On a theorem of W. F. Eberlein

by

V. PTÁK (Praha)

Several years ago Eberlein [2] proved the following beautiful and important theorem:

Let X be a complete normed linear space. Let $A\subset X$ be countably compact in the weak topology of the space X. Then the weak closure of A is weakly compact.

This result has been strengthened by the author [7] in the following manner:

I. *Let X be a complete convex topological linear space. Let $A\subset X$ be pseudocompact*[1]) *in the weak topology of the space X. In such a case the closed symmetrical convex envelope of the set A is weakly compact.*

The proof of this result relies, however, on the original result of Eberlein and is, besides, not very simple. The author believes that, in view of the importance of this theorem, it will not be without some interest to give another simple proof, which at the same time admits a geometrical interpretation. This is exactly the purpose of the present note. Here we limit ourselves to the proof of the following theorem:

II. *Let X be a complete convex topological linear space. Let $A\subset X$ be pseudocompact in the weak topology of the space X. Then the weak closure of A is weakly compact.*

Unfortunately enough, the author has not succeeded in extending the elementary method mentioned above also to the proof of the stronger assertion (I).

We begin with some remarks concerning terminology and notations. At the same time we recapitulate some well known definitions and results of the theory of convex topological linear spaces.

As far as topology is concerned, terminology coincides with that of N. Bourbaki. Completeness is taken in the sense of A. Weil.

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[1]) A completely regular topological space T is said to be *pseudocompact* if every continuous function on T is bounded on T.

Let X be a convex topological linear space. We consider only real spaces in this note. A real function r defined on X will be called a *linear function on X*, if

$$r(\lambda_1 x_1+\lambda_2 x_2)=\lambda_1 r(x_1)+\lambda_2 r(x_2)$$

for any two $x_1\epsilon X$, $x_2\epsilon X$ and any two real numbers λ_1,λ_2. A linear function on X will be called a *linear functional on X* if it is continuous on X.

Let X and Y be two convex topological linear spaces. We shall say that the spaces X and Y are *dual to each other* [1] if, for every $x\epsilon X$ and for every $y\epsilon Y$ a real number xy is defined so that the following conditions are fulfilled:

(1) Let y_0 be a fixed element of Y. Then xy_0, taken as a function of x, becomes a linear functional on X, and every linear functional on X may be obtained in this manner for a suitable $y_0\epsilon Y$.

(1′) Let x_0 be a fixed element of X. Then x_0y, taken as a function of y, becomes a linear functional on Y, and every linear functional on Y may be obtained in this manner for a suitable $x_0\epsilon X$.

(2) Let y_0 be a fixed element of Y. If $xy_0=0$ for every $x\epsilon X$, then $y_0=0$.

(2′) Let x_0 be a fixed element of X. If $x_0y=0$ for every $y\epsilon Y$, then $x_0=0$.

Now let a convex topological linear space X be given. Let us denote by Y the linear space of all linear functionals defined on X. We shall write xy for the value of the functional y at the point x. Then it is possible (see *e. g.* [1]) to define a topology on Y in such a way that Y becomes a convex topological linear space and that X and Y are dual to each other. In the general case, this topology is not uniquely determined.

Let X and Y be dual convex topological spaces. If $A\subset X$, let us denote by A^* the set

$$A^*=\underset{y}{E}[y\epsilon Y,|Ay|\leqslant 1].$$

Here, of course, $|Ay|\leqslant 1$ means that $|ay|\leqslant 1$ for every $a\epsilon A$.

Clearly the set A^* is a closed symmetrical convex subset of Y for every $A\subset X$.

It is easy to see [4] that the set A^{**} is the least closed symmetrical convex subset of X which contains A. It will be called the *closed symmetrical envelope* of A.

Let a convex topological linear space X be given. The term *neighbourhood of zero* is taken to mean a symmetrical convex and closed neighbourhood of zero.

Further we shall use some definitions and results concerning complete convex topological linear spaces, which have been obtained recently by the author in [6].

Let X and Y be dual convex topological linear spaces. A linear function r defined on Y will be called an *almost continuous functional* on Y, if, for every neighbourhood of zero U in X, the restriction of r on U^* is continuous.

