aient des solutions non nulles telles que les séries

$$\sum_{k}^{\infty} |x_k|^2, \qquad \sum_{l}^{\infty} |y_l|^2$$

soient convergentes.

Au cas où les deux suites p_k, q_k sont infinies, le théorème prend une forme purement arithmétique, grâce au théorème de E. Schmidt:

Si p_k et q_k désignent deux suites partielles de la suite $0,1,2,\ldots$, disjointes et infinies, on aura soit

$$\lim_{r\to\infty} \frac{\begin{vmatrix} a_{11} & \ldots & a_{r1} & a_{1i} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1r} & \ldots & a_{rr} & a_{ri} \\ a_{1j} & \ldots & a_{rj} & 0 \end{vmatrix}}{\begin{vmatrix} a_{11} & \ldots & a_{r1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{1r} & \ldots & a_{rr} \end{vmatrix}} = \delta_i^j,$$

soit

$$\lim_{r\to\infty} \frac{\begin{vmatrix} \beta_{11} & \ldots & \beta_{r1} & a_{i1} \\ \cdot & \cdot & \cdot & \cdot \\ \beta_{1r} & \ldots & \beta_{rr} & a_{ir} \\ a_{j1} & \ldots & a_{jr} & 0 \end{vmatrix}}{\begin{vmatrix} \beta_{11} & \ldots & \beta_{r1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \beta_{1r} & \ldots & \beta_{rr} \end{vmatrix}} = \delta_i^j,$$

ou bien

$$a_{kl} = \frac{1}{p_k^2 - q_l^2}, \qquad a_{ij} = \sum_{s=1}^{\infty} a_{is} a_{js}, \qquad \beta_{ij} = \sum_{s=1}^{\infty} a_{si} a_{sj}.$$

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The limiting distributions of sums of arbitrary independent and equally distributed r-point random variables

by

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1. Let us consider a sequence X_n $(n=1,2,\ldots)$ of random variables, where X_n is for each value of n a sum of n independent and equally distributed r-point $(r\geqslant 2)$ random variables Y_{nk} $(k=1,2,\ldots,n)$. Let A_n and $B_n\neq 0$ be sequences of real constants and let the sequence $F_n(z)$ of distribution functions of the random variables

$$\zeta_n = \sum_{k=1}^{n} \frac{Y_{nk}}{B_n} - A_n \tag{1.1}$$

converge with $n\to\infty$ to a distribution function $F(z)$. We can ask what limiting distribution functions are possible. The answer to this question is given in § 2, namely by theorems 2.1-2.3 which are proved in § 4. Some notions and theorems used in the proofs of the theorems given in this paper are quoted in § 3. In theorems 5.1 and 5.2, given in § 5, certain specified sequences A_n and B_n are considered, and sufficient conditions for the convergence of the sequence $F_n(z)$ to a given distribution function $F(z)$ are found. Finally the question of extending the results of § 2 to the case when the random variables Y_{nk} can, with positive probability, take infinitely many values is discussed in § 6.

2. THEOREM 2.1. *Let X_n $(n=1,2,\ldots)$ be defined by the formula*

$$X_n = \sum_{k=1}^{n} Y_{nk}, \tag{2.1}$$

where the random variables Y_{nk} $(k=1,2,\ldots,n)$ are, for each n, independent and equally distributed according to the distribution law

$$P(Y_{nk}=a_{nl}) = p_{nl} \qquad (l=1,2,\ldots,r), \tag{2.2}$$

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where $r \geqslant 2$,

$$0 \leqslant p_{nl} \leqslant 1, \quad \sum_{l=1}^{r} p_{nl} = 1,$$

and a_{nl} *and* p_{nl} *are arbitrary functions of* n.

