En comparant (8) et (9), il vient, pour $q=k-1$,

$$\frac{A_{2k-1}}{B_{2k-1}}=\frac{1}{2}\left(\frac{A_{k-1}}{B_{k-1}}+c\frac{B_{k-1}}{A_{k-1}}\right).$$

Cela prouve que le $(k-1)^{\text{ième}}$ réduit de $\sqrt{c}$, substitué au lieu de x_n dans la formule (1) donne pour x_{n+1} le $(2k-1)^{\text{ième}}$ réduit de $\sqrt{c}$. D'après la proposition (I), les itérations postérieures donneront toujours un réduit de $\sqrt{c}$.

6. Considérons encore le développement du nombre $\sqrt{13}$:

$$3(1,1,1,1,6);$$

la période est ici impaire. Les réduits initiaux sont

$$3,\ 4,\ \frac{7}{2},\ \frac{11}{3},\ \frac{18}{5},\ \frac{119}{33},\ \frac{137}{38},\ \frac{256}{71},\ \frac{393}{109},\ \frac{649}{180},\ \frac{4287}{1189},\ \frac{4936}{1369},$$
$$\frac{9223}{2558},\ \frac{14159}{3927},\ldots$$

En prenant pour x_n successivement les valeurs des cinq réduits initiaux, on trouve pour x_{n+1}, moyennant (1),

$$\frac{11}{3},\ \frac{35}{8},\ \frac{101}{28},\ \frac{119}{33},\ \frac{649}{180},\ \frac{14159}{3927}.$$

D'après la proposition (I), le quatrième des réduits, c'est-à-dire le nombre 18/5 reproduit, moyennant (1), un réduit de $\sqrt{13}$. Or, on voit que les réduits antérieurs 3 et 11/3, ainsi que 119/33, reproduisent encore des réduits. Il serait peut-être intéressant de trouver des règles générales qui permettraient d'indiquer tous les réduits jouissant de telles propriétés.

On a new method of solving homogeneous systems of linear difference equations with constant coefficients

by J. Čermák (Brno)

1. In 1889 appeared a paper of E. Weyr, *O theorii forem bilinearných* [1]. It contains an original theory of matrices and its applications in different branches of mathematics.

Using Weyr's theory I shall present here a new method of solving homogeneous systems of linear difference equations with constant coefficients. I shall show that general solution of the above mentioned system is given and — what is especially remarkable — can be expressed by explicit formulas, if a reduced normal system of vectors, relative to the matrix of coefficients of the system is known. The reduced normal system of vectors is a slightly modified concept of the normal system of vectors of Weyr's theory.

I want to point out here that complete solutions of the above mentioned system were given in course of time by many mathematicians. Particularly neat were those of L. Stickelberger [2], O. Perron [3] and J. Kaucký [4]. O. Perron also obtained explicit formulas for solution; of course his method is quite different from mine.

The idea of the method presented here is due to O. Borůvka, who has derived in a similar way the general solution (hitherto not published) of homogeneous systems of linear differential equations with constant coefficients. It may be mentioned that independently of him, in a somewhat different manner but also starting from a theorem of Weyr's theory, M. Kumorovitz has given the general solution of the same system of differential equations [5].

It was J. Kaucký who called my attention to the difference equations.

2. Let A be a square matrix[1]) with elements in the field of complex numbers and a one of its characteristic roots. Then zero is the characte-

[1]) Square matrices of order n will be denoted by capital Latin characters, vectors in n-dimensional vector space by small Gothic or bold face type Latin characters with possible superscripts and identified with one-column matrices. E is unit-matrix. The determinant of A will be written $|A|$.

ristic root of the matrix $A-aE$. If a is of multiplicity α, let

$$\alpha_1,\quad \alpha_1+\alpha_2,\quad \alpha_1+\alpha_2+\alpha_3,\quad \ldots,\quad \alpha_1+\alpha_2+\ldots+\alpha_r=\alpha,\ldots$$

be the nullities[2]) of the succesive powers

$$A-aE,\quad (A-aE)^2,\quad \ldots,\quad (A-aE)^r,\quad \ldots, \tag{1}$$

where r is the first integer giving the maximum nullity α[3]).