Perhaps it will not be useless to add two remarks to this definition. First of all, it might seem at first glance that the notion of an almost continuous functional depends on the choice of the topology in Y. This is only apparent, however; it is easy to see that the family of all almost continuous functionals on Y remains the same for any choice of the topology in Y (subject to the condition that X and Y be dual to each other).

On the other hand, it is to be noted that, given a convex topological linear space Y and a linear function r defined on Y, the expression "r is an almost continuous linear functional on Y" has no sense in itself. In fact the role played by the space Y in the foregoing definition is only an auxiliary one. As we have seen, its topology is irrelevant, and its elements are uniquely determined by the space X, anyhow. We see thus that, roughly speaking, to obtain the family of all almost continuous functionals on Y, it is sufficient to know the space X only.

We have shown in [6] that the completion of an arbitrary convex topological linear space X consists of all almost continuous linear functionals on Y. Especially: a convex topological linear space X is complete if and only if every almost continuous functional on Y is continuous.

The proof of these facts is given in [6]. To this paper the reader is referred as far as other properties of almost continuous functionals are concerned.

1. Auxiliary results. In this section we intend to collect some preliminary remarks, which will be used in the proof of our theorem. Most of them are nearly obvious and proofs will be added for the sake of completeness only.

Let T be a completely regular topological space. Let r be a function defined on T. It is sometimes useful to consider functions r which fulfil the following *weakened condition of continuity*:

Let t_0 be a limit point[2]) of the sequence $t_n \epsilon T$. In such a case the number $r(t_0)$ is a limit point of the sequence $r(t_n)$.

It is easy to see that this condition is equivalent to the postulate that $r(\bar{S}) \subset \overline{r(S)}$ be fulfilled for every countable $S \subset T$. For the sake of brevity, functions on T fulfilling this condition will be called *countably continuous functions*. The following lemma is obvious:

[2]) A point t_0 of a topological space is said to be a *limit point* of a sequence t_n, if, for every neighbourhood U of t_0, the inclusion $t_n \epsilon U$ is fulfilled for infinitely many n.

(1.1) *Let T be a completely regular topological space. Let r be a countably continuous function on T. Let $W \subset T$ be countably compact. Then r is bounded on W and attains its maximum value at a suitable point $w \epsilon W$.*

Proof. Suppose first that r is not bounded on W. Then there exist $w_n \epsilon W$ such that $|r(w_n)| \geqslant n$. The set W being countably compact, the sequence w_n has at least one limit point $w_0 \epsilon W$. It follows that the number $r(w_0)$ is a limit point of the sequence $r(w_n)$, which is impossible. This contradiction proves that r is bounded on W. Now, if $\mu = \sup r(W)$, a sequence $w_n \epsilon W$ can be found such that $r(w_n) > \mu - 1/n$. Let $w_0 \epsilon W$ be a limit point of the sequence w_n. The value of r at w_0, being a limit point of the sequence $r(w_n)$, cannot be different from μ since, in this case, μ is the only limit point of the sequence $r(w_n)$.

Let X and Y be dual convex topological linear spaces. Let r be a linear function defined on Y. We shall call r a *weakly countably continuous functional* if r is countably continuous in the weak topology of the space Y.

The interest of functionals fulfilling this weakened form of continuity lies in the fact that, in some supplementary conditions, it is possible to prove their continuity. This is shown in the following simple lemma:

(1.2) *Let X and Y be dual convex topological linear spaces. Let r be a weakly countably continuous functional on Y. Suppose that, for every neighbourhood of zero U in X, there exists a countable set $H \subset X$ such that the relation $ry = 0$ holds for every $y \epsilon U^*$ which fulfills $Hy = 0$. Then r is an almost continuous functional on Y.*

Proof. Let us take an arbitrary neighbourhood of zero U in X. We shall prove that r is weakly continuous on U^*.