If, for certain sequences of constants A_n *and* $B_n \neq 0$, *the sequence* $F_n(z)$ *of distribution functions of the variables* ζ_n, *defined by* (1.1), *satisfies, for all continuity points of the function* $F(z)$, *the relation*

$$\lim_{n\to\infty} F_n(z) = F(z), \tag{2.3}$$

where $F(z)$ *is a non-singular distribution function, then* $F(z)$ *is necessarily a distribution function of a sum of independent variables, namely of* s $(0 \leqslant s \leqslant r-2)$ *Poisson variables*[1]) *and* v $(v=0$ *or* $1)$ *normal variables or of a sum of* $r-1$ *Poisson variables.*

From theorem 2.1 it follows immediately that for $r=2$ the function $F(z)$ is necessarily a distribution function of a normal variable or of a Poisson variable. A special case of this consequence of theorem 2.1, when $a_{n1}=a_1$ and $a_{n2}=a_2$ are constants independent of n, was given by Kozuliajev [2].

We shall give here some other interesting special cases of theorem 2.1.

THEOREM 2.2. *If formula* (2.2) *in theorem* 2.1 *is of the form*

$$P(Y_{nk}=a_l)=p_{nl} \qquad (l=1,2,\ldots,r), \tag{2.4}$$

where $a_{l_1} \neq a_{l_2}$ *when* $l_1 \neq l_2$, *then the numbers* s *and* v *in the conclusion of theorem* 2.1 *cannot both be different from* 0.

Thus if the $a_{nl}=a_l$ are independent of n for each n and l, then the limiting non-singular distribution function $F(z)$ is necessarily a distribution function of a normal variable or of a sum of s Poisson variables where $1 \leqslant s \leqslant r-1$.

THEOREM 2.3. *If the formula* (2.2) *in theorem* 2.1 *is of the form*

$$P(Y_{nk}=a_{nl})=p_l \qquad (l=1,2,\ldots,r), \tag{2.5}$$

where $p_l \neq 0$, *then in the conclusion of theorem* 2.1 *we shall have* $s=0$ *and* $v=1$.

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[1]) We say that the random variable Y is a *Poisson variable*, if for each value of $j=0,1,\ldots$ the equality

$$P(Y=aj+b)=\frac{e^{-\lambda}\lambda^j}{j!}$$

holds, where $a\neq 0$, $\lambda>0$ and b are real constants.

Thus if, for each value of n and l, we have $p_{nl}=p_l$ where $p_l \neq 0$, then the function $F(z)$ is a distribution function of a normal variable.

3. We shall need here certain notions and theorems from the theory of infinitely divisible distributions and their applications. The proofs of those theorems are given in the excellent monography of Gnedenko and Kolmogorov [1].

THEOREM 3.1[2]). *The logarithm of the characteristic function* $\varphi(t)$ *of an infinitely divisible distribution is uniquely determined by the formula (given by P. Lévy)*

$$\log\varphi(t) = i\gamma t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0}\left(e^{izt}-1-\frac{izt}{1+z^2}\right)dM(z) + \int_{0}^{\infty}\left(e^{izt}-1-\frac{izt}{1+z^2}\right)dN(z), \tag{3.1}$$

where γ *and* σ *are real constants,* $M(z)$ *and* $N(z)$ *are non-decreasing functions, defined in the intervals* $(-\infty,0)$ *and* $(0,+\infty)$ *respectively, satisfying the relations*

$$M(-\infty)=N(+\infty)=0, \tag{3.2}$$

$$\int_{-1}^{0} z^2 dM(z) + \int_{0}^{1} z^2 dN(z) < \infty. \tag{3.3}$$

Let us consider double sequences of random variables

$$\xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n} \qquad (n=1,2,\ldots),$$

where the ξ_{nk} in each row are independent.

Definition. *The random variables* ξ_{nk} *are called asymptotically constant if, for an arbitrary* $\varepsilon>0$, *the following relation is satisfied:*

$$\lim_{n\to\infty}\max_{1\leqslant k\leqslant k_n} P(|\xi_{nk}-m_{nk}|>\varepsilon)=0, \tag{3.4}$$

where m_{nk} *is the median of the random variable* ξ_{nk}.

THEOREM 3.2[3]). *The random variables* ξ_{nk} *are asymptotically constant if and only if the following relation holds:*

$$\lim_{n\to\infty}\max_{1\leqslant k\leqslant k_n}\int_{-\infty}^{\infty}\frac{z^2}{1+z^2}\,dF_{nk}(z+m_{nk})=0, \tag{3.5}$$

where $F_{nk}(z)$ *is the distribution function of* ξ_{nk}.

