On linear methods of summability

by

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The earliest systematic applications of functional analysis to various problems of the theory of summability were due to S. Mazur and S. Banach. In their investigations (see [12],[5]) [1]) the problematic part was restricted to questions of the consistency of Toeplitz methods corresponding to matrices on which some restricting hypotheses were imposed.

In the autumn of 1932 the authors of this paper carried out some research work with the aim of generalizing the applicability of the methods of functional analysis to the theory of summability. Their main results were published without proof in the C. R. de l'Académie des Sciences [13]. The most important observation consisted in finding out that the Banach spaces do not present a sufficient tool of research, and that the fields of the summability methods constitute more general linear spaces, to which the authors gave the name of *B_0-spaces*.

At the time when functional analysis was flourishing in the school grouped around S. Banach, called the "school of Lwów", investigations of the general properties of B_0-spaces and, later, of polynomial operations and other problems of functional analysis appeared to us as the most urgent task of research. This delayed the publication of a complete paper on the applications of B_0-spaces to the theory of summability, and the outbreak of World War II made it impossible. Various absorbing activities in the period immediately after the war made us put off the publication until now.

Meanwhile, the B_0-spaces, forming a particular case of linear topological spaces, were studied by numerous mathematicians working in the domain of functional analysis. The applications of the methods of functional analysis to the theory of summability also attracted attention and several mathematicians obtained results similar to ours (Agnew [3], Brudno [6], Darewsky [7], [8], Wilansky [18], [19], and others). Especially the school of Tübingen obtained many interesting results

[1]) The numbers in brackets refer to the bibliography at the end of this paper.

in the general theory of summability. In the papers of Zeller [21], [22], [23] one may find some of our theorems partly in a generalized form. Zeller applies B_0-spaces in his research, partly in a manner similar to ours of 1932.

The present paper has been drawn up with slight alterations after some notes dating from the period of our collaboration during the year 1932. Besides certain modifications of an editorial character, there are also some more important changes, consisting in the omission of proofs of some theorems which we believe to be given a more adequate presentation from the methodical point of view in the works of Zeller; moreover, we have added the part (β) of theorem 11 in 4.4, which we proved in 1935. In sections 2 and 3 we prove some theorems dealing, among other problems, with bounded sequences summable by a linear method of summability; in our note [13] those results were given only in part. The consistency theorem for bounded sequences (see [13], théorème 6) is not to be found in the papers of Zeller. It has been found independently and proved again, without the knowledge of our note, by Brudno [6]. The proof of Brudno, however, is more complicated than ours, and the ideas of functional analysis do not appear in it.

At one time the authors intended to transfer their investigations to *continual methods of summability*, *i. e.* methods based upon linear operations transforming sequences into functions. This has been done recently by two pupils of the former of us (cf. Altman [4], Włodarski [20]).

This paper is restricted to the real case; the extension of its results to the summability of sequences with complex terms does present no difficulties.

1. In this section we shall first introduce certain definitions and notations. The terminology concerning the B- (Banach) and B_0-spaces is that used in the monograph of Banach [5] and in our paper [14]. Small italics $x, y, u, v, \ldots$ stand for sequences of real numbers; e_n denotes the *n-th unit sequence*, *i. e.* the sequence whose n-th term is equal to 1, and the remaining terms are equal to 0; e stands for the sequence all elements of which are equal to 1. By T, T_b, T_c, T_0 we denote the sets of all sequences $x=\{t_n\}$, of bounded sequences, convergent sequences and sequences convergent to 0, respectively. With the usual definitions of addition and multiplication by scalars they are linear spaces. The norm being defined as

$$\|x\| = \sup_n |t_n|,$$

T_b, T_c, and T_0 are B-spaces; if we denote in T the n-th pseudonorm as $\|x\|_n = |t_n|$, T becomes a B_0-space.

Infinite matrices $(a_{in}), (b_{in}), (c_{in}), \ldots$ and corresponding methods of summability are denoted by capital italics $A, B, C, \ldots$

The expressions "the method $A=(a_{in})$" and "the method corresponding to the matrix (a_{in})" mean the same, namely "the linear method of summability generated by aid of the matrix (a_{in})" — as below.

The series

$$A_i(x) = \sum_{n=1}^{\infty} a_{in} t_n$$

are called the *transforms* of the sequence $x=\{t_n\}$ (corresponding to the methods A); we shall also write

$$A_{ik}(x) = \sum_{n=1}^{k} a_{in} t_n \qquad (i, k = 1, 2, \ldots).$$

If for a matrix $A=(a_{in})$ and a sequence $x=\{t_n\}$ all the transforms $A_i(x)$ are meaningful and there exists the finite

$$\lim_{i\to\infty} A_i(x) = A(x),$$

then the sequence x is called *summable to $A(x)$ by the method A*, or briefly *A-summable to $A(x)$*. Hence every matrix leads to *a method of summability* defining a generalized limit $A(x)$ in a certain class of sequences. The set A^* of all sequences summable by the method A is called the *field of summability of the method A*; by A_0^* we denote the subset of A^* consisting of those sequences which are A-summable to 0. If $T_c \subset A^*$ or $T_0 \subset A^*$, the method A will be termed *convergence preserving* or *convergence preserving for null sequences*, respectively. If, moreover,

$$A(x) = \lim_{n\to\infty} t_n \quad \text{for every} \quad x \in T_c \text{ or } x \in T_0,$$

the method A will be called respectively *permanent* or *permanent for null sequences*. If, given two methods $A=(a_{in})$ and $B=(b_{in})$, we have $A^* \subset B^*$, then the method B is called *not weaker than A*; if $x \in A^*$ and $x \in B^*$ and $A(x)=B(x)$, the methods are called *consistent for the sequence x*. If $T_b A^* \subset B^*$, the method B is called *not weaker than A for bounded sequences*; if $T_b A^* \subset B^*$ and $A(x)=B(x)$ for every $x \in T_b A^*$, the methods A and B are termed *consistent for bounded sequences*. Similar definitions are used with regard to convergency or convergency to 0 instead of boundedness. The methods A and B are said to be *consistent* if they are consistent for every sequence $x \in A^* B^*$. Two methods with a common field of summability and consistent for every x are called *equivalent*. Method $A=(a_{in})$ is called *identical* if $a_{in}=0$ for $i \neq n$, $a_{ii}=1$; method A is *row-finite* if, in every row $a_{i1}, a_{i2}, \ldots$, almost all terms are equal to 0;

if $a_{in}=0$ for $n>i$ and $a_{ii}\neq 0$, the method is called *normal*. If the system of equations

$$\sum_{n=1}^{\infty} a_{in}t_n=0 \qquad (i=1,2,\ldots)$$

has only one solution, $t_n=0$, method $A=(a_{in})$ is called a *U-method*. If method A has the above property under the supplementary hypothesis of $\{t_n\}\epsilon T_b$, it is called a *U-method for bounded sequences*.

1.1. Let A be the method of summability corresponding to the matrix (a_{in}). The following properties play an essential role in the theory of summability:

(α) *there exists* $\lim\limits_{i\to\infty} a_{in}=a_n$ *for* $n=1,2,\ldots,$

(α') *there exists* $\lim\limits_{i\to\infty} a_{in}=0$ *for* $n=1,2,\ldots,$

(β) $\sup\limits_i \sum\limits_{n=1}^{\infty}|a_{in}|<\infty,$

(γ) *there exists* $\lim\limits_{i\to\infty}\sum\limits_{n=1}^{\infty} a_{in}=a.$

The fulfilment of (α) and (β) implies immediately

$$\sum_{n=1}^{\infty}|a_n|<\infty.$$

It is well known that:

method A is convergence preserving if and only if the conditions (α), (β) *and* (γ) *are satisfied*;

method A is convergence preserving for null sequences if and only if the conditions (α) *and* (β) *are satisfied* (see [16]);

method A is permanent if and only if the conditions (α'), (β), (γ) *are satisfied and* $a=1$;

method A is permanent for null sequences if and only if the conditions (α') *and* (β) *are satisfied* (see [11], [17]).

Suppose now that we are given a second method of summability B, corresponding to the matrix (b_{in}), and let the conditions (α) and (γ) define the numbers b_n and b with regard to method B analogously to the numbers a_n and a with regard to method A. If the methods A and B are convergence preserving (convergence preserving for null sequences), then these methods are consistent for convergent sequences (consistent for null sequences) if and only if $a_n=b_n$ for $n=1,2,\ldots$ and $a=b$ ($a_n=b_n$ for $n=1,2,\ldots$).

After Zeller, we adopt the following notation for the convergence preserving method:

$$\chi(A)=\lim_{i\to\infty} A_i(e)-\sum_{k=1}^{\infty}\lim_{i\to\infty} A_i(e_k)=a-\sum_{k=1}^{\infty}a_k.$$

The following lemmata will be used in the sequel.

1.2. *Let the matrix (a_{in}) be row-finite. The system of equations*

$$\sum_{n=1}^{\infty} a_{in}t_n=u_i \qquad (i=1,2,\ldots)$$

has a solution if and only if, for every system $\lambda_1,\lambda_2,\ldots,\lambda_p$ of reals,

$$(*) \qquad \sum_{i=1}^{p}\lambda_i a_{in}=0 \qquad (n=1,2,\ldots)$$

implies

$$\sum_{i=1}^{p}\lambda_i u_i=0.$$

This is the well-known theorem of Toeplitz (see, for instance, [5], p. 51).

1.2.1. *A necessary and sufficient condition that every bounded sequence be summable by method $A=(a_{in})$ is that the conditions* (α), (β), *and* (γ) *of* 1.1 *and the following one be satisfied:*

$$(\delta) \qquad \lim_{i\to\infty}\sum_{n=1}^{\infty}|a_{in}-a_n|=0.$$

This theorem is due to Schur [16].

1.2.2. *Let the matrix (a_{in}) satisfy the condition* (α') *of* 1.1 *and the following*:

$$\lim_{n\to\infty} a_{in}=0 \quad \text{for} \quad i=1,2,\ldots,$$

$$\sup_n |a_{in}|=1 \quad \text{for} \quad i=1,2,\ldots;$$

then

(a) *there exists a sequence $\{i_k\}$ of indices and a bounded sequence $\{\lambda_k\}$ such that the sequence composed of the terms*

$$u_n=\sum_{k=1}^{\infty}a_{i_kn}\lambda_k$$

is bounded and divergent,

(b) $\sup\limits_n \sum\limits_{k=1}^{\infty}|a_{i_kn}|<\infty.$

Under the hypotheses of the lemma it is possible to define successively two increasing sequences of indices i_k and n_k with the properties

$$|a_{i_k n}|<2^{-(k+1)} \quad \text{for} \quad 1\leqslant n\leqslant n_k \quad \text{and} \quad n_{k+1}<n \quad (k=1,2,\ldots).$$

Then

$$\sum_{k=1}^{\infty}|a_{i_k n}|<2 \quad \text{for} \quad n=1,2,\ldots$$

The sequence

$$u_n^{\lambda}=\sum_{k=1}^{\infty} a_{i_k n}\lambda_k$$

diverges for at least one bounded sequence $\lambda=\{\lambda_k\}$, for in the contrary case $\lim\limits_n a_{i_k n}=0$ would, in virtue of 1.2.1, imply $\sum\limits_{k=1}^{\infty}|a_{i_k n}|\to 0$ as $n\to\infty$, which is impossible.