Let H be a countable subset of X which fulfils the above condition with respect to U. Let h_n be a sequence which contains all elements of H. We are going to show that, for every $\varepsilon > 0$, a natural m can be found such that

$$y \epsilon U^*, \qquad |h_i y| \leqslant \frac{1}{m} \quad \text{for} \quad i = 1, 2, \ldots, m \quad \text{implies} \quad |ry| < \varepsilon.$$

Suppose that this assertion is not true. Then there exists a positive σ and a sequence $y_n \epsilon U^*$ such that, for every natural n, the following inequalities are fulfilled:

$$|h_i y_n| \leqslant \frac{1}{n} \qquad (i = 1, 2, \ldots, n), \qquad |ry_n| \geqslant \sigma.$$

The set U^* being weakly compact, there exists a point $y_0 \epsilon U^*$ which is a weak limit point of the sequence y_n. Now, r is weakly countably continuous. It follows that $|ry_0| \geqslant \sigma$.

Now, let i be a fixed natural number and let $\varepsilon > 0$. The point y_0 being a weak limit point of the sequence y_n, a natural $m \geqslant \max(1/\varepsilon, i)$ can be found such that $|h_i(y_m - y_0)| \leqslant \varepsilon$. Since $m \geqslant i$, we have $|h_i y_m| \leqslant 1/m \leqslant \varepsilon$, so that $|h_i y_0| \leqslant 2\varepsilon$.

Here, however, both i and ε are arbitrary. It follows that $h_i y_0 = 0$ for all i and, in view of our assumption, $ry_0 = 0$. We have thus obtained a contradiction which proves that r is almost continuous.

Let X be a convex topological linear space in its weak topology. According to a well known theorem [5], [6], the complete closure of X coincides with the space R of all linear functions defined on the space Y dual to X. Let A be an algebraic basis of the space Y, *i. e.* a set $A \subset Y$ consisting of linearly independent elements and such that every $y \in Y$ can be expressed as a linear combination of a finite number of elements of A. Let S denote the linear space of all "sequences" of real numbers $s = \{s^a\}$ indexed by A.

In S, let us introduce the topology of a Cartesian product of real lines. To every $r \in R$ let us assign the "sequence" $s^a = ra$. It is easy to see that the mapping of R upon S obtained in this manner is an isomorphism both in the sense of algebra and topology.

Now if B is a bounded subset of X, we shall have for every $a \in A$

$$\sigma(a) = \sup |Ba| < \infty.$$

It follows that the set B is mapped into a Cartesian product of line segments. We have thus proved that, given an arbitrary bounded $B \subset X$, the set $\bar{B}$ (closure in R) is compact. Following Hewitt [3], we shall call a completely regular topological space T *pseudocompact* if every continuous function on T is bounded on T.

(1.3) *Let X and Y be dual convex topological linear spaces. Let $B \subset X$ be pseudocompact in the weak topology of the space X. Let $r \in R$ be contained in the weak closure of the set B. Then r is weakly countably continuous.*

Proof. Our lemma will be proved if we show that for every sequence $y_n \in Y$ a point $b \in B$ can be found such that $ry_n = by_n$ for all n.

To see that, let us take a sequence $y_n \in Y$ and let y_0 be a weak limit point of y_n. We are to show that ry_0 is a limit point of the sequence ry_n. Suppose we have found a $b \in B$ such that $by_0 = ry_0$ and $by_n = ry_n$ for all natural n. Since b is a continuous functional on Y, the number $ry_0 = by_0$ is a limit point of the sequence $ry_n = by_n$. Hence it is sufficient to prove the following lemma:

(1.4) *Let X and Y be dual convex topological linear spaces. Let $B \subset X$ be pseudocompact in the weak topology of the space X. Let $r \in R$ be contained in the weak closure of the set B. Let y_n be a sequence of points of Y. Then there exists a point $b \in B$ such that $ry_n = by_n$ for all n.*

Proof. For every natural n and every $b \in B$ let $f_n(b) = |(b-r)y_n|$. Clearly f_n is a sequence of continuous functions on B. The space B being pseudocompact, there exists a sequence of positive numbers β_n such that $b \in B$ implies $|f_n(b)| \leqslant \beta_n$. Now

$$f(b) = \sum \frac{1}{2^n \beta_n} f_n(b)$$

is clearly a non-negative continuous function on B. It is easy to see that

$$\inf_{b \in B} f(b) = 0.$$

Let us suppose now that $f(b) > 0$ for all $b \in B$. Let us define a function $g(b)$ on B by the relation $g(b)f(b) = 1$. Now the value $f(b)$ can be arbitrarily small, so that g cannot be bounded on B. This, however, is not possible, since g is clearly continuous. This contradiction proves the existence of a $b \in B$, such that $f(b) = 0$. This point b evidently fulfills the relation $ry_n = by_n$ for all n.

