[7] — *Weak compactness in convex topological linear spaces*, Čechosl. Mat. Žurnal 79 (1954), p. 175-186.

[8] J. Dieudonné, *Sur un théorème de Šmulian*, Archiv der Math. 3 (1952), p. 436-440.

[9] A. Grothendieck, *Critères de compacité dans les espaces fonctionnels généraux*, Am. Journ. of Math. 74 (1952), p. 168-186.

(Reçu par la Rédaction le 14. 11. 1953)

On a class of operations over the space of continuous vector valued functions

by

W. ORLICZ (Poznań)

1. By $\omega(u)$ we shall denote a non decreasing function, defined for $u\geqslant 0$, positive for $u>0$, vanishing for $u=0$ and such that $\lim\limits_{u\to 0}\omega(u)=0$. We shall say that the *function* $\omega(u)$ *satisfies the condition* (m) if

(a) $$\omega(uv)\leqslant c\,\omega(u)\,\omega(v),$$

(b) $$\frac{\omega(u)}{u}\to 0 \quad \text{as} \quad u\to\infty.$$

The condition (m) implies

(b′) $$\frac{\omega(u)}{u}\to\infty \quad \text{as} \quad u\to 0.$$

Indeed, by (a)

$$\frac{\omega(u)}{u}>\frac{\omega(1)}{cu\,\omega(1/u)}.$$

The functions $\omega(u)=u^{a}$ or $\omega(u)=u^{a}(|\ln u|+1/a)$, where $0<a<1$, satisfy the condition (m).

X will denote a Banach space. $C(X)$ will stand for the Banach space of continuous X-valued functions $x(t)$ defined in $\Delta=\langle a,b\rangle$ under the usual definitions of addition and multiplication by scalars and with the norm

$$\|x\|_C=\max_{\Delta}\|x(t)\|.$$

By $C(X)_p$ we shall denote the space of continuous X-valued functions $x(t)$ defined for $-\infty<t<\infty$ and of period p; $C(X)_p$ may become, as above, Banach space (if we define the norm by the above formula with $\Delta=\langle 0,p\rangle$).

Given a function $\omega(u)$, we denote for $x(t)$ belonging to $C(X)$ or $C(X)_p$

$$\sup_{t,h} \frac{\|x(t+h)-x(t)\|}{\omega(|h|)} = \mu,$$

where in the case if $x \in C(X)$ the supremum is taken for $t \in \Delta, a-t \leqslant h \leqslant b-t$, and if $x \in C(X)_p$, for arbitrary t, h.

By $L_\omega(X)$ or $L_\omega(X)_p$ respectively we shall denote the linear space of functions of $C(X)$ or $C(X)_p$ respectively, for which $\mu < \infty$. Under the usual definitions of addition and multiplication by scalars, and with the norm

$$\|x\|_\omega = \max_{\Delta} \|x(t)\| + \mu,$$

they are Banach spaces.

If $\omega(u) = u^a$, $0 < a \leqslant 1$, we shall write $L_a(X)$ instead of $L_\omega(X)$, and $\|x\|_a$ instead of $\|x\|_\omega$. The functions of $L_a(X)$ with $0 < a < 1$ are said to *satisfy the Hölder condition with the exponent* a, the functions of $L_1(X)$ are said to *satisfy the Lipschitz condition*. In the last case the constant k, such that for $a-t \leqslant h \leqslant b-t$

$$\|x(t+h)-x(t)\| \leqslant k|h|,$$

is called the *Lipschitz constant*. Analogous terminology will be used for spaces of periodic functions.

Obviously $C(X) \supset L_a(X) \supset L_\beta(X)$ if $a < \beta$.

By $C_0(X)$ or $C_0(X)_p$ we shall denote a complete subspace of $C(X)$ or $C(X)_p$ respectively. If we restrict the functions $x(t)$ to run over the space $C_0(X)$ then we obtain a complete subspace $C_0(X)L_\omega(X)$ of the space $L_\omega(X)$.

We shall say that the *space* $C_0(X)_p$ *is translation-invariant* if $x(t) \in C_0(X)_p$ implies, for every τ, $x(t+\tau) \in C_0(X)_p$.