Numbers $\alpha_1,\alpha_2,\ldots,\alpha_r$ are positive integers and are called characteristic numbers of A relative to the characteristic root a. It can be shown that they satisfy the inequalities

$$\alpha_1\geqslant\alpha_2\geqslant\alpha_3\geqslant\ldots\geqslant\alpha_r.$$

Further it can be shown that there exists a system of vectors relative to the characteristic root zero of $A-aE$ consisting of $\alpha_1+\alpha_2+\ldots+\alpha_r=\alpha$ (linearly) independent vectors,

$$\begin{array}{l}
\mathfrak{a}^{1,1},\ldots,\mathfrak{a}^{1,\alpha_r},\\
\mathfrak{a}^{2,1},\ldots,\mathfrak{a}^{2,\alpha_r},\mathfrak{a}^{2,\alpha_r+1},\ldots,\mathfrak{a}^{2,\alpha_{r-1}},\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\mathfrak{a}^{r,1},\ldots,\mathfrak{a}^{r,\alpha_r},\mathfrak{a}^{r,\alpha_r+1},\ldots,\mathfrak{a}^{r,\alpha_{r-1}},\mathfrak{a}^{r,\alpha_{r-1}+1},\ldots,\mathfrak{a}^{r,\alpha_1}),
\end{array} \tag{2}$$

which is called *the normal system of vectors relative to the characteristic root a of A* and has the following noteworthy quality: $A-aE$ transforms every vector (2), except those standing in the last row, into the vector whose symbol is just below it, and vectors in the last row into zero vectors.

Thus the following formulas are valid:

$$\left.\begin{array}{ll}
(A-aE)\,\mathfrak{a}^{\mu,\nu}=\mathfrak{a}^{\mu+1,\nu} & \text{for}\quad 1\leqslant\mu\leqslant r-1\\
(A-aE)\,\mathfrak{a}^{\mu,\nu}=0 & \text{for}\quad \mu=r
\end{array}\right\}\quad (\nu=1,2,\ldots,\alpha_{r-\mu+1}). \tag{3}$$

---

[2]) If the rank of A is h, then $n-h$ is called the *nullity of A*.

[3]) It can be shown that in the row of matrices (1) exists the first matrix such that its nullity is α, and all those following have also the nullity α.

[4]) Naturally, if, for instance, $\alpha_r=\alpha_{r-1}$, the symbols $\mathfrak{a}^{\mu,\alpha_r+1},\ldots$ for $\mu=2,3,\ldots,r$ cannot be read in the above schema.

The concept of the normal system of vectors relative to a characteristic root of a matrix is well known in the modern algebra but it doesn't figure usually under the name "normal system". See for instance R. Courant, *Methoden der mathematischen Physik I*, Berlin 1931, p. 37 or И. М. Гельфанд, *Лекции по линейной алгебре*, Москва-Ленинград 1951, p. 157-160.

We can construct such a normal system of vectors in the following way:

For vectors in the first row

$$\mathfrak{a}^{1,1},\ldots,\mathfrak{a}^{1,\alpha_r} \tag{4}$$

we take α_r arbitrary independent vectors which $(A-aE)^r$ transforms into zero vectors and $(A-aE)^{r-1}$ into independent vectors. Vectors in the second row

$$\mathfrak{a}^{2,1},\ldots,\mathfrak{a}^{2,\alpha_r},\mathfrak{a}^{2,\alpha_r+1},\ldots,\mathfrak{a}^{2,\alpha_{r-1}} \tag{5}$$

we obtain as follows: We transform vectors (4) by $(A-aE)$ and thus get the first α_r vectors (5); we see that $(A-aE)^{r-1}$ transforms these vectors into zero vectors. If $\alpha_r<\alpha_{r-1}$, let us choose further arbitrary vectors, $\alpha_{r-1}-\alpha_r$ in number, in such a way that all vectors (5) are independent and that $(A-aE)^{r-1}$ transforms them into zero vectors and $(A-aE)^{r-2}$ into independent vectors. Vectors in the third row

$$\mathfrak{a}^{3,1},\ldots,\mathfrak{a}^{3,\alpha_r},\mathfrak{a}^{3,\alpha_r+1},\ldots,\mathfrak{a}^{3,\alpha_{r-1}},\mathfrak{a}^{3,\alpha_{r-1}+1},\ldots,\mathfrak{a}^{3,\alpha_{r-2}} \tag{6}$$

we obtain as follows: We transform vectors (5) by $(A-aE)$ and thus get the first α_{r-1} vectors (6); we see that $(A-aE)^{r-2}$ transforms these vectors into zero vectors. If $\alpha_{r-1}<\alpha_{r-2}$, let us choose further arbitrary vectors, $\alpha_{r-2}-\alpha_{r-1}$ in number, in such a way that all vectors (6) are independent and that $(A-aE)^{r-2}$ transforms them into zero vectors and $(A-aE)^{r-3}$ into independent vectors. And in this way we go on. Weyr's theory guarantees that competent calculations shall be carried out.