To simplify some formulae which we shall need in the following section, it will be convenient to introduce the following notation: if $g(y)$ is a linear function defined on Y, let $Z(g)$ be the set of all $y \in Y$ for which $g(y) = 0$.

2. Weakly compact sets. In this section we shall give the non-trivial part of the proof of our theorem. We shall begin with some remarks.

Let X and Y be dual convex topological linear spaces. Let us denote by R the space of all linear functions defined on Y. The space R will be taken in the weak topology corresponding to Y. The space X, taken in its weak topology, is thus imbedded in R.

Now let $B \subset X$ be pseudocompact in the weak topology of the space X. Since B is clearly bounded in X, it follows from the considerations of the preceding paragraph that the closure of B in R is compact.

It follows that the weak closure of B in X will be compact if and only if the closure of B in R is contained already in X. In other words, our theorem will be proved if we show that every $r \in R$ which lies in the closure of B is a continuous linear functional on Y, since, in such a case, r coincides with an element of X.

Now let X be a complete space. It is shown in [6] that, in this case, the family of all continuous linear functionals on Y coincides with that of all almost continuous linear functionals on Y.

We see thus that the proof of our theorem is reduced to the proof of the following assertion:

(2.1) *Let X be a convex topological linear space. Let $B\subset X$ be pseudocompact in the weak topology of the space X. Let $r\in R$ be contained in the weak closure of the set B. Then r is an almost continuous functional on Y.*

Proof. First of all it follows from lemma (1.3) that r is weakly countably continuous.

We have seen in lemma (1.2) that a weakly countably continuous functional can be shown to be almost continuous if certain supplementary conditions are fulfilled.

In view of these remarks the proof of our theorem will be concluded if we succeed in proving the following assertion:

For every neighbourhood of zero U in X, there exists a sequence $b_n\in B$ with the following property: if $y\in U^*$ and $b_n y=0$ for all n, then $ry=0$.

To prove this, let us take an arbitrary neighbourhood of zero U in X. The construction of the sequence b_n will proceed by a simple induction.

(1) Let b_1 be an arbitrary point of B. The set $U^*\cap Z(b_1)$ being weakly compact, a point $y_1\in U^*\cap Z(b_1)$ can be found such that

$$ry_1=\max ry,\qquad y\in U^*\cap Z(b_1).$$

This follows easily from lemma (1.1) if we take into account the fact that r is weakly countably continuous.

(2) According to (1.4) there exists a point $b_2\in B$ such that $b_2y_1=ry_1$. Now, using a similar argument as in the preceding case, we conclude that there exists a point $y_2\in U^*\cap Z(b_1)\cap Z(b_2)$ such that

$$ry_2=\max ry,\qquad y\in U^*\cap Z(b_1)\cap Z(b_2).$$

$(n+1)$ Let us suppose now that the elements $b_1,\dots,b_n$, $y_1,\dots,y^n$ have been defined so that the following relations are fulfilled:

$$b_iy_j=ry_j\qquad\text{for}\quad j<i,$$

$$b_iy_j=0\qquad\text{for}\quad i\leqslant j,$$

$$ry_j=\max ry,\qquad y\in U^*\cap Z(b_1)\cap\ldots\cap Z(b_j).$$

First of all let us choose $b_{n+1}\in B$ such that

$$b_{n+1}y_j=ry_j\qquad\text{for}\quad j=1,2,\dots,n.$$

To obtain y_{n+1}, we recollect that r attains its maximum value on the weakly compact set $U^*\cap Z(b_1)\cap\ldots\cap Z(b_{n+1})$. Now it is sufficient to take $y_{n+1}\in U^*\cap Z(b_1)\cap\ldots\cap Z(b_{n+1})$ such that

$$ry_{n+1}=\max ry,\qquad y\in U^*\cap Z(b_1)\cap\ldots\cap Z(b_{n+1}).$$

This completes the induction.