LEMMA. *If for every* $\tau \in \langle \tau', \tau'' \rangle$ *the function* $x(\tau; t)$ *belongs to* $C_0(X)$ [*to* $C_0(X)_p$] *and* $x(\tau; :)$ *depends continuously on the parameter* τ, *then the function*

$$y(t) = \int_{\tau'}^{\tau''} x(\tau; t)\, d\tau$$

also belongs to $C_0(x)$ [*to* $C_0(X)_p$]. *The integral is taken in the sense of Riemann-Graves.*

Proof. Given a partition $\pi: \tau' = \tau_0 < \tau_1 < \ldots < \tau_n = \tau''$, let us write

$$z(\pi, t) = \sum_{i=1}^{n} x(\tau_{i-1}; t)(\tau_i - \tau_{i-1}).$$

The function of two variables $x(\tau; :) = x(\tau, t)$ is uniformly continuous in $\langle \tau', \tau'' \rangle \times \Delta$. Let $\Omega(\delta)$ be the modulus of continuity of $x(\tau, t)$; then, for every partition π,

$$\|z(\pi, t') - z(\pi, t'')\| \leqslant (\tau'' - \tau')\,\Omega(\delta),$$

if $|t' - t''| \leqslant \delta$. Given a normal sequence of partitions $\{\pi_n\}$, we see that $z_n = z(\pi_n, t) \to y(t)$ uniformly in $\langle a, b \rangle$, for $z(\pi_n, t) \to y(t)$ at every $t \in \langle a, b \rangle$, and the functions $z(\pi_n, t)$ are uniformly continuous. Hence

$$\|z_n - y\|_C \to 0, \qquad y \in C_0(X).$$

THEOREM 1. *Let the space* $C_0(X)_p$ *be translation-invariant; then it is possible to define linear operations* $T_n(x)$ *from* $C_0(X)_p$ *to* $C_0(X)_p L_1(X)_p$ *is such a manner that:*

(a) *if* $x \in C_0(X)_p L_\omega(X)_p$, $\omega = \omega(u)$ *being fixed, then the functions* $x_n(t) = T_n(x)$ *satisfy the Lipschitz condition with the constant*

(1) $$k_n = Bn\,\omega\left(\frac{1}{n}\right),$$

(b) *for* $n = 1, 2, \ldots$

(2) $$\|x - x_n\|_C \leqslant A\,\omega\left(\frac{1}{n}\right),$$

(c) *the constants* A, B *in* (1) *and* (2) *do not depend on* n.

It is possible to define $T_n(x)$ *such that*

$$A = \|x\|_\omega, \qquad B = \|x\|_\omega.$$

The theorem remains true if we remove the condition of translation-invariance of the space and replace the space $C_0(X)_p$ *by the space* $C(X)$. *In this case we may set* $A = 2\|x\|_\omega$, $B = \|x\|_\omega$.

Proof. The particular case where X is the space of real numbers and $C_0(X)_p$ is identical with $C(X)_p$ is well known. In this case the operations $T_n(x)$ may be defined in several ways, *e. g.* by means of singular integrals. This device may be adapted to the space $C_0(X)_p$ satisfying the translation condition.

As an example we present three kinds of the introduction of $T_n(x)$, taking $p = 2\pi$.

1. Let us write

$$x_n(t) = T_n(x) = n \int_t^{t+1/n} x(\tau)\, d\tau = n \int_0^{1/n} x(\tau + t)\, d\tau.$$

By our Lemma $T_n(x)\epsilon C_0(X)_p$. Since $x_n'(t)=n\left[x\left(t+\frac{1}{n}\right)-x(t)\right]$, we get

$$\|x_n'(t)\|\leqslant n\|x\|_\omega\,\omega\left(\frac{1}{n}\right),\qquad \|x_n'(t)\|\leqslant 2n\|x\|_C.$$

Hence $x_n(t)$ satisfies the Lipschitz condition with the constant (1), where $B=\|x\|_\omega$ and $T_n(x)$ is a continuous operation from $C_0(X)_p$ to $C_0(X)_pL_1(X)$ moreover

$$\|x(t)-x_n(t)\|\leqslant n\int_0^1\|x(t)-x(t+\tau)\|\,d\tau\leqslant\|x\|_\omega\,\omega\left(\frac{1}{n}\right).$$

This implies the condition (b) with $A=\|x\|_\omega$.