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[2]) [1], § 18.

[3]) [1], § 20.

THEOREM 3.3 [4]. *If, for a certain sequence of real constants A_n, the sequence of distribution functions of the random variables*

$$\zeta_n = \sum_{n=1}^{k_n} \xi_{nk} - A_n \tag{3.6}$$

converges to a distribution function, then there exists such a real number $C<\infty$ that

$$\sum_{k=1}^{k_n} \int_{-\infty}^{\infty} \frac{z^2}{1+z^2}\, dF_{nk}(z+m_{nk}) < C. \tag{3.7}$$

THEOREM 3.4 [5]. *Let ζ_n be given by (3.6), where the variables ξ_{nk} are asymptotically constant and independent. In order that, for a certain sequence A_n of real constants, the sequence $F_n(z)$ of distribution functions of ζ_n should converge to a distribution function, the following conditions are necessary and sufficient:*

1. *the existence of non-decreasing functions $M(z)$ and $N(z)$ defined in the intervals $(-\infty, 0)$ and $(0, +\infty)$ respectively, satisfying the relations $M(-\infty)=N(+\infty)=0$, and such that in the continuity points of $M(z)$ and $N(z)$ the following equations hold:*

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{-\infty}^{z} dF_{nk}(z+m_{nk}) = M(z) \qquad (z<0), \tag{3.8}$$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{z}^{\infty} dF_{nk}(z+m_{nk}) = -N(z) \qquad (z>0); \tag{3.9}$$

2. *the existence of such a real number $\sigma \geqslant 0$ that*

$$\lim_{\varepsilon\to 0} \underline{\lim}_{n\to\infty} \sum_{k=1}^{k_n} \left\{ \int_{|z|<\varepsilon} z^2\, dF_{nk}(z+m_{nk}) - \left[\int_{|z|<\varepsilon} z\, dF_{nk}(z+m_{nk}) \right]^2 \right\} \tag{3.10}$$
$$= \lim_{\varepsilon\to 0} \overline{\lim}_{n\to\infty} \sum_{k=1}^{k_n} \left\{ \int_{|z|<\varepsilon} z^2\, dF_{nk}(z+m_{nk}) - \left[\int_{|z|<\varepsilon} z\, dF_{nk}(z+m_{nk}) \right]^2 \right\} = \sigma^2.$$

The constants A_n can be chosen according to the formula

$$A_n = \sum_{k=1}^{k_n} \int_{|z|<\tau} z\, dF_{nk}(z+m_{nk}) + \sum_{k=1}^{k_n} m_{nk} - \gamma(\tau), \tag{3.11}$$

where $-\tau$ and τ are arbitrary continuity points of $M(z)$ and $N(z)$ respectively, and $\gamma(\tau)$ is an arbitrary number.

[4] [1], § 23.
[5] [1], § 25.

The logarithm of the characteristic function $\varphi(t)$ of the limiting distribution $F(z)$ is given by the formula (3.1), where the functions $M(z)$ and $N(z)$ and the constants γ and σ are given by the formulae (3.8)-(3.11).

4. In this section we shall give the proofs of theorems 2.1-2.3.

Proof of theorem 2.1. Let the sequence $F_n(z)$ satisfy the relation (2.3). Let us set

$$\xi_{nk} = \frac{Y_{nk}}{B_n} \qquad (k=1,2,\ldots,n), \tag{4.1}$$

$$b_{nl} = \frac{a_{nl}}{B_n} \qquad (l=1,2,\ldots,r), \tag{4.2}$$

and let m_n denote the median of the random variable ξ_{nk} for $k=1,2,\ldots,n$. Formulae (1.1) and (2.2) can be written, respectively, in the forms

$$\zeta_n = \sum_{k=1}^{n} \xi_{nk} - A_n, \tag{4.3}$$

$$P(\xi_{nk}=b_{nl}) = p_{nl} \qquad (k=1,2,\ldots,n;\ l=1,2,\ldots,r). \tag{4.4}$$

We shall show that the variables ξ_{nk} are asymptotically constant. Indeed, from the assumed convergence of $F_n(z)$ and from theorem 3.3 follows the existence of such a real number $C<\infty$ that

$$n \int_{-\infty}^{\infty} \frac{z^2}{1+z^2}\, dF_{nk}(z+m_n) < C.$$

This formula implies

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{z^2}{1+z^2}\, dF_{nk}(z+m_n) = 0;$$

hence, according to theorem 3.2, the variables ξ_{nk} are asymptotically constant.