1.3. Let method A be convergence preserving or convergence preserving for null sequences (these hypotheses are not essential for the argument in 1.3-1.4.2). The fields of summability A^* and A_0^* form a linear subspace of the space T; in A^* and A_0^* we introduce the pseudonorms for $n=1,2,\ldots$,

(1) $$\|x\|_n^1=|t_n|,$$

(2) $$\|x\|_n^2=\sup_k|A_{nk}(x)|,$$

(3) $$\|x\|^3=\sup_n|A_n(x)|;$$

then we set

$$\|x\|^1=\sum_{n=1}^{\infty}\frac{1}{2^n}\,\frac{\|x\|_n^1}{1+\|x\|_n^1},$$

$$\|x\|^2=\sum_{n=1}^{\infty}\frac{1}{2^n}\,\frac{\|x\|_n^2}{1+\|x\|_n^2}.$$

Obviously $x=0$ if $\|x\|_n^1=0$ for $n=1,2,\ldots$

1.3.1. *The spaces A^* and A_0^* are B_0-spaces under the norm*

(4) $$\|x\|=\|x\|^1+\|x\|^2+\|x\|^3;$$

the convergence generated by the norm in these spaces is equivalent to the convergence generated by the totality of the pseudonorms $\|x\|^3$ and $\|x\|_n^j$, where $j=1,2$, $n=1,2,\ldots$

1.3.2. *If the method A is row-finite, then A^* and A_0^* are B_0-spaces, under the norm*

(4′) $$\|x\|=\|x\|^1+\|x\|^3;$$

the convergence in these spaces, generated by the norm (4′), is equivalent to the convergence generated by the totality of the pseudonorms $\|x\|^3$ and $\|x\|_n^1$ where $n=1,2,\ldots$

1.3.3. *If the U-method A is row-finite, then A^* and A_0^* are Banach spaces under the norm $\|x\|^3$.*

The proof of 1.3.1 and 1.3.2 presents no difficulties (see Zeller [21]). The case 1.3.3 requires somewhat deeper argument. The pseudonorm $\|x\|^3$ is in this case a (homogeneous) norm, since $A_n(x)=0$ $(n=1,2,\ldots)$ implies $x=0$.

Now we must prove that the spaces A^* and A_0^* are complete. For this purpose it suffices to prove that $x_k\in A^*$ $(x_k\in A_0^*)$ together with

$$\sup_i|A_i(x_k)-u_i|\to 0$$

implies the existence of a sequence $x_0=\{t_n^0\}$ such that $u_i=A_i(x_0)$ for $i=1,2,\ldots$ Let the numbers $\lambda_1,\lambda_2,\ldots,\lambda_p$ satisfy the condition (*) of 1.2; then

$$\sum_{i=1}^{p}\lambda_i A_i(x_k)=0 \quad \text{for} \quad k=1,2,\ldots,$$

whence

$$\sum_{i=1}^{p}\lambda_i u_i=0.$$

In virtue of 1.2 the system of equations $u_i=A_i(z)$ has precisely one solution.

1.3.4. *For row-finite methods (row-finite U-methods) the norm (4) is equivalent to the norm (4′) of 1.3.2 (to the norm $\|x\|^3$).*

This is implied by the well-known theorem of Banach ([5], p. 41, Théorème 7).

1.3.5. Let the U-method A be row-finite. Then $x_k=\{t_n^k\}$ and $\|x_k\|^3\to 0$ implies

$$\lim_k t_n^k=0 \quad \text{for} \quad n=1,2,\ldots$$

An analogous property have the norms (4) of 1.3.1 and (4′) of 1.3.2 as immediately follows from their definitions.

By aid of the above norms in A^* and A_0^* we may define the metric in the usual way by the formula $d(x,y)=\|x-y\|$; we shall say that *this metric is implied by the norm* $\|\ \|$.

1.4. Let method A be convergence preserving or convergence preserving for null sequences. Denote by U the set of double sequences $u=\{u_{in}\}$ for which there exists $\lim\limits_i\lim\limits_n u_{in}$. It is a B_0-space ([14], (I),

p. 190); under the usual definition of addition and multiplication by scalars and with the pseudonorms

$$\|u\|_1 = \sup_i |\lim_{n\to\infty} u_{in}|,$$

$$\|u\|_{i+1} = \sup_n |u_{in}| \quad \text{for} \quad i=1,2,\ldots$$

Let us associate with the element $x=\{t_n\}\epsilon A^*$ the element (y,z,u) of the product $W=T\times T_c\times U$ where $y=\{t_n\}$, $z=\{A_i(x)\}$, $u=\{A_{in}(x)\}$. This correspondence obviously establishes an isomorphical mapping of A^* onto a subset R of W, which is closed in virtue of 1.3-1.3.4. The space W is separable since such are its components (for the separability of U see, for instance, [14], (I), p. 191-193). Hence:

1.4.1. *The spaces A^* and A_0^* are separable.*

1.4.2. It is not difficult to establish the general form of linear functionals in W; therefore by the above described mapping of A^* onto W we may determine the general form of linear functionals in A^* and A_0^* (see [21]).

Let us also notice that if the series

$$\xi(x) = \sum_{n=1}^{\infty} a_n t_n$$

converges for every $x\epsilon A^*$ $(x\epsilon A_0^*)$, then $\xi(x)$ is a linear functional in A^* (A_0^*).

2. In this section we shall deal with the problems of consistency of methods of summability. Given two methods of summability A and B and a class of sequences $\overline{T}$ one may ask under what hypotheses about A and B the equality $A(x)=B(x)$ results for every $x\epsilon S=\overline{T}A^*B^*$. We shall answer this problem in certain simple cases.

2.1. *Let $x_i=\{t_n^i\}$ denote sequences convergent to 0 such that*

$$\lim_{i\to\infty} t_n^i = t_n \quad \textit{for} \quad n=1,2,\ldots, \quad \lim_{n\to\infty} t_n = 0,$$

and

$$\sup_n |t_n^i| \leqslant K < \infty \quad \textit{for} \quad i=1,2,\ldots$$

Then, for every $\varepsilon>0$ and every positive integer p, there exist numbers $a_1, a_2, \ldots, a_k$ such that

$$(\alpha) \qquad \begin{aligned} a_i &\geqslant 0 \quad \textit{for} \quad i=1,2,\ldots,k, \\ a_i &= 0 \quad \textit{for} \quad i=1,2,\ldots,p, \end{aligned}$$

$$(\beta) \qquad \sum_{i=1}^{k} a_i = 1,$$

$$(\gamma) \qquad \sup_n \Big|\sum_{i=1}^{k} a_i t_n^i - t_n\Big| < \varepsilon.$$

The hypotheses imply the existence of increasing sequences of indices $\{n_i\}$ and $\{m_i\}$ such that

$$\sup_{n>n_i} |t_n^{m_l} - t_n| < \frac{\varepsilon}{2} \quad \text{for} \quad l=1,2,\ldots,i-1 \qquad (m_0=1),$$

$$\sup_{n\leqslant n_i} |t_n^{m_i} - t_n| < \frac{\varepsilon}{2}.$$

Let us choose $r>(4K+\varepsilon p)/\varepsilon$ and set

$$a_i = \begin{cases} \dfrac{1}{r-p} & \text{when} \quad i=m_l, \ l>p, \\ 0 & \text{when} \quad i\leqslant m_p, \ i\neq m_l, \ l>p. \end{cases}$$

For $k=m_r$ the conditions (α) and (β) are satisfied, and, as can easily be seen,

$$\Big|\sum_{i=1}^{k} a_i t_n^i - t_n\Big| = \Big|\sum_{l=p+1}^{r} a_{\ l}(t_n^{m_l} - t_n)\Big| < \frac{\varepsilon}{2} + \frac{2K}{r-p} < \varepsilon.$$

We have not used the concepts of functional analysis either in the formulation or in the proof of the above lemma. In the terms of functional analysis the hypotheses of Lemma 2.1 are equivalent to the weak convergence of the elements $x_i=\{t_n^i\}$ to $x=\{t_n\}$ in the space T_0. The lemma asserts that there are numbers a_i satisfying (α) and (β), and such that

$$\Big\|\sum_{i=1}^{k} a_i x_i - x\Big\| < \varepsilon.$$

By a well-known theorem of Mazur such assertion is implied in every Banach space by the weak convergence to a limit element.

2.2. *Let method A be permanent for null sequences. If a bounded sequence $x=\{t_n\}$ is A-summable to 0, then for every p and $\varepsilon>0$ there exist non-negative numbers $a_1, a_2, \ldots, a_s$ such that $a_i\leqslant 1$ and such that the sequence $\bar{x}=\{\bar{t}_n\}$, whose terms are defined as*

$$(1) \qquad \bar{t}_n = \begin{cases} t_n & \textit{for} \quad n=1,2,\ldots,p, \\ a_{n-p} t_n & \textit{for} \quad n=p+1,p+2,\ldots,p+s, \\ 0 & \textit{elsewhere}, \end{cases}$$

satisfies the inequality

$$(2) \qquad \sup_i |A_i(x) - A_i(\bar{x})| < \varepsilon.$$

Let

$$y_m = \left\{ \sum_{n=1}^{m} a_{in} t_n \right\}, \qquad y = \{A_i(x)\}.$$

These sequences satisfy the hypotheses of Lemma 2.1, whence there exist non-negative numbers $\beta_1, \beta_2, \ldots, \beta_{p+s}$ such that $\beta_i = 0$ for $i = 1, 2, \ldots, p$, $\sum_{i=1}^{p+s} \beta_i = 1$ and

$$\left\| \sum_{m=1}^{p+s} \beta_m y_m - y \right\|^3 < \varepsilon. \tag{3}$$

The i-th element of the sequence $\sum_{1}^{p+s} \beta_m y_m$ is equal to

$$(\beta_1 + \ldots + \beta_{p+s}) a_{i1} t_1 + (\beta_2 + \ldots + \beta_{p+s}) a_{i2} t_2 + \ldots + \beta_{p+s} a_{ip+s} t_{p+s}, \tag{4}$$

whence setting $a_n = \beta_{p+n} + \beta_{p+n+1} + \ldots + \beta_{p+s}$ for $n = 1, 2, \ldots, s$ we see, in virtue of (3) and (4), that sequence (1) satisfies the inequality (2).