2. Set

$$k_n(t)=\left(\sin\frac{nt}{2}\operatorname{cosec}\frac{t}{2}\right)^4,\qquad \gamma_n=\int_{-\pi}^{\pi}k_n(t)\,dt=2\pi\frac{n(2n^2+1)}{3}.$$

We define first the Jackson integrals[1]) for $n=1,2,\ldots$ as

$$s_n(t)=S_{2n-2}(x)=\frac{1}{\gamma_n}\int_{-\pi}^{\pi}x(\tau)k_n(\tau-t)\,d\tau=\frac{1}{\gamma_n}\int_{-\pi}^{\pi}x(\tau+t)k_n(\tau)\,d\tau.$$

If $x(t)\epsilon C_0(X)_p$ then, by our Lemma, $s_n(t)\epsilon C_0(X)_p$. The same estimations as in the case of real functions $x(t)$ give

$$\|x-s_n\|_C\leqslant 6\|x\|_\omega\,\omega\left(\frac{1}{n}\right),\tag{3}$$

$$\|s_n\|_C\leqslant\|x\|_C.\tag{3'}$$

If $x(t)$ satisfies the Lipschitz condition with the constant K, then $s_n(t)$ satisfies this condition with the same constant, for

$$\|s_n(t+h)-s_n(t)\|\leqslant\frac{1}{\gamma_n}\int_{-\pi}^{\pi}\|x(t+\tau+h)-x(t+\tau)\|\,k_n(\tau)\,d\tau\leqslant K|t|.$$

Choosing an arbitrary sequence of operations $x_n^*(t)=T_n^*(x)$ satisfying the condition (a), (b) with constants $A=B=\|x\|_\omega$ and setting

$$T_{2n-1}(x)=S_{4n-2}\big(T_{2n-1}^*(x)\big),$$

$$T_{2n}(x)=S_{4n-2}\big(T_{2n}^*(x)\big)\quad\text{ for }\quad n=1,2,\ldots,$$

[1]) Concerning 2 see, for instance, И. П. Натансон, *Конструктивная теория функций*, Москва 1949, p. 111-119.

we see that the functions $T_{2n}(x)$ and $T_{2n-1}(x)$ satisfy the Lipschitz condition with the constants $\|x\|_\omega 2n\omega(1/2n)$ or $\|x\|_\omega(2n-1)\omega\big(1/(2n-1)\big)$ respectively. Since

$$\|x-T_{2n}^*(x)\|_C\leqslant\|x\|_\omega\,\omega\left(\frac{1}{2n}\right),$$

we infer by (3') that

$$\|S_{4n-2}(x)-T_{2n}(x)\|_C\leqslant\|x\|_\omega\,\omega\left(\frac{1}{2n}\right),$$

and this, together with (3), leads to

$$\|x-T_{2n}(x)\|_C\leqslant 7\|x\|_\omega\,\omega\left(\frac{1}{2n}\right).$$

Analogously

$$\|x-T_{2n-1}(x)\|_C\leqslant 7\|x\|_\omega\,\omega\left(\frac{1}{2n-1}\right).$$

Hence we can set $A=7\|x\|_\omega$, $B=\|x\|_\omega$. Let us observe that $k_n(t)$ is a trigonometric polynomial ot the form

$$k_n(t)=\sum_{i=0}^{2n-2}c_i\cos it.$$

The representation of $T_{2n}(x)$ and $T_{2n-1}(x)$ by aid of the Jackson integral shows that these operations may be written as a trigonometrical polynomial of degree $4n-2$

$$\sum_{i=0}^{4n-2}(x_i\cos it+y_i\sin it),$$

where

$$x_i=c_i\int_{-\pi}^{\pi}T_m^*(x)\cos i\tau\,d\tau,\qquad y_i=c_i\int_{-\pi}^{\pi}T_m^*(x)\sin i\tau\,d\tau,$$

and are linear operations from $C_0(X)_p$ to X.

3. Replacing $C_0(X)_p$ by $C(X)$, we can define $T_n(x)$ as a polygonal function $x_n(t)$ assuming for

$$t_i=a+\frac{i}{n},\qquad i=0,1,\ldots,\qquad m=E[n(b-a)],$$

the value $x(t_i)$, linear in the intervals $\langle t_{i-1},t_i\rangle$ for $i=1,\ldots,m-1$, and in the interval $(a+m/n,b)$ equal to $x(t_{m-1})$ if $a+m/n\neq b$.

In this case we can choose $A=3\|x\|_\omega/2$, $B=\|x\|_\omega$. If $\omega(u)$ satisfies the condition (m), we can replace the coefficient $3/2$ above by $1/2+c\omega(1/2)$.

Analogously we can define $x_n(t)$ in the space $C(X)_p$.