Let us now observe that the sequences p_{nl} $(l=1,\ldots,r)$ are limited. It is thus possible to find — following the method of Cantor — such a subsequence n_α of indices that all the sequences $p_{n_\alpha l}$ $(l=1,\ldots,r)$ will be convergent. Clearly the subsequence $F_{n_\alpha}(z)$ converges with $n_\alpha\to\infty$ to $F(z)$ given by (2.3). In future we shall consider only this subsequence n_α of indices, but for the sake of simplicity we shall suppose — without limiting the generality of our considerations — that the following relations hold:

$$\lim_{n\to\infty} p_{nl} = p_l \qquad (l=1,2,\ldots,r). \tag{4.5}$$

Each of the sequences np_{nl} contains a subsequence divergent to $+\infty$ or a subsequence convergent to a finite number λ_l. Following again the method of Cantor we can choose such a subsequence n_β of indices that all the sequences $n_\beta p_{n_\beta l}$ are either convergent to finite numbers λ_l or divergent to $+\infty$. We shall consider only this subsequence n_β, but for the sake of simplicity we shall assume that for $l=1,\ldots,r$ one of the following equalities holds:

$$\lim_{n\to\infty} np_{nl}=\lambda_l \qquad (\lambda_l<\infty), \tag{4.6}$$

$$\lim_{n\to\infty} np_{nl}=\infty. \tag{4.7}$$

We shall now prove two lemmata.

Lemma 1. *If the relation* (4.6) *holds and* $\lambda_l>0$, *then the sequence* $b_{nl}-m_n$ *is limited.*

Proof. Let the assumption of this lemma hold for a certain value of l. Let us suppose that some subsequence $b_{n_il}-m_{n_i}$ tends to ∞. Then if n_i is sufficiently large, the relation

$$P(\xi_{n_ik}-m_{n_i}\geqslant z)=\int_z^\infty dF_{n_ik}(z+m_{n_i})\geqslant p_{n_il}$$

is satisfied for an arbitrary large number z. From this formula and from (3.9) it follows that

$$N(z)=-\lim_{n_i\to\infty} n_i\int_z^\infty dF_{n_ik}(z+m_{n_i})\leqslant -\lim_{n_i\to\infty} n_ip_{n_il}=-\lambda_l. \tag{4.8}$$

As $\lambda_l>0$ and z can be a bitrarily large, this formula is contrary to the relation $N(+\infty)=0$.

We can show in a similar way that no subsequence of the sequence $b_{nl}-m_n$ can be divergent to $-\infty$. Lemma 1 is thus proved.

Let us now consider the values of l for which the relation (4.6) holds, where $\lambda_l>0$. Since, for the l considered, the sequences $b_{nl}-m_n$ are limited, it is possible to find such a subsequence n_γ that for $n_\gamma\to\infty$ we have $b_{n_\gamma l}-m_{n_\gamma}\to b_l$. For the sake of simplicity we assume that for the l considered the following relation holds:

$$\lim_{n\to\infty}(b_{nl}-m_n)=b_l. \tag{4.9}$$

Lemma 2. *If the relation* (4.7) *holds, then the following equality is satisfied:*

$$\lim_{n\to\infty}(b_{nl}-m_n)=0. \tag{4.10}$$

Proof. Let us suppose that for a certain value of l the relation (4.7) holds but (4.10) does not hold, *i. e.* that in (4.9) we have $b_l\neq 0$. For instance let $b_l<0$. Let z be an arbitrary value satisfying the double inequality $b_l<z<0$. In view of the relation (4.9) we see that

$$\lim_{n\to\infty} n\int_{-\infty}^z dF_{nk}(z+m_n)\geqslant \lim_{n\to\infty} np_{nl}=\infty. \tag{4.11}$$

From this formula and from (3.8) it follows that $M(z)=\infty$. But this result is impossible. Lemma 2 is thus proved.