Remark. Given an increasing sequence of indices $\{n_k\}$, the index s may be chosen in such a way that $p+s$ will be an element of the sequence $\{n_k\}$.

This follows by a similar argument — it suffices to replace, in the above proof, y_n by y_{n_m}.

2.3. *If method $A = (a_{in})$ satisfies the condition* (β) *of* 1.1, *then there exists a normal method $B = (b_{in})$ satisfying the same condition* (β) *and equivalent to method A for bounded sequences.*

Let us choose an increasing sequence of indices $\{m_i\}$ in such a way that

$$\sum_{n=m_i}^{\infty} |a_{in}| < \frac{1}{2^i} \quad \text{for} \quad i = 1, 2, \ldots,$$

and define the matrix (b_{in}) as

$$b_{in} = \begin{cases} 0 & \text{for} \quad i < m_1, \; n = 1, 2, \ldots \; \text{and} \; i \neq n, \\ 1 & \text{for} \quad i = n < m_1, \\ a_{in} & \text{for} \quad m_j \leqslant i < m_{j+1}, \; n < m_j, \\ 2^{-i} & \text{for} \quad m_j \leqslant i = n < m_{j+1}, \\ 0 & \text{elsewhere.} \end{cases}$$

The method $B = (b_{in})$ has the desired properties.

2.4. Theorem 1. *Let the methods A and B be permanent for null sequences, let B be not weaker than A for bounded sequences A-summable to* 0. *Then every bounded sequence A-summable to* 0 *is B-summable to* 0.

By 2.3 we may suppose that the methods A and B are normal ones. Let $x = \{t_n\}$ be a bounded sequence belonging to A_0^*. By 2.2 there exist sequences of the form

$$(*) \quad \begin{aligned} x_1 &= a_1 t_1, a_2 t_2, \ldots, a_{n_1} t_{n_1}, 0, 0, \ldots, \\ x_2 &= t_1, t_2, \ldots, t_{n_1}, a_{n_1+1} t_{n_1+1}, a_{n_1+2} t_{n_1+2}, \ldots, a_{n_2} t_{n_2}, 0, 0, \ldots, \\ x_3 &= t_1, t_2, \ldots, t_{n_2}, a_{n_2+1} t_{n_2+1}, a_{n_2+2} t_{n_2+2}, \ldots, a_{n_3} t_{n_3}, 0, 0, \ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

such that $0 \leqslant a_n \leqslant 1$ and

$$\|x_n - x\|^3 < \frac{1}{2^n} \quad \text{for} \quad n = 1, 2, \ldots$$

Let $\lambda = \{\lambda_n\}$ be an arbitrary bounded sequence; the series

$$x_\lambda = \lambda_1 x_1 + \sum_{n=2}^{\infty} \lambda_n (x_n - x_{n-1}) \tag{5}$$

converges in the space A_0^*, whence x_λ is A-summable to 0. Since from the convergence implied by the norm $\|x\|^3$ follows the convergence of every "coordinate" of the sequence separately, therefore the formula $(*)$ implies that $x_\lambda = \{t_n^\lambda\}$ is of the form

$$(**) \quad \begin{aligned} t_1^\lambda &= \lambda_1 a_1 t_1 + \lambda_2 (1 - a_1) t_1, \\ t_2^\lambda &= \lambda_1 a_2 t_2 + \lambda_2 (1 - a_2) t_2, \\ &\ldots\ldots\ldots\ldots \\ t_{n_1}^\lambda &= \lambda_1 a_{n_1+1} t_{n_1+1} + \lambda_2 (1 - a_{n_1}) t_{n_1}, \\ t_{n_1+1}^\lambda &= \lambda_2 a_{n_1+1} t_{n_1+1} + \lambda_3 (1 - a_{n_1+1}) t_{n_1+1}, \\ t_{n_1+2}^\lambda &= \lambda_2 a_{n_1+2} t_{n_1+2} + \lambda_3 (1 - a_{n_1+2}) t_{n_1+2}, \\ &\ldots\ldots\ldots\ldots \\ t_{n_2}^\lambda &= \lambda_2 a_{n_2} t_{n_2} + \lambda_3 (1 - a_{n_2}) t_{n_2}, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

whence the sequence x_λ is bounded.

The functionals

$$B_i(x) = \sum_{n=1}^{\infty} b_{in} t_n$$

being continuous in A_0^*, we have

$$B_i(x_\lambda) = \lambda_1 B_i(x_1) + \sum_{n=2}^{\infty} \lambda_n B_i(x_n - x_{n-1}).$$

The sequence x_λ being B-summable by hypothesis, method C, corresponding to the matrix whose i-th row is composed of the elements

$$B_i(x_1), B_i(x_2-x_1), B_i(x_3-x_2), \ldots,$$

is such that every bounded sequence $\{\lambda_n\}$ is C-summable. Therefore this matrix satisfies the condition (δ) of 1.2.1, whence

$$\lim_{i\to\infty}\Big(|B_i(x_1)|+\sum_{n=2}^{\infty}|B_i(x_n-x_{n-1})|\Big)=0;$$

thus

$$B_i(x)=B_i(x_1)+\sum_{n=2}^{\infty}B_i(x_n-x_{n-1})\to 0.$$

THEOREM 1'. *Let the methods $A^1, A^2, \ldots, B$ be permanent for null sequences and let $\prod_k A_0^{k*}T_b \subset B^*$. Then every bounded sequence A^i-summable to 0 for $i=1,2,\ldots$ is B-summable to 0.*

As in proof of Theorem 1, we may suppose that the methods $A^1, A^2, \ldots$ $\ldots, B$ are normal. Let $\|\ \|_k^3$ denote the norm $\|\ \|^3$ in the space A_0^{k*} and let the sequence $x=\{t_n\}$ be A^i-summable to 0 for $i=1,2,\ldots$ Applying 2.2 we can prove that there exist numbers $0\leqslant a_n\leqslant 1$ and sequences x_n of the form $(*)$ such that

$$\|x_n-x\|_k^3<2^{-n} \quad \text{for } k=1,2,\ldots n, \quad n=1,2,\ldots$$

Again as in the proof of Theorem 1, we observe that the series (5) converges in the space A_0^{k*} for every bounded sequence, whence x_λ is A^k-summable to 0 for $k=1,2,\ldots$ This implies, as before, that x is B-summable to 0.

THEOREM 2. *Let the methods A and B be convergence preserving and consistent for convergent sequences and let $\chi(A)\neq 0$. If method B is not weaker than A for bounded sequences, then these methods are consistent for bounded sequences* (see [13], Théorème 6).

Denote by $\bar{A}$ and $\bar{B}$ respectively the methods corresponding to the matrices $(a_{in}-a_n)$ and $(b_{in}-b_n)$ respectively, where $a_n=\lim\limits_i a_{in}$, $b_n=$ $=\lim\limits_i b_{in}$, and, given a bounded A-summable sequence x, let

$$c=\frac{A(x)-\sum\limits_{n=1}^{\infty}a_n t_n}{\chi(A)}.$$

Then the sequence $\{t_n-c\}$ is $\bar{A}$-summable to 0. Since both methods, $\bar{A}$ and $\bar{B}$, are obviously permanent for null sequences and every bounded $\bar{A}$-summable sequences is $\bar{B}$-summable, the sequence $\{t_n-c\}$ is $\bar{B}$-summable to 0. Finally, $a_n=b_n$, $\chi(A)=\chi(B)$ implies $B(x)=A(x)$.

2.5. *Let method A be permanent. If for every bounded A-summable sequence $x=\{t_n\}$ it follows that the "translated" sequence $x^*=\{t_{n+1}\}$ is also A-summable, then $A(x)=A(x^*)$.*

The hypotheses imply that every bounded A-summable sequence is B-summable, where $B=(b_{in})$ and $b_{i1}=0$, $b_{in}=a_{in-1}$ for $i=1,2,\ldots,$ $n=2,3,\ldots$ Hence we can apply Theorem 2.

2.5.1. In connection with Theorem 2.5 let us observe that there exist permanent methods A with the following property: the A-summability of the sequence x always implies the same for the translated sequence x^*, but $A(x)=A(x^*)$ does not hold for every $x\in A^*$.

Choose $a>1$ and define the methods $A=(a_{in})$ and $B=(b_{in})$ so that

$$A_1(x)=\frac{1}{a-1}t_1,$$

$$A_{n+1}(x)=\frac{a}{a-1}t_n-\frac{1}{a-1}t_{n+1}+\frac{1}{a^n}t_n \qquad \text{(method } A\text{)},$$

$$B_1(x)=\frac{1}{a-1}t_1,$$

$$B_{n+1}(x)=\frac{a}{a-1}t_n-\frac{1}{a-1}t_{n+1} \qquad \text{(method } B\text{)}.$$

Suppose that the sequence $x=\{t_n\}$ is B-summable. An easy computation gives

$$t_n=(a-1)a^n\left[\frac{B_1(x)}{a}-\frac{B_2(x)}{a^2}-\ldots-\frac{B_n(x)}{a^n}\right],$$

whence setting

$$z_n=(a-1)\sum_{k=1}^{\infty}\frac{B_{n+k}(x)}{a^k},$$

$$c=(a-1)\left(\frac{B_1(x)}{a}-\frac{B_2(x)}{a^2}-\frac{B_3(x)}{a^3}-\ldots\right),$$

we see that $t_n=z_n+ca^n$, and that the sequence $\{z_n\}$ converges. On the other hand, method B is permanent and the sequence $\{a^n\}$ is B-summable to 0, whence every sequence $\{z_n+ca^n\}$ with convergent $\{z_n\}$ and $c=\mathrm{const}$ is B-summable; thus the field of method B consists exclusively of the sequences of such form. Further, for $n=1,2,\ldots$

$$A_1(x)=B_1(x),$$

$$A_{n+1}(x)=(a-1)\left[\frac{B_1(x)}{a}-\frac{B_2(x)}{a^2}-\ldots-\frac{B_n(x)}{a^n}\right]+B_{n+1}(x).$$

It follows, *e. g.* in virtue of 3.52, that the methods A and B are equivalent; hence A^* consists exclusively of the sequences of form $\{z_n+ca^n\}$, therefore, x being any A-summable sequence, the translated sequence x^* is also A-summable. The sequence $x=\{a^n\}$ is A-summable to 1, the translated sequence $x^*=\{a^{n+1}\}$, however, is A-summable to $a\neq 1$.