Let $a,b,\ldots,f$ denote all distinct characteristic roots of A and $\alpha,\beta,\ldots,\varphi$ their multiplicities. Then to every characteristic root there corresponds a normal system of vectors and these normal systems contain altogether $\alpha+\beta+\ldots+\varphi=n$ vectors. The set of these n vectors is called *the normal system of vectors relative to the matrix A*, and it can be shown that all these vectors are independent.

The following remark is important for our purpose:

Remark I. If A is real, *i. e.* elements of A are real numbers, then its characteristic roots are either real or occur in conjugate pairs. Then it can easily be shown that to conjugate pairs of complex roots correspond normal systems consisting of conjugate pairs of vectors.

Suppose now that the characteristic root a of A of multiplicity α is $\neq 0$, and form the matrix

$$\frac{1}{a}\,(A-aE). \tag{7}$$

The matrix (7) has evidently α-multiple characteristic root zero, and it is easy to see that Weyr's characteristic numbers relative to this root are the same as the characteristic numbers relative to the characteristic root a of A and that the normal system of vectors relative to this root is obtained from the normal system (2), for instance, in the following way:

For vectors in the first row we directly choose vectors (4), for vectors in the second row we take vectors (5) multiplied by the scalar $1/a$; generally, we then take for vectors in the k-th row the vectors from the k-th row of the system (2) multiplied by the scalar $(1/a)^{k-1}$.

The normal system of vectors relative to the characteristic root zero of the matrix (7) will be called the *reduced normal system of vectors relative to the characteristic root a of A* and denoted by

$$\begin{array}{l} \boldsymbol{a}^{1,1},\ldots,\boldsymbol{a}^{1,\alpha_r}, \\ \boldsymbol{a}^{2,1},\ldots,\boldsymbol{a}^{2,\alpha_r},\boldsymbol{a}^{2,\alpha_r+1},\ldots,\boldsymbol{a}^{2,\alpha_{r-1}}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \boldsymbol{a}^{r,1},\ldots,\boldsymbol{a}^{r,\alpha_r},\boldsymbol{a}^{r,\alpha_r+1},\ldots,\boldsymbol{a}^{r,\alpha_{r-1}},\boldsymbol{a}^{r,\alpha_{r-1}+1},\ldots,\boldsymbol{a}^{r,\alpha_1}. \end{array} \tag{8}$$

There are valid formulas similar to formulas (3):

$$\begin{array}{ll} \frac{1}{a}(A-aE)\,\boldsymbol{a}^{\mu,\nu}=\boldsymbol{a}^{\mu+1,\nu} & \text{for}\quad 1\leqslant\mu\leqslant r-1, \\ \frac{1}{a}(A-aE)\,\boldsymbol{a}^{\mu,\nu}=0 & \text{for}\quad \mu=r, \end{array} \qquad (\nu=1,2,\ldots,\alpha_{r-\mu+1}). \tag{9}$$

Suppose now that all distinct characteristic roots of A, $a,b,\ldots,f$ of multiplicities $\alpha,\beta,\ldots,\varphi$ are different from zero. Then to every characteristic root there corresponds a reduced normal system of vectors and these systems contain altogether $\alpha+\beta+\ldots+\varphi=n$ vectors. The set of these n vectors will be called *the reduced normal system of vectors relative to A*. Since the reduced normal system relative to A consists either of the vectors of the normal system relative to A or of their scalar multiples, it follows from the independence of the vectors of the normal system that the vectors of the reduced normal system are likewise independent.

From remark I easily follows another important

Remark II. If A is real and has a conjugate pair of complex roots, then vectors of corresponding reduced normal systems occur likewise in conjugate pairs.

3. Consider now the system of linear homogeneous difference equations in the normal form

$$u_i(x+1)=\sum_{j=1}^{n} a_{ij}u_j(x) \qquad (i=1,2,\ldots,n), \tag{10}$$

where a_{ij} are constants, x independent variable.

A set of n particular solution

$$u_{1k}(x),u_{2k}(x),\ldots,u_{nk}(x), \qquad (k=1,2,\ldots,n), \tag{11}$$

will be identified with vectors $\boldsymbol{u}^1(x),\boldsymbol{u}^2(x),\ldots,\boldsymbol{u}^n(x)$ and denoted by the matrix $U(x)$, so that the columns of $U(x)$ are precisely those particular solutions (11).