Now the proof of theorem 2.1 will easily be deduced from the lemmata 1-2 and theorem 3.4. Indeed, the formulae (3.8) and (3.9) will take the form

$$M(z)=\lim_{n\to\infty} n\int_{-\infty}^z dF_{nk}(z+m_n) \qquad (z<0), \tag{4.12}$$

$$-N(z)=\lim_{n\to\infty} n\int_z^\infty dF_{nk}(z+m_n) \qquad (z>0). \tag{4.13}$$

The values of the functions $M(z)$ and $N(z)$ are determined by the positive and finite limits λ_l of the sequences np_{nl}. Indeed, the equality $\lambda_l=\infty$ implies the relation (4.10); if, on the other hand, $\lambda_l=0$, then λ_l adds nothing to the functions $M(z)$ and $N(z)$.

Let the equality (4.6), where $\lambda_l>0$, be satisfied for s values of l. It will be noted that s can be equal at most to $r-1$. Indeed, at least one of the sequences p_{nl} must converge to $p_l\neq 0$ and thus, for this l, the relation (4.6) will not be satisfied. Let the values b_{l_j} $(j=1,\ldots,s)$, defined by (4.9), be ordered as follows:

$$b_{l_1}<b_{l_2}<\ldots<b_{l_w}<0<b_{l_{w+1}}<\ldots<b_{l_s}.$$

Then formulae (4.12) and (4.13) imply that

$$\begin{aligned}
M(z)&=0 && (z\leqslant b_{l_1}),\\
&\cdots\cdots\cdots\cdots\\
M(z)&=\lambda_{l_1}+\ldots+\lambda_{l_w} && (b_{l_w}<z<0),\\
&\cdots\cdots\cdots\cdots\\
-N(z)&=\lambda_{l_{w+1}}+\ldots+\lambda_{l_s} && (0<z\leqslant b_{l_{w+1}}),\\
&\cdots\cdots\cdots\cdots\\
-N(z)&=0 && (b_{l_s}<z).
\end{aligned}$$

Substituting these formulae into (3.1) we get

$$\log\varphi(t)=\left(\gamma-\sum_{j=1}^s\frac{b_{l_j}\lambda_{l_j}}{1+b_{l_j}^2}\right)it-\frac{\sigma^2t^2}{2}+\sum_{j=1}^s\lambda_{l_j}\left(e^{ib_{l_j}t}-1\right). \tag{4.14}$$

Formula (4.14) is the logarithm of a characteristic function of a sum of independent s Poisson variables and v normal variables, where $0 \leqslant s \leqslant r-2$ and $v=1$ if $\sigma \neq 0$ and $v=0$ if $\sigma=0$. If, on the other hand, $s=r-1$, the equality $\sigma=0$ holds. Indeed, let $s=r-1$, then $r-1$ among the r sequences p_{nl} will converge to 0, and thus one sequence, say p_{nl_r}, satisfies the relation

$$\lim_{n\to\infty} p_{nl_r} = 1. \tag{4.15}$$

We see that

$$\lim_{n\to\infty} [n(1-p_{nl_r})] = \lim_{n\to\infty} [np_{nl_1} + \ldots + np_{nl_{r-1}}] = \lambda_1 + \ldots + \lambda_{r-1}. \tag{4.16}$$

Here formula (3.10) will take the form

$$\lim_{n\to\infty} [n(b_{nl_r} - m_n)^2 p_{nl_r}(1-p_{nl_r})] = \sigma^2. \tag{4.17}$$

Lemma 2 and formulae (4.15) and (4.16) imply that $\sigma=0$. Theorem 2.1 is thus proved.