The methods A and B, constructed above, yield a simple example of two permanent and non-consistent methods of summability with a common field.

2.6. Let method A be permanent; we shall say that the *sequence* $x_0=\{t_n^0\}\epsilon A^*$ *has property* (p) if, given any $\varepsilon>0$ and positive integer r, there exists a convergent sequence $x=\{t_n\}$ such that

$$(\alpha)\qquad \Big|\sum_{n=1}^{\infty}a_{in}(t_n^0-t_n)\Big|<\varepsilon \quad \text{for} \quad i=1,2,\ldots,$$

$$(\beta)\qquad |t_n^0-t_n|<\varepsilon \quad \text{for} \quad n=1,2,\ldots,r,$$

$$(\gamma)\qquad \Big|\sum_{n=1}^{m}a_{in}(t_n^0-t_n)\Big|<\varepsilon \quad \text{for} \quad i=1,2,\ldots,r,\ \ m=1,2,\ldots$$

It is apparent that $x_0=\{t_n^0\}\epsilon A^*$ *has property* (p) *if and only if there exist convergent sequences* x_n *such that* $\|x-x_n\|\to 0$, *the norm being defined by formula* (4) *of* 1.3.

In order that every sequence of A^* *have property* (p) *it is necessary and sufficient that the set of convergent sequences be dense in* A^* *in the metric implied by the norm* (4) *of* 1.3.1.

Replacing in the above definition of property (p) the permanency of method A by the permanency for null sequences, A^* by A_0^*, and the convergent sequence $\{t_n\}$ by the sequence convergent to 0, we obtain the definition of *property* (p_0).

By 2.2 and by the definition of the norm (4) of 1.3.1 it follows that

2.6.1. *If method* A *is permanent for null sequences (permanent), then every bounded sequence of* A_0^* (A^*) *has property* (p_0) (*property* (p)).

2.6.2. THEOREM 3. *Let the method* $A=(a_{in})$ *be permanent, and let method* $B=(b_{in})$ *be not weaker than* A. *Then a necessary and sufficient condition for the existence of a method* C *with* $A^*=C^*$ *and* $C(x)=B(x)$ *is* $\chi(B)\neq 0$.

THEOREM 4. *Let the method* $A=(a_{in})$ *be permanent.*

(a) *If the sequence* $x_0\epsilon A^*$ *has property* (p), *then* $A(x_0)=B(x_0)$ *for every permanent method* B *not weaker than* A.

(b) *If the sequence* $x_0\epsilon A^*$ *does not have property* (p), *then for every real* c *there exists a permanent method* B *such that* $B^*=A^*$ *and* $B(x_0)=c$.

THEOREM 4'. *Let the method* $A=(a_{in})$ *be permanent for null sequences.*

(a) *If the sequence* $x_0\epsilon A_0^*$ *has property* (p_0), *then* $A(x_0)=B(x_0)=0$ *for every method* B *permanent for null sequences and such that* $A_0^*\subset B^*$.

(b) *If the sequence* $x_0\epsilon A_0^*$ *does not have property* (p_0), *then for every real* c *there exists a method* B *permanent for null sequences, such that* $A_0^*\subset B^*$ *and* $B(x_0)=c$.

2.6.3. *A permanent method* A *is consistent with every permanent method not weaker than* A *if and only if the set of convergent sequences is dense in* A^* *in the metric implied by the norm* (4) *of* 1.3.1.

The above theorems, 3 and 4, have been given by the authors without proof in [13]. The method which we used in the course of our research in 1932 consisted in principle in determining the form of linear functionals in the space A^* in order to construct effectively method C. Since the same method of proof was discovered independently some years ago by Zeller [21], whose exposition is somewhat clearer and more general, we omit the proofs of these theorems, referring the reader to paper [21] of Zeller. Theorem 6.2 of paper [21] of Zeller presents a slight generalization of our theorem 3, and his theorem 6.3 generalizes our theorem 4 to the case of convergence preserving methods. Theorem 4' may be proved analogously.

2.7. The sequence $\{u_n\}$ is said to be *orthogonal to the sequence* $\{v_n\}$ if $\sum_1^{\infty}|u_n|<\infty$ and $\sum_1^{\infty}u_nv_n=0$ (absolute convergence of the last series not being required).

2.7.1. *Let the method* $A=(a_{in})$ *be permanent and let* $x_0\epsilon A^*$. *Then for every* $\varepsilon>0$ *there exists a convergent sequence* x *satisfying the inequality*

$$\sup_i|A_i(x_0)-A_i(x)|<\varepsilon,$$

if and only if every sequence orthogonal to the columns of the matrix (a_{in}) *is orthogonal to the sequence* $\{A_i(x_0)\}$.

Necessity. Let the sequence $\{u_i\}$ be orthogonal to the columns of the matrix (a_{in}). The functional

$$(6)\qquad \xi(y)=\sum_{i=1}^{\infty}u_iv_i$$

defined for $y=\{v_i\}\epsilon T_c$ is linear in T_c. Let $y_n=\{A_i(e_n)\}$; then $\xi(y_n)=0$ for $n=1,2,\ldots$ The sequence $z_n=y_1+y_2+\ldots+y_n$ converges weakly in T_c to the element $y^*=\{A_i(e)\}$. Indeed, we have

$$\sup_i|A_i(e_1)+\ldots+A_i(e_n)|\leqslant K \quad \text{for} \quad n=1,2,\ldots,$$

$$\lim_{n\to\infty}[A_i(e_1)+\ldots+A_i(e_n)]=A_i(e) \quad \text{for} \quad i=1,2,\ldots$$

Weak convergence of the sequence $\{z_n\}$ implies $\xi(y_1+y_2+\dots y_n)\to\xi(y^*)$, whence $\xi(y^*)=0$. Let $\bar{x}$ denote a linear combination of the elements $e, e_1, e_2, \dots, e_n, \dots$ and let $\bar{y}=\{A_i(\bar{x})\}$. The elements $\bar{x}$ form a set dense in T_c; whence by (β) of 1.1 the elements $\bar{y}$ form a dense set in the subset $Z\subset T_c$ consisting of the sequences $y=\{A_i(x)\}$, where x is a convergent sequence. We have proved above that $\xi(\bar{y})=0$ for every $\bar{y}$, whence $\xi(y)=0$ for every $y\,\epsilon\, Z$. By hypothesis the sequence $y_0=\{A_i(x_0)\}$ belongs to the closure (in the space T_c) of the set Z; consequently

$$\xi(y_0)=\sum_{i=1}^{\infty}u_iA_i(x_0)=0.$$

S u f f i c i e n c y. Let every sequence orthogonal to the columns of the matrix (a_{in}) be orthogonal to the sequence $\{A_i(x_0)\}$, and suppose that x_0 does not have the property mentioned in the theorem. Let Z have the same meaning as above; then the distance of the element $y_0=\{A_i(x_0)\}$ from the set Z (in the space T_c) is positive. Therefore there exists a linear functional in T_c of the form

$$\xi(y)=\sum_{i=1}^{\infty}u_iv_i+u\lim_{i\to\infty}v_i,$$

where $y=\{v_i\}\,\epsilon\, T_c$, $\sum\limits_1^{\infty}|u_i|<\infty$, such that $\xi(y)=0$ for $y\,\epsilon\, Z$ and $\xi(y_0)\neq 0$. Hence, writing $y_n=\{A_i(e_n)\}$, we have

$$\xi(y_n)=\sum_{i=1}^{\infty}u_iA_i(e_n)=\sum_{i=1}^{\infty}u_ia_{in}=0$$

for $n=1,2,\dots$, and arguing in the same way as in the proof of necessity we infer that

$$\sum_{i=1}^{\infty}u_iA_i(e)=0.$$

On the other hand,

$$\xi(y^*)=\sum_{i=1}^{\infty}u_iA_i(e)+u=0,$$

whence $u=0$. Orthogonality of the sequence $\{u_i\}$ to the columns of the matrix (a_{in}) gives

$$\sum_{i=1}^{\infty}u_iA_i(x_0)=0,$$

and then $u=0$ implies $\xi(y_0)=0$, which is contradictory.

THEOREM 5. *Let the U-method A be permanent and row-finite. Then the sequence $x_0\,\epsilon\, A^*$ has the property* (p) *if and only if every sequence orthogonal to the columns of the matrix (a_{in}) is orthogonal to the sequence $\{A_i(x_0)\}$.*

In the case under consideration the convergence implied by the norm $\|\ \|^3$ is equivalent to that implied by the norm (4) of 1.3.1 — as follows by 1.3.3. We infer immediately that the sequence $x_0=\{t_n^0\}\,\epsilon\, A^*$ has the property (p) if and only if, given any positive ε, there exists a convergent sequence $\{t_n\}$ satisfying the condition (α) of 2.6. We apply now 2.7.1.

THEOREM 5'. *Let the method A be permanent and row-finite. Then the sequence $x_0\,\epsilon\, A^*$ has the property* (p) *if and only if for $r=1,2,\dots$ orthogonality of the sequence x_0 to the r-th, $(r+1)$-th, $(r+2)$-th,... column of the matrix (a_{in}) implies its orthogonality to the sequence*

$$\Big\{\sum_{n=r}^{\infty}a_{in}t_n^0\Big\}.$$

For row-finite matrices the convergences implied by the norms (4) and (4') of 1.3.1 are equivalent, therefore $x_0\,\epsilon\, A^*$ has the property (p) if and only if for every $\varepsilon>0$ and every positive integer r there exists a convergent sequence $x=\{t_n\}$ satisfying the conditions (α) and (β) of 2.6. Let the method $A^r=(a_{in}^r)$ arise from the method A by replacing the first r columns of the matrix (a_{in}) by 0's. It is easy to see that if for every $\varepsilon>0$ and every A^r there exists a convergent sequence $\{t_n\}$ (depending on r) and satisfying the condition (α) of 2.6 with a_{in} replaced by a_{in}^r, then, given any $\varepsilon>0$, there exists a convergent sequence $\{t_n\}$ such that for the method A the conditions (α) and (β) of 2.6 are satisfied. The converse is also true. It is sufficient to apply Lemma 2.7.1.

3. The structure of the fields of summability may present many singularities. *E. g.* the field of summability defined in 2.5.1 is composed exclusively of all linear combinations of convergent sequences and one unbounded sequence (cf. [7] containing some more general theorems of this kind). Therefore the study of various properties of fields of summability, aimed at explaining their structural properties, is not without interest for the general theory of summability. This section is devoted to various problems of this kind.