THEOREM 2. *If there exist functions $x_n(t)\in C_0(X)_p$ satisfying the Lipschitz condition with the constant* (1), *and if the inequality* (2) *is satisfied for $n=1,2,\ldots$, then*

$$x(t)\in C_0(X)_p L_\omega(X)_p$$

for every $\omega(u)$; moreover,

$$\|x\|_\omega \leqslant \|x\|_C+\lambda,$$

where $\lambda=2\max[A+B, A+B|\Delta|]$.

The theorem remains true if we replace $C_0(X)_p$ and $L_\omega(X)_p$ by $C_0(X)$ and $L_\omega(X)$ respectively.

Proof. By our hypothesis

$$\|x(t)-x_{2^n}(t)\|_C\leqslant A\,\omega\left(\frac{1}{2^n}\right),$$

$$\|x_{2^n}(t')-x_{2^n}(t'')\|\leqslant B\,2^n\omega\left(\frac{1}{2^n}\right)|t'-t''|.$$

Given $|h|\in(0,1)$ let us choose n so that

$$\frac{1}{2^{n-1}}>|h|\geqslant\frac{1}{2^n}.$$

Then the inequality

$$\|x(t+h)-x(t)\|\leqslant\|x(t+h)-x_{2^n}(t+h)\|+\|x(t)-x_{2^n}(t)\|+\|x_{2^n}(t+h)-x_{2^n}(t)\|$$

$$\leqslant 2A\,\omega\left(\frac{1}{2^n}\right)+B2^n\omega\left(\frac{1}{2^n}\right)|h|\leqslant 2(A+B)\,\omega(|h|)$$

is satisfied. If $1\leqslant|h|\leqslant|\Delta|$, the last inequality with $n=0$ leads to

$$\|x(t+h)-x(t)\|\leqslant 2(A+B|\Delta|)\,\omega(|h|).$$

Thus setting $\lambda=2\max[A+B, A+B|\Delta|]$, we get

$$\|x\|_\omega\leqslant\|x\|_C+\lambda.$$

THEOREM 2'. *Let $\omega(u)$ satisfy the condition* (m), *and let $C^n=C^n(X)_p$, be a linear subset of $C(X)_p$ whose functions have the following properties:*

(*) *the class C^n is contained in C^{n+1},*

(**) *the functions of C^n satisfy the Lipschitz condition with the constant $k'_n=Bn\|x\|_C$, B being independent of n.*

If for $n=1,2,\ldots$ there exists a function $x_n(t)\in C^n$ belonging to the given $C_0(X)_p$, and satisfying the inequality (2), *then*

$$x(t)\in C_0(X)_p L_\omega(X)_p,$$

and

$$\|x\|_\omega\leqslant\|x\|_C+\lambda,$$

where

$$\lambda=AL_1(\omega)+ABL_2(\omega)+BL_3(\omega,x). \tag{o}$$

Here $L_1(\omega), L_2(\omega)$ are constants depending only on $\omega(u)$, while $L_3(\omega,x)$ depends only on $\omega(u)$ and $\|x\|_C$.

The theorem remains true when we replace $C_0(X)_p$ and $L_\omega(X)_p$ by $C_0(X)$ and $L_\omega(X)$ respectively.

Proof. We shall prove the theorem for the space $C_0(X)$; for the space $C_0(X)_p$ the proof runs in the same way.

Suppose there exist functions $x_n(t)$ satisfying the hypotheses. Given $a>1$, let us set

$$y_0(t)=x_1(t),\qquad y_n(t)=x_{a^n}(t)-x_{a^{n-1}}(t)\quad\text{for}\quad n=1,2,\ldots$$

By (2)

$$x(t)=\sum_{n=0}^{\infty}y_n(t)$$

and the series on the right-hand side converges uniformly in $\langle b,c\rangle$ for sufficiently large t, which results from the estimations below. Let m be an index; since $\omega(u)$ satisfies the condition (m), $n\geqslant m$ implies

$$\omega\left(\frac{1}{a^n}\right)=\omega\left(\frac{1}{a^m}\,\frac{1}{a^{n-m}}\right)\leqslant c^{n-m}\left[\omega\left(\frac{1}{a}\right)\right]^{n-m}\omega\left(\frac{1}{a^m}\right),$$

and for $n\leqslant m$

$$\omega\left(\frac{1}{a^n}\right)=\omega\left(\frac{1}{a^m}\,a^{m-n}\right)\leqslant c^{m-n}\,\omega\left(\frac{1}{a^m}\right)[\omega(a)]^{m-n}.$$