Proof of theorem 2.2. Let formula (2.4) hold and let the sequence $F_n(z)$ of distribution functions of ζ_n defined by (1.1) converge to a distribution function $F(z)$. Then the relation

$$\lim_{n\to\infty} (b_{nl} - m_n) = 0 \tag{4.18}$$

holds either for all $l=1,\ldots,r$ or only for one l. Indeed, for at least one value of l the relation (4.18) must hold, since at least one of the sequences p_{nl} tends to $p_l \neq 0$. Let us now suppose that for two values of l, say l_1 and l_2, the equality

$$\lim_{n\to\infty} \left(\frac{a_{l_1}}{B_n} - m_n\right) = \lim_{n\to\infty} \left(\frac{a_{l_2}}{B_n} - m_n\right) = 0 \tag{4.19}$$

is satisfied and hence

$$\lim_{n\to\infty} \frac{a_{l_1} - a_{l_2}}{B_n} = 0. \tag{4.20}$$

As $a_{l_1} \neq a_{l_2}$, the last equality holds only when $|B_n| \to \infty$. Then it follows from (4.19) that $m_n \to 0$, and thus equality (4.18) holds for $l=1,2,\ldots,r$.

Consequently, if equality (4.18) holds for all l, then the functions $M(z)$ and $N(z)$ will be given by the formulae

$$M(z) \equiv 0, \qquad N(z) \equiv 0,$$

respectively, and the limiting non-singular distribution will be normal. If equality (4.18) holds only for one value of l, say l_r, then the formula (3.10) will take the form (4.17) and in the same way as above it can be shown that $\sigma=0$. The function $F(z)$ is then a distribution function of a sum of s independent Poisson variables, where $1 \leqslant s \leqslant r-1$.

Proof of theorem 2.3. This proof is very easy. Let formula (2.5) hold and let the sequence $F_n(z)$ converge to a non-singular distribution function $F(z)$. From lemma 2 we see that for all values of l the relation (4.18) holds. Thus we have $M(z) \equiv 0$ and $N(z) \equiv 0$. As $F(z)$ is a non-singular distribution function, it follows $\sigma \neq 0$. The limiting distribution is thus normal.

5. Theorem 5.1. *Let X_n be a sequence of sums of random variables Y_{nk}:*

$$X_n = \sum_{k=1}^{n} Y_{nk},$$

where the Y_{nk} are, for each n, independent and equally distributed according to the distribution law

$$\begin{aligned} &P(Y_{nk} = a_{n1}) = p_{n1}, \\ &P(Y_{nk} = a_{n2}) = p_{n2} = 1 - p_{n1}, \end{aligned} \tag{5.1}$$

and where a_{n1}, a_{n2} and p_{n1} are arbitrary functions of n.

Let $F_n(z)$ be the sequence of distribution functions of the standardized variables

$$\zeta_n = \frac{X_n - E(X_n)}{\sqrt{D(X_n)}}. \tag{5.2}$$

Then:

1° if the relation

$$\lim_{n\to\infty} np_{n1}(1-p_{n1}) = \infty, \tag{5.3}$$

holds, the sequence $F_n(z)$ will satisfy the relation

$$\lim_{n\to\infty} F_n(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-z^2/2} dz; \tag{5.4}$$

2° if the relations

$$\lim_{n\to\infty} p_{n1} = p_1, \tag{5.5}$$

$$\lim_{n\to\infty} np_{n1}(1-p_{n1}) = \lambda, \tag{5.6}$$

hold, where $\lambda > 0$, and if there exists such an integer n_0 that for $n > n_0$ all the differences $a_{n1} - a_{n2}$ have the same sign, $F_n(z)$ converges with $n \to \infty$ to a distribution function $F(z)$ of a Poisson variable in all continuity points of $F(z)$.

THEOREM 5.2. *Let X_n be a sequence defined by*

$$X_n = \sum_{k=1}^{n} Y_{nk},$$

where Y_{nk} $(k=1,\ldots,n)$ are independent random variables, equally distributed according to the law

$$P(Y_{nk}=a_l)=p_{nl} \qquad (l=1,2,\ldots,r), \tag{5.7}$$

and where

$$0 \leqslant p_{nl} \leqslant 1, \qquad \sum_{l=1}^{r} p_{nl}=1.$$

Let $F_n(z)$ be the sequence of distribution functions of the variables ξ_n defined by (5.2). *Then:*

1° *if the relation*

$$\lim_{n\to\infty} n[p_{n1}p_{n2}+p_{n1}p_{n3}+\ldots+p_{n(r-1)}\,p_{nr}]=\infty \tag{5.8}$$

holds, the sequence $F_n(z)$ will satisfy the relation

$$\lim_{n\to\infty} F_n(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-z^2/2}\,dz; \tag{5.9}$$