In matrix notation the system (10) is expressed by

$$\boldsymbol{u}(x+1)=A\boldsymbol{u}(x),$$

where A is the matrix of coefficients a_{ij}.

Let us observe that we can suppose, without loss of generality, that $|A|\neq 0$ and followingly all characteristic roots of A are $\neq 0$ [5]).

From the theory of systems of the type (10) it is then well known that the solutions make up an n-dimensional vector space. The base of this space is called *the fundamental system of solutions* which is a set of n solutions $\boldsymbol{u}^1(x),\boldsymbol{u}^2(x),\ldots,\boldsymbol{u}^n(x)$ such that $|U(x)|\neq 0$ for all x. The general solution, and hence every solution, is then a linear combination of the solutions of the fundamental system, whose coefficients are periodic functions with the period 1.

Thus, in order to find all solutions of (10), it is sufficient to find a base, *i. e.* a fundamental system of solutions.

Now (10) can be written in the form

$$\Delta\boldsymbol{u}(x)=(A-E)\,\boldsymbol{u}(x), \qquad \Delta\boldsymbol{u}(x)=\boldsymbol{u}(x+1)-\boldsymbol{u}(x). \tag{10a}$$

Let us set

$$\boldsymbol{u}(x)=\lambda^x\boldsymbol{y}(x) \tag{12}$$

and seek to determine the constant λ and the vector $\boldsymbol{y}(x)$ (whose components are suitable functions of the independent variable x) so that this will be a solution of (10).

Substituting (12) in (10a) and dividing by λ^{x+1} we obtain

$$\Delta\boldsymbol{y}(x)=\frac{1}{\lambda}(A-\lambda E)\,\boldsymbol{y}(x). \tag{13}$$

---

[5]) See P. Funk [6].

Thus if the vector (12) is a solution of (10), then $\boldsymbol{y}(x)$ is a solution of (13) which is also a system of linear homogeneous difference equations with constant coefficients.

I shall show that the fundamental system of solutions of (13) is determined and can be expressed by explicit formulas when the characteristic roots and a reduced normal system of vectors relative to A are known. In other words, to solve system (13), and hence also (10), it is sufficient to find the characteristic roots of A and a certain reduced normal system of vectors relative to A.

First of all we shall prove the following

LEMMA I. *Let a be an arbitrary characteristic root of A of multiplicity α and* (8) *a certain reduced normal system relative to it. Then to the root a correspond α solutions of* (10) *of the form*

$$\boldsymbol{u}^{\mu,\nu}=a^x\Big\{\boldsymbol{a}^{\mu,\nu}+\frac{x}{1!}\boldsymbol{a}^{\mu+1,\nu}+\frac{x(x-1)}{2!}\boldsymbol{a}^{\mu+2,\nu}+\dots \tag{14}$$
$$\dots+\frac{x(x-1)\dots\big(x-(r-\mu-1)\big)}{(r-\mu)!}\boldsymbol{a}^{r,\nu}\Big\}$$

for $\mu=1,2,\dots,r-1,\ \nu=1,2,\dots,\alpha_{r-\mu+1}$, *and*

$$\boldsymbol{u}^{\mu,\nu}=a^x\boldsymbol{a}^{\mu,\nu}\quad \textit{for}\quad u=r,\quad \nu=1,2,\dots,\alpha_1.$$

Proof. Let us coordinate to every vector $\boldsymbol{a}^{\mu,\nu}$ the vector

$$\boldsymbol{y}^{\mu,\nu}=\boldsymbol{a}^{\mu,\nu}+\frac{x}{1!}\boldsymbol{a}^{\mu+1,\nu}+\frac{x(x-1)}{2!}\boldsymbol{a}^{\mu+2,\nu}+\dots+\frac{x(x-1)\dots\big(x-(r-\mu-1)\big)}{(r-\mu)!}\boldsymbol{a}^{r,\nu}$$

for $\mu=1,2,\dots,r-1$, $\nu=1,2,\dots,\alpha_{r-\mu+1}$, and the vector

$$\boldsymbol{y}^{\mu,\nu}=\boldsymbol{a}^{\mu,\nu}\quad\text{for}\quad \mu=r,\quad \nu=1,2,\dots,\alpha_1.$$

