumptions of* (I), *then at least one of them vanishes a. e. in* $[0,\frac{1}{2}T]$.

It is evident that (II) follows from (I). To prove it *vice versa*, denote respectively by $[0,t_1]$ and $[0,t_2]$ the largest intervals in which f and g vanish. Then

[1]) a. e. = almost everywhere

$$h(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau=\int_{t_2}^{t-t_1} f(t-\tau)g(\tau)\,d\tau=\int_0^u f(t_1+u-\tau)g(t_2+\tau)\,d\tau$$

for $t=t_1+t_2+u$. If (II) holds, then h cannot vanish in any right neighbourhood of t_1+t_2. Thus, we have $t_1+t_2\geqslant T$.

3. We shall prove Titchmarsh's theorem in the form (II).

From
$$\int_0^t f(t-\tau)g(\tau)\,d\tau=0$$
it follows that
$$I_a=\int_0^T e^{a(T-t)}\,dt\int_0^t f(t-\tau)g(\tau)\,d\tau=0.$$

The iterated integral I_a can be written as a double integral
$$I_a=\iint_A e^{a(T-t)}f(t-\tau)g(\tau)\,d\tau;$$
the domain of integration A is the triangle defined by the inequalities $0\leqslant\tau\leqslant t\leqslant T$. By the substitution $t=T-u-v$, $\tau=\frac{1}{2}T-v$, we get
$$I_a=\iint_B e^{a(u+v)}f(\tfrac{1}{2}T-u)\,g(\tfrac{1}{2}T-v)\,du\,dv,$$
where the domain of integration B is the triangle defined by the inequalities $0\leqslant u+v$, $u\leqslant\frac{1}{2}T$, $v\leqslant\frac{1}{2}T$. We may write
$$\iint_{B+C}=\iint_B+\iint_C,$$
where C is the triangle $-\frac{1}{2}T\leqslant u$, $-\frac{1}{2}T\leqslant v$, $u+v\leqslant 0$ and $B+C$ is the square $-\frac{1}{2}T\leqslant u\leqslant\frac{1}{2}T$, $-\frac{1}{2}T\leqslant v\leqslant\frac{1}{2}T$.

Since $\iint_B=I_a=0$, we have
$$\iint_{B+C} e^{au}f(\tfrac{1}{2}T-u)\,e^{av}g(\tfrac{1}{2}T-v)\,du\,dv=\iint_C e^{a(u+v)}f(\tfrac{1}{2}T-u)\,g(\tfrac{1}{2}T-v)\,du\,dv.$$

If $a>0$, the coefficient $e^{a(u+v)}$ in the last integral is less then 1. Thus

$$(1)\qquad \left|\int_{-T/2}^{T/2} e^{au}f(\tfrac{1}{2}T-u)\,du\int_{-T/2}^{T/2} e^{av}g(\tfrac{1}{2}T-v)\,dv\right| \leqslant\iint_C |f(\tfrac{1}{2}T-u)\,g(\tfrac{1}{2}T-v)|\,du\,dv=M^2\qquad (a>0).$$

Denote by $\{\alpha_n\}$ the sequence of all positive integers such that

$$\Big|\int_{-T/2}^{T/2} e^{\alpha_n u} f(\tfrac{1}{2}T-u)\,du\Big| \leqslant M \qquad (n=1,2,\ldots)$$

and by $\{\beta_n\}$ the sequence of all positive integers such that

$$\Big|\int_{-T/2}^{T/2} e^{\beta_n v} g(\tfrac{1}{2}T-v)\,dv\Big| \leqslant M \qquad (n=1,2,\ldots).$$

By (1), one at least of the relations

$$\sum_{n=1}^{\infty}\frac{1}{\alpha_n}=\infty \qquad \text{or} \qquad \sum_{n=1}^{\infty}\frac{1}{\beta_n}=\infty$$

must hold. Suppose the first does so.

Since

$$\Big|\int_{0}^{T/2} e^{\alpha_n u} f(\tfrac{1}{2}T-u)\,du\Big| \leqslant M+\Big|\int_{-T/2}^{0} f(\tfrac{1}{2}T-u)\,du\Big| = N \qquad (n=1,2,\ldots),$$

we have, by the Theorem on bounded moments, $f(\frac{1}{2}T-t)=0$ a. e. in $[0,\frac{1}{2}T]$, that is $f(t)=0$ a. e. in $[0,\frac{1}{2}T]$. Thus, the theorem (II) and, consequently, the theorem (I) are proved.

References.

[1] M. M. Crum, *On the resultant of two functions*, The Quarterly Journal of Mathematics, Oxford Series 12, N° 46 (1941), p. 108-111.

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[4] J. G.-Mikusiński and C. Ryll-Nardzewski, *A theorem on bounded moments*, this volume.

[5] E. C. Titchmarsh, *The zeros of certain integral functions*, Proceedings of the London Mathematical Society 25 (1926), p. 283-302.

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Remarks on a moment problem

by

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J. G.-Mikusiński[2] recently gave an elementary proof of the following generalization of Lerch's theorem:

If $f(t)$ is integrable over the finite interval $0\leqslant a<b$ and if for some $\delta>0$ and every $\varepsilon>0$

$$(1) \qquad \int_a^b t^{n\delta} f(t)\,dt = O[(a+\varepsilon)^{n\delta}],$$

then $f(t)=0$ almost everywhere in (a,b).

He raised the question of whether the theorem can be extended by replacing the arithmetic progression $\{n\delta\}$ by a more general sequence $\{\lambda_n\}$. I shall show that the theorem can be proved by less elementary methods, one of which leads to a generalization of the desired kind.

By a change of variable we can make $\delta=1$ in (1), and we suppose this done. We remark first that if $f(t)$ is non-negative the conclusion is immediate, since if $f(t)$ does not vanish almost everywhere in a neighbourhood of b, we have[3]

$$(2) \qquad \overline{\lim_{n\to\infty}}\Big|\int_a^b t^n f(t)\,dt\Big|^{1/n} = b.$$

[1] Fellow of the John Simon Guggenheim Memorial Foundation.

[2] J. G.-Mikusiński, *Remarks on the moment problem and a theorem of Picone*, Colloquium Mathematicum 2 (1951), p. 138-141.

[3] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge 1934, p. 143.