Further, the following inequalities are true:

$$\|y_n(t)\|\leqslant\|x(t)-x_{a^n}(t)\|+\|x(t)-x_{a^{n-1}}(t)\|\leqslant 2A\,\omega\left(\frac{1}{a^{n-1}}\right),$$

$$\sum_{n=m+1}^{\infty}\|y_n(t)\|\leqslant 2A\sum_{n=m+1}^{\infty}\omega\left(\frac{1}{a^{n-1}}\right)\leqslant 2A\,\omega\left(\frac{1}{a^m}\right)\sum_{i=0}^{\infty}c^i\left[\omega\left(\frac{1}{a}\right)\right]^i, \tag{4}$$

$$\|y_n(t')-y_n(t'')\|\leqslant Ba^n\|y_n\|_C|t'-t''|\leqslant 2ABaa^{n-1}\omega\left(\frac{1}{a^{n-1}}\right)|t'-t''|,$$

$$\sum_{n=1}^{m}\|y_n(t')-y_n(t'')\|\leqslant 2ABa\sum_{n=1}^{m}a^{n-1}\omega\left(\frac{1}{a^{n-1}}\right)|t'-t''|$$

$$\leqslant 2ABac^m[\omega(a)]^m\,\omega\left(\frac{1}{a^m}\right)\sum_{n=1}^{m}\left[\frac{a}{c\,\omega(a)}\right]^{n-1}|t'-t''|, \tag{5}$$

where $m=1,\ldots$ and, finally,

$$(5')\qquad \|y_0(t')-y_0(t'')\| \leqslant B\big(A\,\omega(1)+\|x\|_C\big)|t'-t''| \leqslant Bc^m[\omega(a)]^m\omega\left(\frac{1}{a^m}\right)\left(A+\frac{\|x\|_C}{\omega(1)}\right)|t'-t''|.$$

Choose a so large that

$$c\omega\left(\frac{1}{a}\right)<1,\qquad \frac{c\omega(a)}{a}<1,\qquad a>|\varDelta|;$$

this is possible in virtue of the postulate (b) in the condition (m). Given $|h|\in(0,1)$ choose m so that

$$\frac{1}{a^{m-1}}>|h|\geqslant\frac{1}{a^m}.$$

Then

$$\|x(t+h)-x(t)\|\leqslant\sum_{n=0}^{m}\|y_n(t+h)-y_n(t)\|+\sum_{n=m+1}^{\infty}\|y_n(t+h)\|+\sum_{n=m+1}^{\infty}\|y_n(t)\|,$$

whence the estimations (4), (5), and (5′) lead to

$$\|x(t+h)-x(t)\|\leqslant|h|\,B\left(A+\frac{\|x\|_C}{\omega(1)}\right)c^m[\omega(a)]^m\omega\left(\frac{1}{a^m}\right) +2|h|\,ABac^m[\omega(a)]^m\omega\left(\frac{1}{a^m}\right)\frac{\left(\frac{a}{c\omega(a)}\right)^m}{\frac{a}{c\,\omega(a)}-1}+4A\,\omega\left(\frac{1}{a^m}\right)\frac{1}{1-c\omega\left(\frac{1}{a}\right)},$$

and since $\omega(1/a^m)\leqslant\omega(|h|)$, therefore

$$\|x(t+h)-x(t)\|\leqslant\left[2AB\frac{a^2}{\frac{a}{c\omega(a)}-1}+ABa+A\frac{4}{1-c\omega\left(\frac{1}{a}\right)}+\frac{Ba}{\omega(1)}\|x\|_C\right]\omega(|h|).$$

Set

$$(6)\qquad L_1(\omega)=\frac{4}{1-c\omega\left(\frac{1}{a}\right)},\qquad L_2(\omega)=\frac{2a^2}{\frac{a}{c\omega(a)}-1}+a,\qquad L_3(\omega,x)=\frac{a\|x\|_C}{\omega(1)}.$$

If $1\leqslant|h|\leqslant|\varDelta|$, then, applying the inequality (4) with $m=0$ we obtain from (5′)

$$\|x(t+h)-x(t)\|\leqslant\|y_0(t+h)-y_0(t)\|+4A\,\omega(1)\frac{1}{1-c\,\omega\left(\frac{1}{a}\right)} \leqslant\left(\left(A+\frac{\|x\|_C}{\omega(1)}\right)B|\varDelta|+4A\frac{1}{1-c\,\omega(1/a)}\right)\omega(|h|),$$

and since

$$|\varDelta|\leqslant L_2(\omega),\qquad \frac{\|x\|_C|\varDelta|}{\omega(1)}\leqslant L_3(\omega,x),$$

we see that for every $|h|\leqslant|\varDelta|$ the inequality

$$\|x(t+h)-x(t)\|\leqslant\lambda\omega(|h|)$$

holds with λ defined by the formula (o), whence $\|x\|_\omega\leqslant\|x\|_C+\lambda$.