3.1. *The space T is isomorphic with every closed linear set $R\subset T$ of infinite dimension.*

Denoting by $x=\{t_n\}$ a variable element of the space T we define the functionals $\xi_n(x)=t_n$. By a well-known procedure we can choose from the sequence $\{\xi_n(x)\}$ a subsequence $\{\xi_{n_i}(x)\}$ such that these functionals are linearly independent over R, and that every functional $\xi_n(x)$ is representable over R as a linear combination of the functionals $\xi_{n_i}(x)$. The sequence $\{\xi_{n_i}(x)\}$ is infinite — for, if it were composed of m elements it would follow that R contains not more than m linearly independent elements (it can even be proved that precisely m elements) which is impossible. Indeed, let $x_i=\{t_n^i\}\in R$ for $i=1,2,\ldots,m+1$ and let $l_1,l_2,\ldots,l_{m+1}$, denote arbitrary but different indices. Since

$$\mu_1\xi_{l_1}(x)+\mu_2\xi_{l_2}(x)+\ldots+\mu_{m+1}\xi_{l_{m+1}}(x)=0$$

if $x\in R$, $\sum_1^{m+1}|\mu_i|>0$, $\xi_{l_j}(x_i)=t_{l_j}^i$, it follows that all $\det(t_{l_j}^i)=0$ $(i,j=1,2,\ldots,m+1)$, whence the rank of the infinite matrix $t_1^i,t_2^i,\ldots,t_n^i,\ldots$, where $i=1,2,\ldots,m+1$, does not exceed m. It follows easily that there exist λ_i, non vanishing simultaneously, such that

$$\lambda_1x_1+\lambda_2x_2+\ldots+\lambda_{m+1}x_{m+1}=0.$$

This being so, let us consider the set S of the sequences $\{\xi_{n_i}(x)\}$ with $x\in R$. We shall prove that this set is closed (in the space T). Suppose that sequences $\{\xi_{n_i}(x^p)\}$ converge to the sequence $\{\vartheta_i\}$ as $p\to\infty$, *i. e.* that

$$\lim_{p\to\infty}\xi_{n_i}(x^p)=\vartheta_i \quad \text{for} \quad i=1,2,\ldots,$$

i. e.

$$\lim_{p\to\infty}t_{n_i}^p=\vartheta_i \quad \text{if} \quad x^p=\{t_n^p\}.$$

Denote the functionals not appearing in the sequence $\{\xi_{n_i}(x)\}$ by $\xi_{m_1}(x),\xi_{m_2}(x),\ldots$; these functionals may be represented as

$$\xi_{m_k}(x)=\sum_{i=1}^{r_k}a_i^k\xi_{n_i}(x),$$

the coefficients a_i^k being uniquely determined. Then

$$t_{m_k}^p=\xi_{m_k}(x^p)=\sum_{i=1}^{r_k}a_i^k\xi_{n_i}(x^p)\to\sum_{i=1}^{r_k}a_i^k\vartheta_i \quad \text{as} \quad p\to\infty.$$

It follows that the sequence $\{x^p\}$ converges in T to an element x which belongs to R, the set R being closed; moreover $\xi_{n_i}(x)=\vartheta_i$, whence $\{\vartheta_i\}\in S$. We prove now that $T=S$. In the contrary case there would exist a non-trivial linear functional ξ in T such that $\xi(y)=0$ for $y=\{t_i\}\in S$; the general form of ξ being

$$\xi(x)=\sum_{i=1}^{k}a_it_i \qquad (x=\{t_i\}),$$

we must have

$$\sum_{i=1}^{k}a_i\xi_{n_i}(x)=0$$

for $x\in R$. This, however, is impossible, since $a_1,\ldots,a_k$ do not vanish simultaneously and the functionals $\xi_{n_i}(x)$ are linearly independent over R.

It follows that the set R consists of all sequences $\{t_n\}$ such that $t_{n_1},t_{n_2},\ldots$ are arbitrary numbers and

$$(1) \qquad t_{m_k}=\sum_{i=1}^{r_k}a_i^kt_{n_i} \quad \text{for} \quad k=1,2,\ldots$$

Associating with the sequence $\{\vartheta_i\}\in T$ the sequence $\{t_n\}$ such that $t_{n_i}=\vartheta_i$ and t_{m_i} are defined by (1) we get a linearly-isomorphical mapping between T and R.

Remark. From the above proof it follows that under the hypotheses of Lemma 3.1 there exists a sequence of indices $\{n_i\}$ such that for every sequence $\{\vartheta_n\}$ there exists an element $\{t_n\}\in R$ such that $t_{n_i}=\vartheta_i$.

3.2. Method A will be said *to have a rate of growth* if there exists a sequence $\{\vartheta_n\}$ such that $\vartheta_n\neq 0$ and $\{t_n\}\in A^*$ implies $|\vartheta_nt_n|=O(1)$.

Method A will be said *to have a strict rate of growth* if there is a sequence $\{\vartheta_n\}$ such that 1° $|\vartheta_nt_n|=O(1)$ for every $\{t_n\}\in A^*$, $\vartheta_n\neq 0$ and 2° if $|\sigma_nt_n|=O(1)$ for every $\{t_n\}\in A^*$, then there is a constant N such that $|\sigma_n|\leqslant N|\vartheta_n|$ for $n=1,2,\ldots$

3.2.1. *Every convergence preserving row-finite U-method has a strict rate of growth.*

By the theorem 1.2 of Toeplitz the set S of sequences $\{u_i\}$ of the form

$$u_i=\sum_{n=1}^{\infty}a_{in}t_n, \qquad \{t_n\}\in T,$$

is a linear closed subspace of the space T. Let the operation U map the element $u=\{u_i\}$ of S upon the element $t=\{t_i\}$ of T; this operation is additive and continuous (the simplest proof of that is by applying the closed-graph theorem of Banach [5], p. 41). Hence t_n are linear functionals of the variable u and therefore representable in the form

$$(2) \qquad t_n=\sum_{m=1}^{\infty}c_{nm}u_m \quad \text{for} \quad n=1,2,\ldots,$$

where $c_{nm}=0$ for almost all m's; this follows by the theorem on the extension of linear functionals in the B_0-spaces and by the general form of linear functionals in the space T. Write

$$\vartheta_n=\Big(\sup\Big|\sum_{m=1}^{\infty}c_{nm}A_m(x)\Big|\Big)^{-1},$$

the supremum being extended over all elements $x\in A^*$ for which

$$\sup_m|A_m(x)|\leqslant 1.$$

Method A being convergence preserving, ϑ_n is defined for every n, moreover $\vartheta_n \neq 0$, for $\vartheta_n^{-1} \leqslant \sum_m |c_{nm}|$. If $x=\{t_n\} \in A^*$, then in virtue of formula (2)

$$|\vartheta_n t_n| \leqslant \sup_m |A_m(x)| \quad \text{for} \quad n=1,2,\ldots,$$

and this implies that 1° is satisfied.

Suppose now that the sequence $\{\sigma_n\}$ is such that $|\sigma_n t_n| = O(1)$ for every $\{t_n\} \in A^*$; let us define linear functionals in A^*:

$$\xi_n(x) = \sigma_n \cdot \sum_{m=1}^{\infty} c_{nm} A_m(x) = \sigma_n t_n.$$

The sequence $\{\xi_n(x)\}$ is bounded in A^*; A being a U-method, A^* is a Banach space under the norm $\| \ \|^3$ whence there exists a constant $N>0$ such that

$$|\xi_n(x)| \leqslant N \sup_m |A_m(x)|$$

for $x \in A^*$, and there results

$$|\sigma_n| \sup_{|A_m(x)| \leqslant 1} \Big| \sum_{m=1}^{\infty} c_{nm} A_m(x) \Big| = |\sigma_n \vartheta_n^{-1}| \leqslant N,$$

whence 2° is also satisfied.

3.2.2. *A row-finite convergence preserving method $A=(a_{in})$ has a rate of growth if and only if the system of equations*

$$\sum_{n=1}^{\infty} a_{in} t_n = 0 \qquad (i=1,2,\ldots) \tag{3}$$

has a finite number of linearly independent solutions (see [13], théorème 2).

Necessity. Let R denote the set of all solutions $\{t_n\}$ of the equations (3); R is obviously linear and closed in T. The remark which follows the Lemma 3.1 implies that if method A has a rate of growth, then R is finitely dimensional.

Sufficiency. Suppose that the system of equations (3) has precisely p linearly independent solutions $\{t_n^1\}, \{t_n^2\}, \ldots, \{t_n^p\}$; then there exist indices $n_1, n_2, \ldots, n_p$ such that $\det(t_{n_i}^j) \neq 0$, $i,j=1,2,\ldots,p$. Let $B=(b_{in})$ denote the matrix arising from method A by adding to the matrix (a_{in}) p initial rows b_{in} where

$$b_{in} = \begin{cases} 0 & \text{for} \quad n \neq n_i, \\ 1 & \text{for} \quad n = n_i \end{cases}$$

for $i=1,2,\ldots,p$. The system of equations

$$\sum_{1}^{\infty} b_{in} t_n = 0 \qquad (i=1,2,\ldots)$$

has only the trivial solution, whence B is a row-finite U-method, moreover $A^* = B^*$. We apply now 3.2.1.

By 3.2.1 and the proof of 3.2.2 there follows

THEOREM 6. *A row-finite and convergence preserving method has a rate of growth if and only if it is equivalent to a row-finite U-method.*

In the sequel we shall apply the following lemma:

3.3. *Let X be a B_0-space with the norm $\|x\|$ and let $\|x\|_i$ denote pseudonorms in X such that $\|x_n\| \to 0$ if and only if, for every i, $\|x_n\|_i \to 0$. Let the sequence $\{\xi_n(x)\}$ of linear functionals in X be bounded everywhere. Then*

(α) *the functionals are equicontinuous,*

(β) *there exists an index k and a constant K such that*

$$|\xi_n(x)| \leqslant K \sup(\|x\|_1, \|x\|_2, \ldots, \|x\|_k) \quad \textit{for } n=1,2,\ldots$$

Equicontinuity follows from a general theorem ([15], p. 153). By (α) there exists a $\varrho>0$ such that $\|x\| \leqslant \varrho$ implies $|\xi_n(x)| \leqslant 1$. Now, for k sufficiently large and r sufficiently small, the inequalities $\|x\|_1 \leqslant r, \|x\|_2 \leqslant r, \ldots, \|x\|_k \leqslant r$ imply $\|x\| \leqslant \varrho$ — there follows (β) with $K=1/r$.

From 3.3 it follows immediately that

3.3.1. *If the sequence $\{\xi_n(x)\}$ of linear functionals in a B_0-space X is bounded everywhere and converges (converges to 0) in a set dense in this space, then the sequence converges (converges to 0) everywhere and its limit is a linear functional in X* (see [15]).