2° *if the relations*

$$\lim_{n\to\infty} p_{nl}=p_l \qquad (l=1,2,\ldots,r), \tag{5.10}$$

$$\lim_{n\to\infty} np_{nl}=\lambda_l \tag{5.11}$$

$$\lim_{n\to\infty} n[p_{n1}p_{n2}+p_{n1}p_{n3}+\ldots+p_{n(r-1)}\,p_{nr}]=\lambda \tag{5.12}$$

hold, where $\lambda>0$, then the sequence $F_n(z)$ converges with $n\to\infty$ to a distribution function $F(z)$ of a sum of s Poisson variables in all continuity points of $F(z)$, where $1\leqslant s\leqslant r-1$.

We shall give only the proof of the last theorem, as the proof of theorem 5.1 follows the same lines.

Proof of theorem 5.2. Let us write

$$\eta_{nk}=\frac{Y_{nk}-E(Y_{nk})}{\sqrt{nD(Y_{nk})}}.$$

Here we have

$$E(Y_{nk})=a_1p_{n1}+a_2p_{n2}+\ldots+a_rp_{nr}, \tag{5.13}$$

$$D(Y_{nk})=p_{n1}p_{n2}(a_1-a_2)^2+p_{n1}p_{n3}(a_1-a_3)^2+\ldots +p_{n(r-1)}p_{nr}(a_{r-1}-a_r)^2. \tag{5.14}$$

The random variable η_{nk} takes, with the probability p_{nl}, the value c_{nl} given by the formula

$$c_{nl}=\frac{a_lq_{nl}-a_1p_{n1}-\ldots-a_{l-1}p_{n(l-1)}-a_{(l+1)}p_{n(l+1)}-\ldots-a_rp_{nr}}{\sqrt{nD(Y_{nk})}}, \tag{5.15}$$

where $q_{nl}=1-p_{nl}$.

1° Let the relation (5.8) hold. Then, for all values of l,

$$\lim_{n\to\infty} c_{nl}=0. \tag{5.16}$$

From the last relation it follows that the sequence m_n of medians of the variables η_{nk} tends to 0, and that for an arbitrary $\varepsilon>0$ the following relation holds:

$$\lim_{n\to\infty}\int_{|z|>\varepsilon} dF_{nk}(z)=0.$$

Thus the variables η_{nk} are asymptotically constant and theorem 3.4 can be applied. Formulae (3.8) and (3.9) give here

$$M(z)\equiv 0, \qquad N(z)\equiv 0, \tag{5.17}$$

and formula (3.10) is of the form

$$\lim_{n\to\infty} n\left[\sum_{l=1}^{r} c_{nl}^2p_{nl}-\left(\sum_{l=1}^{r} c_{nl}\,p_{nl}\right)^2\right]=1. \tag{5.18}$$

Since, for each n, we have $E(\zeta_n)=0$, therefore from theorem 3.4 and from the last two relations we obtain the following equality for the characteristic function $\varphi(t)$ of the limiting distribution function:

$$\log\varphi(t)=-\frac{t^2}{2}.$$

The relation (5.9) follows from the last relation.

2° Let the relations (5.10)-(5.12) hold. Then one and only one of the sequences p_{nl}, say p_{n1}, does not converge to 0. Thus $p_{n1}\to 1$. Then in formula (5.12), for all $l\neq 1$, we shall have

$$\lim_{n\to\infty} np_{n1}p_{nl}=\lambda_l, \tag{5.19}$$

and consequently

$$\lim_{n\to\infty} np_{nl}=\lambda_l, \tag{5.20}$$

$$\lim_{n\to\infty} np_{nl_1}p_{nl_2}=0 \qquad (l_1\neq 1;\ l_2\neq 1). \tag{5.21}$$

Then formulae (5.14), (5.15) and (5.21) imply that

(5.22) $$\lim_{n\to\infty} D(X_n)=\lim_{n\to\infty} nD(y_{nk})=\sum_{l=2}^{r}\lambda_l(a_l-a_1)^2,$$

where $\lambda_l\neq 0$ at least for one value of $l\geqslant 2$.