Remark. The theorem 2 belongs to the domain of the classical approximation problems of D. Jackson and S. Bernstein. In the classical problematics $x_n(t)$ is supposed to be a real polynomial of degree n and $\omega(u)=u^a$, while in our case more general Lipschitzian vector valued functions are admitted and $\omega(u)$ are slightly more general. Our method does not differ esentially from the classical procedure.

Similarly to the real case we can choose in the space $C(X)_{2\pi}$ as C^{n+1} the class of trigonometric polynomials of degree $\leqslant n$ and of the form

$$x(t)=\sum_{i=0}^{n}(x_i\cos it+y_i\sin it),$$

where $x_i,y_i\in X$. Indeed, as may easily be seen, an analogue of the classical theorem of S. Bernstein holds:

$$\|x'(t)\|_C\leqslant n\,\|x\|_C.$$

If we choose as C^n the class of the polynomials of degree $\leqslant 2n-2$, then they contain the Jackson polynomials $s_n(t)=S_{2n-2}(x)$, defined in 2, of the proof of theorem 1 and taking $x_n(t)=S_{2n-2}(x)$ with $x\in C_0(X)_{2\pi}$ we obtain as equence of functions $x_n(t)$ belonging to $C_0(X)_{2\pi}$ (the translation-invariance being assumed), which can be used as a sequence of approximating functions in Theorem 2′.

The method used in the proof of Theorem 2′ may also be applied to prove the following theorem:

THEOREM 3. *Let $\omega(u)$ satisfy the condition* (m) *and let $y_n(t)$ denote functions from $C_0(X)_p$ satisfying the following conditions*:

(*) *There is a constant $A>0$ such that*

$$\|y_n(t)\|_C\leqslant A\,\omega\left(\frac{1}{n}\right)\quad \textit{for}\quad n=1,2,\ldots$$

(**) *$y_n(t)$ satisfy the Lipschitz condition with the constant* (1).

Under these hypotheses every lacunary series of form

$$\sum_{n=1}^{\infty}y_{a^n}(t)$$

converges uniformly in $\langle 0,p\rangle$ *and represents a function in* $C_0(X)_p L_\omega(X)_p$ *if* a *is a positive integer satisfying the conditions*

$$c\omega\left(\frac{1}{a}\right)<1,\qquad \frac{c\omega(a)}{a}<1.$$

An analogous statement holds it we replace $C_0(X)_p$ *by* $C_0(X)$.

As application of Theorem 3 let us consider the following example. Let $\varphi(t)$ be a vector valued function with values in X and of period p, satisfying the Lipschitz condition. If $y_n(t)=\omega(1/n)\varphi(nt)$, then the conditions (*) and (**) are satisfied, and if a satisfied the conditions of the theorem, then the series

$$\sum_{n=1}^{\infty}\omega\left(\frac{1}{a^n}\right)\varphi(a^n t)$$

represents a function of $C(X)_p L_\omega(X)_p$, provided that $\omega(u)$ satisfies the condition (m). In particular, let $\omega(u)=u^\gamma$, $0<\gamma<1$; then the condition (m) with the constant $c=1$ is satisfied and we may apply Theorem 3 with $a=2,3,\ldots$

Choose $0<b<1$, $ab>1$ and let $\gamma=-\ln b/\ln a$; then $0<\gamma<1$ and $b^n=(1/a^\gamma)^n=\omega(1/a^n)$, whence the series

$$x(t)=\sum_{n=1}^{\infty}b^n\varphi(a^n t)$$

belongs to $C(X)_p L_\delta(X)_p$ for $0<\delta\leqslant\gamma$. This result is in a certain sense the best possible. Indeed, for $\varphi(t)=\cos\pi t$ the function $x(t)$ presents for almost every t the following singularity:

$$\lim_{h\to 0}\operatorname{apr}\frac{|x(t+h)-x(t)|}{|h|^\delta}=\infty$$

for $\delta>\gamma$. Analogous singularities may be obtained for more general $\varphi(t)$ under supplementary conditions imposed upon a and b[2]).