3.4. *Let A be a permanent (permanent for null sequences) method, consistent (consistent for sequences A-summable to 0) with every method not weaker than A. Let $C=(c_{in})$ be a permanent (permanent for null sequences) method such that for every $x \in A^*$ $(x \in A_0^*)$ the transforms $C_i(x)$ are bounded. Then method C is not weaker than A.*

It is sufficient to apply 2.6.1 and 3.3.1 setting $\xi_n(x) = C_n(x)$.

We shall prove now a generalization of Theorem 3.4 by means of an argument which avoids the use of the property (p).

3.4.1. *Let the methods $A^1, A^2, \ldots$ be such that:*

(α) *these methods are convergence preserving (convergence preserving for null sequences),*

(β) *every two of these methods are consistent for convergent sequences (for sequences convergent to 0),*

(γ) *if B is a convergence preserving (convergence preserving for null sequences) method such that every sequence A^i-summable to the same limit b (to $b=0$) for $i=1,2,\ldots$ is B-summable, then $B(x)=b$;*

let the method $C=(c_{in})$ be such that

(δ) *every sequence x, summable to the same value (to 0) by all the methods $A^1, A^2, \ldots$, has bounded transforms $C_i(x)$.*

Under these hypotheses every sequence summable to the same value c (to 0) by all the methods A^i is C-summable to c (to 0).

Let X denote the set of all sequences summable by all the methods A^i to the same value; we define the norm in X as

$$\|x\|=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|x\|^n}{1+\|x\|^n},$$

where $\| \|^n$ denotes norm (4) of 1.3.1 in the space A^{n*}. X is a B_0-space — obviously separable, since the spaces A^{n*} are such by 1.4.1 (the simplest proof of this fact is by applying the general principle of [14], (I), p. 191, proposition 1.41). Suppose that the theorem is false; then there exists a sequence $x_0 \epsilon X$ such that the sequence $\{C_i(x_0)\}$ diverges, therefore, for a sequence of indices $\{i_k\}$, $\{C_{i_k}(x_0)\}$ converges to a limit different from c, where $c=A^n(x_0)$ for $n=1,2,\ldots$ The transforms $C_i(x)$ are, in virtue of 3.3.1, linear functionals in X since

$$C_i(x)=\lim_{n\to\infty}C_{in}(x) \quad \text{for} \quad x \epsilon X,$$

and $C_{in}(x)$ are obviously linear functionals in X. Separability of the space X implies, by 3.3.1, the existence of a sequence $\{C_{j_l}(x)\}$ extracted from $\{C_{i_k}(x)\}$, convergent in X. Then, as follows by (γ), the method $\bar{C}=(c_{j_l n})$ is such that $\bar{C}(x_0)=c$, which is contradictory.

3.4.2. In Theorem 3.4, the hypothesis of the consistency of method A with every method not weaker than A is essential. Indeed, suppose that method B is not weaker than A and not consistent with A. All A-summable sequences x have bounded transforms $C_i(x)$, where $C_{2i-1}(x)=A_i(x)$, $C_{2i}(x)=B_i(x)$ — the assertion of Theorem 3.4, however, is not satisfied.

3.5. *Suppose that method A is convergence preserving for null sequences and let a_n have the same meaning as in (α) of 1.1. If there exists a sequence, not converging to 0, which is A-summable to the value $\sum_1^\infty a_n t_n$, then there exist unbounded sequences A-summable to $\sum_1^\infty a_n t_n$.*

Let $a_n=0$ for $n=1,2,\ldots$ Suppose that only bounded sequences are A-summable; taking as C an identical method, we find, in virtue of 2.6.1, that the hypotheses of 3.4 are satisfied, therefore $A_i(x)\to 0$ implies $t_n\to 0$, whence A_0^* consists only of sequences convergent to 0, which contradicts the hypotheses. If not all a_n are equal to 0, let us choose a sequence $\{k_n\}$ of positive integers in such a manner that $\sum_1^\infty |a_n| k_n<\infty$ (this is possible since, by (β) of 1.1, $\sum_1^\infty |a_n|<\infty$). Let C denote the method corresponding to the matrix (c_{in}) where

$$c_{2i-1,n}=a_{in}-a_n,$$

$$c_{2i,n}=\begin{cases} a_{in}-a_n & \text{for } i\neq n,\\ a_{nn}-a_n+\dfrac{1}{k_n} & \text{for } i=n.\end{cases}$$

Then $C^*\subset A^*$ and $C(x)=A(x)-\sum_1^\infty a_n t_n$, $T_b A^*\subset C^*$. Suppose that every sequence A-summable to $\sum_1 a_n t_n$ is bounded. A sequence not converging to 0 and A-summable to $\sum_1^\infty a_n t_n$, which exists by hypothesis, is C-summable to 0. Since method C is permanent for null sequences, there exists in C_0^* an unbounded sequence x, whence $A(x)=\sum_1^\infty a_n t_n$, and this leads to a contradiction.

3.5.1. Theorem 7. *Let method A be convergence preserving. Then there exists an unbounded A-summable sequence if one of the following conditions is satisfied:*

(a) $\chi(A)=0$,

(b) $\chi(A)\neq 0$ *and there exists a bounded divergent A-summable sequence.*

If the sequence $x=\{t_n\}$ converges to t and method A is convergence preserving, then

$$A(x)=\sum_{n=1}^\infty a_n t_n+\chi(A)t,$$

whence in virtue of hypothesis (a) the sequence e is A-summable to $\sum_1^\infty a_n$ and it suffices to apply 3.5. In the case (b) let the sequence $x=\{t_n\}$ be bounded, divergent, and A-summable. Setting

$$c=\frac{A(x)-\sum_{n=1}^\infty a_n t_n}{\chi(A)}$$

we see that the sequence $\{t_n-c\}$ is A-summable to $\sum_1^\infty a_n(t_n-c)$ and it suffices to apply 3.5.

Applying Theorems 3.4.1 and 1′ we can similarly prove

Theorem 7′. (a) *Let the methods $A^1, A^2,\ldots,A^n,\ldots$ be permanent for null sequences. If there exists a sequence, not converging to 0, which is*

A^n-summable to 0 for every n, then there exists an unbounded sequence A^n-summable to 0 for every n.

(b) *Suppose that the methods $A^1, A^2, \ldots, A^n, \ldots$ are convergence preserving and such that every two of them are consistent for convergent sequences. If there exists a bounded sequence A^n-summable for every n to a value not depending on n, then there exists an unbounded sequence A^n-summable to the same value for every n* (see [13], théorème 3, and [23]).

3.5.2. As a simple application of Theorem 7 we shall prove that *If method A is normal, convergence preserving, and such that*

$$\lim_{i\to\infty}\{|a_{ii}|-\sum_{n=1}^{i-1}|a_{in}|\}>0,$$

then $A^* = I^*$ (see [1], [3] and [13]).

In the contrary case there would exist, by Theorem 7, an unbounded A-summable sequence. Choose an increasing sequence $\{i_k\}$ of indices in such a manner that $|t_{i_k}| > \sup(|t_1|, |t_2|, \ldots, |t_{i_{k-1}}|)$ for $k=2,3,\ldots$; then $|t_{i_k}|\to\infty$ and

$$|A_{i_k}(x)| \geqslant |a_{i_k i_k}|\,|t_{i_k}| - \sum_{n=1}^{i_k-1}|a_{i_k n}t_n| \geqslant |t_{i_k}|\,(|a_{i_k i_k}| - \sum_{n=1}^{i_k-1}|a_{i_k n}|),$$

whence $\overline{\lim_i}|A_i(x)| = \infty$ — which is contradictory.

3.6. *Suppose that method A is permanent for null sequences. If there exists a bounded divergent sequence A-summable to 0, then there exists a sequence of indices $\{m_p\}$ such that for every bounded sequence $\{u_i\}$ there is an element $x=\{t_n\}\in A_0^* T_b$ such that $t_{m_p}=u_p$ for $p=1,2,\ldots$*

By 2.3 we can suppose that method A is a normal one. We shall prove first that if there exists a bounded divergent sequence $\{\bar{t}_n\}$ A-summable to 0, then there is a sequence of indices $\{n_r\}$ with $n_{r_0}=1$ and a divergent sequence $\{t_n^0\}$, A-summable to 0, such that $|t_n^0|\leqslant 1$, $t_{n_r}^0=1$ for $r=0,1,\ldots$ To see this, write $s_k=\sup(|\bar{t}_k|, |\bar{t}_{k+1}|, \ldots)$; then obviously

$$\overline{\lim_k}|\bar{t}_k| = \lim_k s_k = s>0, \qquad \overline{\lim_k}|\bar{t}_k|s_k^{-1} = \overline{\lim_k}|\bar{t}_k|s^{-1}=1.$$

Therefore we can choose a sequence of indices $\{n_r\}$ in such a manner that $\bar{t}_{n_r}$ are of the same sign, $n_{r_0}=1$ and

$$\frac{|\bar{t}_{n_r}|}{s_{n_r}} = 1-z_r \quad \text{for } r=0,1,2,\ldots$$

where

$$\sum_{r=0}^{\infty}|z_r|<\infty, \qquad \sum_{r=0}^{\infty}\left(\frac{1}{s}-\frac{1}{s_{n_r}}\right)<\infty.$$

To define the sequence $\{t_n^0\}$ we set $t_{n_r}^0=\bar{t}_{n_r}\operatorname{sign}\bar{t}_{n_r}s_{n_r}^{-1}+z_r=1$, $t_n^0=\bar{t}_n\operatorname{sign}\bar{t}_{n_r}s_{n_r}^{-1}$ for $n_r<n<n_{r+1}$, $r=0,1,\ldots$

Suppose, for instance that $\operatorname{sign}\bar{t}_{n_r}=1$ for $r=0,1,\ldots$; then

$$v_i = \frac{1}{s}\sum_{n=1}^{\infty}a_{in}\bar{t}_n - \sum_{n=1}^{\infty}a_{in}t_n^0 = \sum_{r=0}^{\infty}\operatorname{sign}\bar{t}_{n_r}\left(\frac{1}{s}-\frac{1}{s_{n_r}}\right)\sum_{n=n_r}^{n_{r+1}-1}a_{in}\bar{t}_n + \sum_{r=0}^{\infty}a_{in_r}z_r.$$

The conditions (α') and (β) of 1.1 are satisfied, whence $v_i\to 0$. Therefore the sequence $\{t_n^0\}$ is A-summable to 0, moreover $|t_n^0|\leqslant 1$, $t_1^0=1$ and $t_{n_r}^0=1$. An analogous statement holds if $\operatorname{sign}\bar{t}_{n_r}=-1$. Applying Lemma 2.2 and the remark belonging to it we can select from the sequence $\{n_r-1\}$ a subsequence which will again be denoted by $\{n_i\}$ and find a sequence x_n of elements of the form (*) of 2.4 (setting therein $t_n=t_n^0$) where $0\leqslant \alpha_i\leqslant 1$, $\alpha_1=1$ and $\alpha_{n_i+1}=1$ for $i=1,2,\ldots$, and

$$\|x_n-x\|^3<\frac{1}{2^n} \quad \text{for } n=1,2,\ldots$$

Now, $\{\lambda_i\}$ being an arbitrary bounded sequence, the series (5) of 2.4 converges in A_0^*, and — as in the proof of Theorem 2.4 — we can write the terms of this series in the form (**) of 2.4. Since $t_{n_i+1}^0=1$, we get

$$t_{n_i+1}^{\lambda} = \lambda_{i+1}\alpha_{n_i+1}t_{n_i+1}^0 + \lambda_{i+2}(1-\alpha_{n_i+1})t_{n_i+1}^0 = \lambda_{i+1}, \qquad t_1^{\lambda}=\lambda_1,$$

whence it suffices to set $m_p=n_p+1$, $u_p=\lambda_{p+1}$, $x=\{t_i^{\lambda}\}$.