Now we are going to show that $\lim ry_n=0$. Clearly ry_n is a non-increasing sequence of non-negative numbers. Let $\varepsilon=\inf ry_n$. Suppose that $\varepsilon>0$.

Let $y_0\in U^*$ be a weak limit point of the sequence y_n. Since $b_iy_n=0$ for $n\geqslant i$, we have $b_iy_0=0$ for all i. For $b\in B$ let

$$\varphi_n(b)=\varepsilon-\min\big(|b(y_n-y_0)|,\varepsilon\big).$$

The functions $\varphi_n(b)$ are continuous functions on B. We have $0\leqslant\varphi_n(b)\leqslant\varepsilon$ and the relation $\varphi_n(b)=0$ holds if and only if $|b(y_n-y_0)|\geqslant\varepsilon$. For $b\in B$ let us define further

$$\varphi(b)=\sum\frac{1}{2^n}\varphi_n(b).$$

The function $\varphi(b)$ is continuous and non-negative. Let us estimate the value $\varphi(b_i)$. For $n<i$ we have $b_iy_n=ry_n\geqslant\varepsilon$, so that

$$b_i(y_n-y_0)=b_iy_n\geqslant\varepsilon,$$

which implies $\varphi_n(b_i)=0$. It follows that $\varphi(b_i)\leqslant\varepsilon/2^{i-1}$. The space B being pseudocompact, we can easily conclude the existence of a point $b\in B$ such that $\varphi(b)=0$ or, which is the same, such that $\varphi_n(b)=0$ for all n.

It follows that $|b(y_n-y_0)|\geqslant\varepsilon$ for all n, which is a contradiction, y_0 being a limit point of y_n in the weak topology. This contradiction shows that ε cannot be positive. We have thus proved that $\lim ry_n=\varepsilon=0$.

Now let us take a point $y\in U^*$ such that $b_iy=0$ for all i. We are going to show that $ry=0$. To see that, let m be an arbitrary natural number. Since $y\in U^*\cap Z(b_1)\cap\ldots\cap Z(b_m)$, we have

$$|ry|\leqslant\max ry,\qquad y\in U^*\cap Z(b_1)\cap\ldots\cap Z(b_m),$$

so that $|ry|\leqslant ry_m$. Since m was arbitrary, we have $ry=0$, which concludes the proof.

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On a class of operations over the space of continuous vector valued functions

by

W. ORLICZ (Poznań)

1. By $\omega(u)$ we shall denote a non decreasing function, defined for $u\geqslant 0$, positive for $u>0$, vanishing for $u=0$ and such that $\lim\limits_{u\to 0}\omega(u)=0$. We shall say that the *function* $\omega(u)$ *satisfies the condition* (m) if

(a) $$\omega(uv)\leqslant c\,\omega(u)\,\omega(v),$$

(b) $$\frac{\omega(u)}{u}\to 0 \quad \text{as} \quad u\to\infty.$$

The condition (m) implies

(b′) $$\frac{\omega(u)}{u}\to\infty \quad \text{as} \quad u\to 0.$$

Indeed, by (a)

$$\frac{\omega(u)}{u}>\frac{\omega(1)}{cu\,\omega(1/u)}.$$

The functions $\omega(u)=u^{a}$ or $\omega(u)=u^{a}(|\ln u|+1/a)$, where $0<a<1$, satisfy the condition (m).

X will denote a Banach space. $C(X)$ will stand for the Banach space of continuous X-valued functions $x(t)$ defined in $\Delta=\langle a,b\rangle$ under the usual definitions of addition and multiplication by scalars and with the norm

$$\|x\|_C=\max_{\Delta}\|x(t)\|.$$

By $C(X)_p$ we shall denote the space of continuous X-valued functions $x(t)$ defined for $-\infty<t<\infty$ and of period p; $C(X)_p$ may become, as above, Banach space (if we define the norm by the above formula with $\Delta=\langle 0,p\rangle$).