THEOREM 4. *Let* $C_0(X)_p$ *be translation-invariant, let* $U(x)$ *be a linear operation from* $C_0(X)_p$ *to* $C_0(X)_p$. *Moreover, let* $U(x)$ *map the space* $C_0(X)_p L_1(X)_p$ *into the space* $C_0(X)_p L_1(X)_p$. *Under these hypotheses* $U(x)$ *has the following properties:*

(α) *for fixed* $\omega(u)$, $x\in C_0(X)_p L_\omega(X)_p$ *implies* $U(x)\in C_0(X)_p L_\omega(X)_p$,

(β) *the operation* $U(x)$ *is linear from the space* $C_0(X)_p L_\omega(X)_p$ *to the space* $C_0(X)_p L_\omega(X)_p$ *and its norm satisfies the inequality*

$$\|U\|_\omega\leqslant\|U\|_C(2s+1)+\|U\|_1(2+\gamma)2s. \tag{7}$$

Here $s=\max(1,p)$ and $\gamma=\sup\limits_{0<u\leqslant 1}u/\omega(u)$ is supposed to be finite.

[2]) See W. Orlicz, *Sur les fonctions satisfaisant à une condition de Lipschitz généralisée (II)*, Studia Mathematica 13 (1953), p. 69-82.

The theorem remains true if we remove the translation-invariance, replace the spaces $C_0(X)_p, L_\omega(X)_p, L_1(X)_p$ *by* $C(X), L_\omega(X), L_1(X)$ *respectively, and multiply the right-hand side in the inequality* (7) *by* 2.

Proof. We prove first that $U(x)$ is linear from $C_0(X)L_1(X)_p$ to $C_0(X)_p L_1(X)_p$. Indeed, let $U(x_n)\to y_0$ and $x_n\to x_0$ (in the sense of the convergence generated by the norm in $C_0(X)_p L_1(X)_p$); then $\|x_n-x_0\|_c\to 0$, $\|U(x_n)-y_0\|_c\to 0$, whence $y_0=U(x_0)$ and it suffices to apply the closed graph theorem of Banach[3]).

Let $T_n(x)$ be linear operations of theorem 1 chosen so that $A=B=\|x\|_\omega$. If $x\in C_0(X)_p L_\omega(X)_p$, then

$$\|x-T_n(x)\|_C\leqslant\|x\|_\omega\,\omega\left(\frac{1}{n}\right).$$

Further, the following inequalities are satisfied:

$$\|U(x)-U(T_n(x))\|_C\leqslant\|U\|_C\|x-T_n(x)\|_C\leqslant\|U\|_C\|x\|_\omega\,\omega\left(\frac{1}{n}\right),$$
$$\|T_n(x)\|_C\leqslant\|x\|_C+\|x\|_\omega n\omega\left(\frac{1}{n}\right)\leqslant\|x\|_\omega\left(1+n\,\omega\left(\frac{1}{n}\right)\right), \tag{8}$$

$$\|T_n(x)\|_1\leqslant\|T_n(x)\|_C+k_n\leqslant\|x\|_\omega\left(1+2n\,\omega\left(\frac{1}{n}\right)\right)\leqslant\|x\|_\omega(2+\gamma)n\omega\left(\frac{1}{n}\right), \tag{9}$$

$$\|U(T_n(x))\|_1\leqslant\|U\|_1\|T_n(x)\|_1. \tag{9'}$$

By Theorem 2, (8) and (9), (9′) imply $U(x)\in C_0(X)_p L_\omega(X)_p$ and

$$\|U(x)\|_\omega\leqslant\|U(x)\|_C+\lambda\leqslant\|U\|_C\|x\|_\omega+2s(\|U\|_C\|x\|_\omega+\|U\|_1(2+\gamma)\|x\|_\omega)$$
$$\leqslant[\|U\|_C(2s+1)+\|U\|_1(2+\gamma)2s]\|x\|_\omega,$$

and this implies the inequality (7).

Remark. We can replace in Theorem 4 the hypothesis of the linearity of $U(x)$ by the hypothesis that $U(x)$, as an operation from $C_0(X)_p$ and from $C_0(X)_p L_1(X)_p$ to itself, satisfies the Lipschitz condition of the following form:

$$\|U(x)\|_C\leqslant K_C\|x\|_C,\qquad \|U(x)\|_1\leqslant K_1\|x\|_1.$$

Then the assertion of Theorem 4 is to be read:

$U(x)$, as an operation from $C_0(X)_p L_\omega(X)_p$ to $C_0(X)_p L_\omega(X)_p$, satisfies a Lipschitz condition of the form

$$\|U(x)\|_\omega\leqslant K_\omega\|x\|_\omega.$$

[3]) S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne, Warszawa 1932, p. 41, théorème 7.