Let us notice that the formulae (**) of 2.4 define a one-to-one linear mapping of bounded sequences onto a linear subset of the space, composed of the sequences of the form $\{t_i^{\lambda}\}\in A_0^*$. Moreover,

$$\sup_i|t_i^{\lambda}| = \sup_i|\lambda_i|,$$

therefore we get the following theorem:

3.6.1. Theorem 8. *Suppose that method A is permanent for null sequences and that there are bounded divergent sequences A-summable to 0. Then the set of bounded sequences A-summable to 0, considered as a linear subset of the space T_b, contains a linear subset equivalent to the space T_b.*

Theorem 8 implies immediately that

3.6.2. *Under the hypotheses of Theorem 8 the set of bounded sequences A-summable to 0, considered as elements of the space T_b, is non-separable in T_b* (see [13], théorème (5), and [2]).

3.7. *If the system of equations*

$$(3') \qquad \sum_{n=1}^{\infty}a_{in}t_n=0 \quad \text{where} \quad \sum_{n=1}^{\infty}|a_{in}|<\infty \qquad (i=1,2,\ldots)$$

has no bounded divergent solution, then it has a finite number of linearly independent solutions in T_0.

Denote by R the set of the solutions of the system (3′) belonging to T_0 and suppose that R is infinitely dimensional. Then for every positive integer k there exists an element $x=\{t_n\}\epsilon R$, different from 0 and such that $t_1=t_2=\ldots=t_k=0$.

Indeed, choose $k+1$ linearly independent elements $x_1,x_2,\ldots,x_{k+1}$ of R and write $x_i=\{t_n^i\}$. Vectors $(t_1^1,t_2^1,\ldots,t_k^1)$, $(t_1^2,t_2^2,\ldots,t_k^2),\ldots,(t_1^{k+1},\ t_2^{k+1},\ldots,t_k^{k+1})$ are linearly dependent, therefore there are numbers $a_1,a_2,\ldots,a_{k+1}$, non-vanishing simultaneously, such that

$$\sum_{i=1}^{k+1} a_i t_n^i=0 \quad \text{for} \quad n=1,2,\ldots,k.$$

The element

$$x=\sum_{i=1}^{k+1} a_i x_i$$

is different from 0 and its first k terms are 0, whence x has the desired properties. Let us choose the elements $x_k=\{t_n^k\}\epsilon R$ in such a manner that $t_n^k=0$ for $1\leqslant n\leqslant k$, $k=1,2,\ldots$, and

$$\sup_n |t_n^k|=1 \quad \text{for} \quad k=1,2,\ldots$$

Applying Lemma 1.2.2 to the matrix (t_n^k) we shall prove that there is a sequence of indices $\{i_k\}$ and a bounded sequence $\{\lambda_k\}$ such that

$$\sup_n \sum_{k=1}^{\infty} |t_n^{i_k}|<\infty \tag{4}$$

and the sequence with terms

$$t_n=\sum_{k=1}^{\infty} \lambda_k t_n^{i_k} \tag{5}$$

diverges. The sequence $\{t_n\}$ is bounded and, as may easily be seen, satisfies the equations (3′), which leads to a contradiction.

3.7.1. *Let $A=(a_{in})$ be a U-method for bounded sequences which is permanent for null sequences. The following condition is necessary and sufficient in order that no bounded divergent sequence be A-summable to 0:*

there is a positive constant c such that for every system $\lambda_1,\lambda_2,\ldots,\lambda_p$ of numbers

$$\sup_i \Big|\sum_{n=1}^{p} a_{in}\lambda_n\Big|\geqslant c\sup_n |\lambda_n|. \tag{6}$$

Necessity. Suppose there exists no constant $c>0$ such that (6) is satisfied for every system $\lambda_1,\lambda_2,\ldots,\lambda_p$ of numbers. Then we can determine sequences $x_k=\{t_n^k\}$ such that $t_n^k=0$ for almost all n's,

$$\sup_n |t_n^k|=1, \qquad \lim_k \sup |A_i(x_k)|=0.$$

We can suppose freely that the limits

$$\lim_k t_n^k=t_n$$

exist for $n=1,2,\ldots$ Taking into account the condition (β) of 1.1, we see, in virtue of

$$\lim_k |A_i(x_k)|=0,$$

that $A_i(x)=0$ for $i=1,2,\ldots$ where $x=\{t_n\}$. The sequence x being bounded, $t_n=0$. Applying Lemma 1.2.2 we can determine a subsequence $\{x_{i_k}\}$ and a bounded sequence $\{\lambda_k\}$ in such a manner that the sequence x with the terms (5) of 3.7 is divergent, that condition (4) of 3.7 is satisfied and, moreover, that

$$\sup_i |A_i(x_{i_k})|<\frac{1}{2^k} \quad \text{for} \quad k=1,2,\ldots \tag{7}$$

The condition (4) of 3.7 implies

$$A_j(x)=\sum_{k=1}^{\infty} \lambda_k A_j(x_{i_k}),$$

and since $\lim_j |A_j(x_{i_k})|=0$, it follows by (7) that the sequence x is A-summable to 0, and this leads to a contradiction.

Sufficiency. Let us denote by R and R_0 respectively the set of all sequences $\{A_i(x)\}$ where $x\epsilon T_bA_0^*$ or $x\epsilon T_0$ respectively. The inequality (6) implies, for every bounded sequence $\{t_n\}$,

$$\sup_i \Big|\sum_{n=1}^{\infty} a_{in}t_n\Big|\geqslant c\sup_n |t_n|.$$

The set R_0 is closed in T_0. Indeed, let $y_k=\{A_i(x_k)\}\epsilon R_0$, $x_k=\{t_n^k\}$, $\sup_i |A_i(x_k)-t_i|\to 0$, $y=\{t_n\}$. Then the inequality

$$\sup_i \Big|\sum_{n=1}^{\infty} a_{in}(t_n^p-t_n^q)\Big|\geqslant c\sup_n |t_n^p-t_n^q|$$

is satisfied, and hence the sequence x_k converges in T_0; denoting by x the limit of this sequence, we see that

$$A_i(x)=\lim_k A_i(x_k)=t_i,$$

whence $y\epsilon R_0$. By 2.2 the set R_0 is dense in R (for the topology of the space T_0); therefore $R_0=R$.

Remark. The hypothesis that A is a U-method is superfluous for the proof of the sufficiency of the condition. This hypothesis is implied by condition (6).

3.7.2. THEOREM 9. *Let the method $A=(a_{in})$ be permanent for null sequences. The following condition is necessary and sufficient for every sequence A-summable to 0 to be either convergent or unbounded:*

There exists a positive constant c and a positive integer r such that for every system $\lambda_1, \lambda_2, \ldots, \lambda_p$ of numbers the inequality

$$\sup_i \Big|\sum_{n=1}^{p} a_{in+r}\lambda_n\Big| \geqslant c \sup_n |\lambda_n| \tag{8}$$

is satisfied.

Necessity. If there exist no bounded divergent sequences A-summable to 0, then the system (3′) of 3.7 has in T_b only a finite number of linearly independent solutions, say k. Hence by the adjunction to the matrix (a_{ik}) of k initial rows — as in Lemma 3.2.2 — we obtain a method B which is both a U-method for bounded sequences and equivalent to A. Choosing r sufficiently large and applying Lemma 3.7.1 to the method B we get (8) for every system $\lambda_1, \lambda_2, \ldots, \lambda_p$.

Sufficiency. The method B corresponding to the matrix $(a_{i\,n+r})$ obviously satisfies the conditions of Lemma 3.7.1, whence the sequences B-summable to 0 are either convergent or unbounded. The method A, as easily seen, has the same property.

A permanent method is called *perfectly inconsistent* if for every divergent sequence $x \epsilon A^*$ there exists a permanent method B not weaker than A and such that $A(x) \neq B(x)$.

THEOREM 10. *A permanent method A is perfectly inconsistent if and only if every A-summable sequence is either convergent or unbounded.*

The necessity of the condition follows from 2.6.1 and Theorem 4 of 2.6.2.

Sufficiency. Let us apply Theorem 9, let r have the same meaning as in that theorem. Let B denote the method corresponding to the matrix whose i-th row is $0, 0, \ldots, 0, a_{i,r+1}, a_{i,r+2}, \ldots$; the set of convergent sequences is closed in B^*. Indeed, let $x_k = \{t_n^k\} \epsilon T_c$, $y = \{t_n\}$; inequality (8) of 3.7.2 implies

$$\sup_{r<i} |t_i^p - t_i^q|\, c \leqslant \sup_i |B_i(x^p) - B_i(x^q)|.$$

If $\|x^p - y\| \to 0$ as $p \to \infty$, then

$$\sup_{r<i} |t_i^p - t_i|\, c \leqslant \sup_i |B_i(x^p) - B_i(y)|,$$

and there follows $\{t_n\} \epsilon T_c$; the method A and B are equivalent, whence it suffices to apply Theorem 4.

4. In this section we shall give some further theorems of structural character.

4.1. *Let $A^1, A^2, \ldots, A^n, \ldots$ be arbitrary methods of summability such that A^{n*} is properly contained in A^{n+1*} for $n=1,2,\ldots$ Then there exists no method B such that*

$$B^* = \sum_{n=1}^{\infty} A^{n*}.$$

This theorem was found independently by Zeller ([21], p. 483) who was the first to publish its proof. As compared with our proof, which is based on theorems concerning linear Borel sets, the proof of Zeller is more elegant and reduces the problem to a general theorem on increasing sequences of B_0-spaces; therefore we omit our proof.