We see further that $q_{n1}\to 0$ and, for $l=2,\ldots,r$, $q_{nl}\to 1$. Then it follows from (5.15) and (5.22) that

(5.23) $$\lim_{n\to\infty} c_{n1}=0,$$

(5.24) $$\lim_{n\to\infty} c_{nl}=\frac{a_l-a_1}{\sqrt{\sum_{l=2}^{r}\lambda_l(a_l-a_1)^2}}=c_l\neq 0,$$

for $l=2,\ldots,r$. Thus the sequence m_n of medians of η_{nk} converges with $n\to\infty$ to 0. Therefore for each $\varepsilon>0$

$$\lim_{n\to\infty} P(|\eta_{nk}-m_n|>\varepsilon)=\lim_{n\to\infty} P(|\eta_{nk}|>\varepsilon)=\lim_{n\to\infty} P(\eta_{nk}\neq c_{n1})=0.$$

The variables η_{nk} are asymptotically constant and theorem 3.4 can be applied. Formula (3.10) is here of the form $\lim_{n\to\infty}[nc_{n1}^2p_{n1}(1-p_{n1})]=\sigma^2$. From the relation $n(1-p_{n1})\to\lambda$ — which is satisfied in view of (5.19) and (5.21) — and from (5.23) it follows that $\sigma=0$.

Since $E(\zeta_n)=0$ for each n, we see from (3.8) and (3.9) that the logarithm of the characteristic function of the limiting distribution function of the sequence $F_n(z)$ is given by the formula

(5.25) $$\log\varphi(t)=-\sum_{j=1}^{s}\lambda_{l_j}c_{l_j}it+\sum_{j=1}^{s}\lambda_{l_j}(e^{ic_{l_j}t}-1),$$

where $1\leqslant s\leqslant r-1$. Assertion 2^0 is thus proved.

6. We discuss briefly the problem of extending of theorems 2.1-2.3 to the case when the equally distributed discrete random variables Y_{nk} can, with positive probability, take infinitely many values.

In the case considered lemmata 1-2 (§ 4) hold. However, such general and simple theorems as those formulated in § 2, where the convergence of the sequence $F_n(z)$ was the only assumption, cannot be obtained here. Difficulties arise when the set of limits b_l of the sequence $b_{nl}-m_n$ has finite density points. We shall not go farther into this matter. We shall only give the following example, showing that if no additional assumptions[6]) concerning the set of the limits b_l are made, then an arbitrary infinitely divisible distribution function can be the limit of a sequence $F_n(z)$ of distribution functions of random variables ζ_n defined by (1.1).

Let us consider an infinitely divisible distribution function for which

$$M(z)=0\quad (z<a),\qquad M(z)=M(b)\quad (b\leqslant z<0),$$

where $a<b$ and

$$N(z)\equiv 0.$$

Let us assume that the function $M(z)$ is continuous in the interval (a,b). We divide this interval into n subintervals by the points

$$a=z_1<z_2<\ldots<z_n<z_{n+1}=b.$$

Let Y_{nk} $(k=1,2,\ldots,n)$ be independent and equally distributed random variables and let the logarithm of the characteristic function of Y_{nk} be equal to

$$\frac{1}{n}\sum_{j=1}^{n}(e^{iz_jt}-1)[M(z_{j+1})-M(z_j)].$$

Let $\varphi_n(t)$ be the characteristic function of the random variable $\zeta_n=\sum_{k=1}^{n}Y_{nk}$. Then

$$\log\varphi_n(t)=\sum_{j=1}^{n}(e^{iz_jt}-1)[M(z_{j+1})-M(z_j)],$$

and the following relation holds:

$$\lim_{n\to\infty}\log\varphi_n(t)=\int_a^b(e^{izt}-1)\,dM(z).$$

However, $M(z)$ is an arbitrary non-decreasing function continuous in the interval (a,b), and thus an arbitrary infinitely divisible distribution can be obtained in the limit.

[6]) The case when the ξ_{nk} are integervalued has been considered by A. Rényi [3].

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