In formula (7) the norms $\|U\|_\omega$, $\|U\|_C$, $\|U\|_1$ are to be replaced by K_ω, K_C, K_1.

2. We shall consider some cases of the spaces $C(X)_p$, $C_0(X)_p$ with the space X properly chosen, leading to some classes of functions considered in the aproximation theory.

A. Let X denote the space of real numbers; then $C(X)_p$ is the space of continuous functions of period p, and $C(X)_p L_\omega(X)_p$ is the space of the functions of period p whose modulus of continuity $\tilde{\omega}(u)$ satisfies the inequality

$$\tilde{\omega}(u)=O(\omega(u)).$$

B. Let X be the space L^r of functions of period p, integrable with the r-th power in $\langle 0,p\rangle$ $(r\geqslant 1)$. As $C_0(X)_p$ let us choose the space of the functions $x(t)$ of the form $x(t)=f(t+v)$ where $f(v)\in L^r$; then evidently $C_0(X)_p$ is translation-invariant.

Since there exists a linearly-isomorphic correspondence between the functions $x(t)$ and $f(v)$ and, moreover, for every t

$$\|x(t)\|=\Big(\int_0^p |f(t+v)|^r dv\Big)^{1/r}=\Big(\int_0^p |f(v)|^r dv\Big)^{1/r}=\|f\|_r,$$

therefore it follows that the space $C_0(x)_p$ is equivalent to the space L^r.

The formula

$$\|x(t+h)-x(t)\|=\Big(\int_0^p |f(v+h)-f(v)|^r dv\Big)^{1/r}$$

implies that $C_0(X)_p L_\omega(X)_p$ is the space of the functions for which the L^r-modulus of continuity $\tilde{\omega}_r(u)$ satisfies the condition $\tilde{\omega}_r(u)=O(\omega(u))$, whence it is identical with the class $L(\omega,r)$ of functions occuring in the theory of Fourier series.

B′. Let $M(u)$ be a monotone, convex and continuous function in $\langle 0,\infty\rangle$, vanishing only for $u=0$ and such that

$$\lim_{u\to 0}\frac{M(u)}{u}=0,\qquad \lim_{u\to\infty}\frac{M(u)}{u}=\infty,\qquad M(2u)=O(M(u)).$$

We choose as X the space L^M corresponding to the function $M(u)$, *i. e.* the space of measurable functions of period p, for which

$$\int_0^p M(|f(v)|)\,dv$$

is finite[4]). The space $C_0(X)_p$ will be defined as in B with $f(v)\in L^M$.

[4]) See W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Acad. Polonaise des Sciences (1932), p. 93-107; W. Orlicz, *Über Räume L^M*, ibidem (1936), p. 93-107.

Now we shall supply an application of theorem 4 to the theory of orthogonal systems. Let $\Phi\{\varphi_i(t)\}$ be an orthonormal system in $\langle a,b\rangle$ and let $\boldsymbol{F}$ be a class of integrable functions. The sequence $\{\lambda_i\}$ is called the *multiplicator of class* $(\boldsymbol{F},\boldsymbol{F})$ if, in case of

$$x(t)\sim\sum_{i=1}^{\infty} a_i\varphi_i(t) \tag{10}$$

being the development of an arbitrary function $x(t)$ of $\boldsymbol{F}$ the series

$$\sum_{i=1}^{\infty}\lambda_i a_i\varphi_i(t) \tag{11}$$

is also a development of a function $y(t)\in F$. Let us denote by C, L_1, L_ω respectively the spaces $C(X), L_1(X), L_\omega(X)$, where X is the space of real numbers.

THEOREM 5. *Let the system $\Phi\{\varphi_i(t)\}$ be complete in C. If $\{\lambda_i\}$ is simultaneously a multiplicator of the class (C,C) and (L_1,L_1), it is also a multiplicator of the class (L_ω,L_ω) with arbitrary $\omega=\omega(u)$,* $\gamma=\sup\limits_{0<u\leqslant 1} u/\omega(u)$ being supposed to be finite.

Proof. Let $U(x)$ be an operation in C, transforming the function $x(t)$ into the function $y(t)$ whose development is (11). The completeness of the system Φ and the closed-graph theorem of Banach imply that $U(x)$ is a linear operation from C to C. It suffices to apply Theorem 4.

(Reçu par la Rédaction le 18. 12. 1953)