4.2. *Let the methods $A^1, A^2, \ldots, A^n, \ldots$ be permanent and such that A^{n+1*} is properly contained in A^{n*} for $n=1,2,\ldots$ Let the method A^n, for every n, be consistent with every method not weaker than it. Then there exists no row-finite method B such that*

$$B^* = \prod_{n=1}^{\infty} A^{n*}. \tag{1}$$

Also this theorem has not been given in our note [13]. It has been found independently by Zeller [22].

Let us denote by $\|x\|_i^{1n}$, $\|x\|_i^{2n}$, and $\|x\|^{3n}$ the pseudonorms (1)-(3) of 1.3 for the method A^n. By 1.3.1 A_n^* is a B_0-space under the norm

$$\|x\|^n = \|x\|^{1n} + \|x\|^{2n} + \|x\|^{3n}$$

where

$$\|x\|^{sn} = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x\|_i^{sn}}{1+\|x\|_i^{sn}} \quad \text{for} \quad s=1,2,\ldots,\ n=1,2,\ldots$$

Therefore B^* (supposing that (1) holds for a row-finite method B) is a B_0-space under the norm

$$\|x\|_0 = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x\|^n}{1+\|x\|^n};$$

in this space the convergence implied by the norm $\|x\|_0$ is equivalent to the convergence implied by all pseudonorms $\|x\|_i^{sn}$ and $\|x\|^{3n}$ simultaneously. By hypothesis the transforms $B_j(x)$ are linear functionals in B^* under the norm $\|x\|_0$. By the proposition (β) of 3.3 there is a constant K and a positive integer k such that

$$|B_j(x)| \leqslant K \sup_{1\leqslant i,\, n\leqslant k} \big(\|x\|_i^{1n}, \|x\|_i^{2n}, \|x\|^{3n}\big) \tag{2}$$

for $j=1,2,\ldots, x \epsilon B^*$. This is true in particular for convergent sequences.

On the other hand, the convergent sequences lie dense in the space A^{k*} (with the norm $\|x\|^* = \sup(\|x\|^1, \|x\|^2, \ldots, \|x\|^k)$). Indeed, the method $D = (d_{in})$ arising by the juxtaposition of all rows of the methods $A^1, A^2, \ldots, A^k$, is equivalent to the method A^k, whence it is consistent with every method, not weaker than it, and it suffices to apply 2.6.3. Since the inequality (1) is satisfied in a set dense in the space A^{k*} with the norm $\|x\|^*$, and since $B_j(x)$ is a linear functional under this norm, (2) holds for every $x \in A^{k*}$ and the sequence $B_j(x)$ converges in a set dense in A^{k*}. By 3.3.1 this would imply $A^{k*} \subset B^*$, which is impossible.

4.3. *There exists no sequence $A^1, A^2, \ldots$ of methods having the following property:*

(α) $\quad I^* = \prod_{n=1}^{\infty} A_0^{n*}$ (see [13]).

By (α) the methods A^n are permanent for null sequences and the sequence e is A^n-summable to 0. It suffices to apply Theorem 7′, part (a).

The last theorem has the following meaning. Let $A(\{p_n\}, \{q_n\})$, where $p_n \to \infty, q_n \to \infty$, be the method of summability whose i-th transform is $t_{p_i} - t_{q_i}$. Cauchy's test of convergence states that the sequence $\{t_n\}$ is convergent if and only if it is summable to 0 by every method $A(\{p_n\}, \{q_n\})$ (which is obviously permanent for null sequences). The set of these methods is uncountable and this is essential. Indeed, 4.3 implies that it is impossible to determine a countable set of generalized conditions of Cauchy's type the totality of which would give a test of convergence.

4.4 THEOREM 11. (α) *Let the method A be convergence preserving and such that $x = \{u_n\} \in A^*$, $y = \{v_n\} \in A^*$ implies $\{u_n v_n\} \in A^*$. Then there exists an increasing sequence of indices $\{k_n\}$ such that the field A^* is identical with the set of the sequences $\{t_n\}$ for which there exists $\lim_n t_{k_n}$.*

(β) *If the method A is permanent and such that $x = \{u_n\} \in A^*$, $y = \{v_n\} \in A^*$, $v_n \neq 0$ $(n = 1, 2, \ldots)$, $A(y) \neq 0$ implies $\{u_n/v_n\} \in A^*$, then the conclusion of* (α) *holds too* (see [13], [23]).

In both cases, from $x = \{t_n\} \in A^*$, $v_n \to 0$ follows $\{t_n v_n\} \in A^*$. In the case (α) this is trivial, and in the case (β) it may be shown as follows. Let $v_n \to 0$; let v be such that $v \neq 0$, $v + v_n \neq 0$ for every n. By hypothesis

$$\{(v_n + v)^{-1}\} \in A^*, \quad \text{whence} \quad \{t_n v_n\} = \{t_n(v_n + v)\} - \{t_n v\} \in A^*.$$

Denote by B the method corresponding to the matrix $(a_{in} t_n)$; in both cases it is convergence preserving and hence, by (β) of 1.1,

$$\sum_{n=1}^{\infty} |a_{in}|\,|t_n| \leqslant K(x) \quad \text{for} \quad i = 1, 2, \ldots \tag{3}$$

and for $x = \{t_n\} \in A^*$. Let

$$a_n = \sup_i |a_{in}| \quad \text{for} \quad n = 1, 2, \ldots$$

and denote successively by $k_1, k_2, \ldots$ all indices for which $a_n \neq 0$. The theorem will follow from the inequality $\inf_n a_{k_n} = \gamma > 0$. In fact, (3) implies $|t_{k_n}| \leqslant K(x)\gamma^{-1}$ and it follows immediately that the method $\bar{A} = \{a_{ik_n}\}$ is such that only bounded sequences are $\bar{A}$-summable. It suffices to make use of Theorem 7.

We shall prove $\gamma > 0$ first in the case (α). In the contrary case there would exist an increasing sequence of indices $\{k_{n_s}\}$ such that $\sum_s a_{k_{n_s}}^{1/3} < \infty$. Let

$$t_n = \begin{cases} a_{k_{n_s}}^{-2/3} & \text{for} \quad n = k_{n_s},\ s = 1, 2, \ldots, \\ 0 & \text{elsewhere.} \end{cases}$$

The sequence $x = \{t_n\}$ is A-summable, for the series

$$\sum_{s=1}^{\infty} a_{ik_{n_s}} a_{k_{n_s}}^{-2/3}$$

converges uniformly in i. By hypothesis the sequence $x_0 = \{t_n^2\}$ is also A-summable, and applying to it the inequality (3) we obtain $|a_{ik_{n_s}}| \leqslant K(x_0) a_{k_{n_s}}^{4/3}$, whence $a_{k_{n_s}} \leqslant K(x_0) a_{k_{n_s}}^{4/3}$, $a_{k_{n_s}} \geqslant [K(x_0)]^{-3}$, which is impossible. Proceeding now to the case (β), suppose that $\gamma = 0$. Then there must exist a sequence of indices $\{k_{n_s}\}$ such that $\sum_s a_{k_{n_s}} < \infty$. Let

$$t_n = \begin{cases} a_{k_{n_s}}^2 & \text{for} \quad n = k_{n_s},\ s = 1, 2, \ldots, \\ 1 & \text{elsewhere.} \end{cases}$$

The sequence $x = \{t_n\}$ is A-summable, for

$$A_i(x) = A_i(e) + \sum_{s=1}^{\infty} a_{ik_{n_s}}(a_{k_{n_s}}^2 - 1),$$

and since $|a_{ik_{n_s}}(a_{k_{n_s}}^2 - 1)| \leqslant a_{k_{n_s}} |a_{k_{n_s}}^2 - 1|$ and

$$\sum_{s=1}^{\infty} a_{k_{n_s}} |a_{k_{n_s}}^2 - 1| < \infty,$$

we obtain

$$\lim_{i \to \infty} \sum_{s=1}^{\infty} a_{ik_{n_s}}(a_{k_{n_s}}^2 - 1) = 0.$$

Now $x_0 = \{t_n^{-1}\} \in A^*$ for $t_n \neq 0$ and $A(x) = 1$; applying (3) to this sequence, we get $|a_{ik_{n_s}}| \leqslant K(x_0) a_{k_{n_s}}^2$ and $a_{k_{n_s}} > [K(x_0)]^{-1}$, which is impossible.

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Sur les méthodes continues de limitation (I)

(Application de l'espace B_0 de Mazur et Orlicz à l'étude des méthodes continues)

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Introduction [1].

La limite ξ d'une suite $\{\xi_n\}$ peut être considérée comme une fonctionnelle qui fait correspondre des valeurs numériques aux suites convergentes; en symbole $\xi=\Lambda(x)$, où $x=\{\xi_n\}$. Cette fonctionnelle satisfait aux conditions suivantes:

1° si $\xi=\Lambda(x)$ et $\eta=\Lambda(y)$, on a $\xi+\eta=\Lambda(x+y)$,

2° si $\xi=\Lambda(x)$ et α est un nombre quelconque, on a $\alpha\xi=\Lambda(\alpha x)$.

Les notations suivantes ont été employées ici:

$$x=\{\xi_n\},\qquad y=\{\eta_n\},\qquad x+y=\{\xi_n+\eta_n\},\qquad \alpha x=\{\alpha\xi_n\}.$$

La première condition exprime que la limite d'une somme de deux suites est égale à la somme de leurs limites. La seconde dit qu'en multipliant une suite par un facteur α, la limite est multipliée par α.

Supposons qu'une fonctionnelle $\Lambda(x)$, satisfaisant aux conditions 1° et 2°, soit définie non seulement pour les suites convergentes mais aussi pour certaines suites divergentes. Cette fonctionnelle peut être considérée comme une généralisation de la notion de limite; les suites auxquelles la fonctionnelle est applicable sont dites *limitables* et la fonctionnelle elle-même représente une *méthode de limitation*.

Une méthode est dite *permanente* lorsque

3° pour les suites convergentes, la fonctionnelle $\Lambda(x)$ est définie et admet la même valeur que la limite au sens habituel.

Steinhaus [8] a démontré qu'il existe des méthodes permanentes qui embrassent toutes les suites convergentes et divergentes. La démonstration s'appuie sur l'axiome de choix. Or, on ne sait construire une telle méthode d'une manière effective, même pour les suites bornées.

[1]) Les résultats de ce travail ont été présentés le 5 mai et de la deuxième partie (ce volume, p. 60-71) le 12 mai 1951 à la Section de Łódź de la Société Polonaise de Mathématique.