By (9) we have

$$\Delta\boldsymbol{y}^{\mu,\nu}=\boldsymbol{a}^{\mu+1,\nu}+\frac{x}{1!}\boldsymbol{a}^{\mu+2,\nu}+\frac{x(x-1)}{2!}\boldsymbol{a}^{\mu+3,\nu}+\dots+\frac{x(x-1)\dots\big(x-(r-\mu-2)\big)}{(r-\mu-1)!}\boldsymbol{a}^{r,\nu}$$
$$=\frac{1}{a}(A-aE)\Big\{\boldsymbol{a}^{\mu,\nu}+\frac{x}{1!}\boldsymbol{a}^{\mu+1,\nu}+\frac{x(x-1)}{2!}\boldsymbol{a}^{\mu+2,\nu}+\dots$$
$$\dots+\frac{x(x-1)\dots\big(x-(r-\mu-2)\big)}{(r-\mu-1)!}\boldsymbol{a}^{r-1,\nu}+\frac{x(x-1)\dots\big(x-(r-\mu-1)\big)}{(r-\mu)!}\boldsymbol{a}^{r,\nu}\Big\}$$
$$=\frac{1}{a}(A-aE)\boldsymbol{y}^{\mu,\nu}$$

for

$$\mu=1,2,\dots,r-1,\quad \nu=1,2,\dots,\alpha_{r-\mu+1},$$

and

$$\Delta\boldsymbol{y}^{\mu,\nu}=0=\frac{1}{a}(A-aE)\boldsymbol{a}^{\mu,\nu}=\frac{1}{a}(A-aE)\boldsymbol{y}^{\mu,\nu}$$

for

$$\mu=r,\quad \nu=1,2,\dots,\alpha_1.$$

Thus every vector $\boldsymbol{y}^{\mu,\nu}$ is a solution of (13). Hence, in view of (12), all vectors (14) are solutions of (10).

Thus to every characteristic root of A there corresponds a set of solutions, their number being always equal to the multiplicity of the respective root. With the help of this lemma we easily obtain the above mentioned result which we shall express by

THEOREM I. *Let all distinct characteristic roots of A be $a,b,\dots,f$ of multiplicities $\alpha,\beta,\dots,\varphi$. All solutions of the form* (14) *corresponding to these characteristic roots make up the fundamental system of solutions of* (10).

Proof. By lemma I, to every characteristic root a $(b,\dots,f)$ correspond α $(\beta,\dots,\varphi)$ solutions of (10). Thus we get altogether $\alpha+\beta+\dots+\varphi=n$ solutions, and it only remains to be shown that these n solutions form the fundamental system. But it is easy to see that the determinant of the matrix of these solutions is equal to the product of the function $\exp(\alpha\log a+\beta\log b+\dots+\varphi\log f)$ and the determinant of the reduced normal system of vectors relative to A, and hence $\neq 0$ for all x since vectors of the reduced normal system are independent.

Thus (10) is completely solved.

If the independent variable in (10) is real and A is real, the question arises whether a real base of solutions can be constructed.

If all characteristic roots of A are real, then in order to find the fundamental system of solutions we can make use of real reduced normal systems of vectors, so that we evidently obtain a real base.

If some characteristic roots of A are complex, they occur in conjugate pairs $a=a_1+ia_2, \bar{a}=a_1-ia_2$. Following remark II such reduced normal system of vectors can be found that to every vector $\boldsymbol{a}^{\mu,\nu}$ of a reduced normal system relative to a corresponds a conjugate vector $\bar{\boldsymbol{a}}^{\mu,\nu}$ of the reduced normal system relative to $\bar{a}$. Thus we see that for every solution $\boldsymbol{u}^{\mu,\nu}$ corresponding, in view of the formulas (14), to the characteristic root a there exists a conjugate solution $\bar{\boldsymbol{u}}^{\mu,\nu}$ corresponding, in view of the same formulas, to $\bar{a}$. Vectors

$$\boldsymbol{u}^{\mu\nu 1}=\frac{1}{2}(\boldsymbol{u}^{\mu\nu}+\bar{\boldsymbol{u}}^{\mu\nu}),\qquad \boldsymbol{u}^{\mu\nu 2}=\frac{1}{2i}(\boldsymbol{u}^{\tau\nu}-\bar{\boldsymbol{u}}^{\mu\nu}) \tag{15}$$

are real, independent and are of course solutions of (10). They are given by the formulas

$$\boldsymbol{u}^{\mu\nu 1} = \varrho^x (\boldsymbol{q}^{\mu\nu 1} \cos\psi x - \boldsymbol{q}^{\mu\nu 2} \sin\psi x),$$

$$\boldsymbol{u}^{\mu\nu 2} = \varrho^x (\boldsymbol{q}^{\mu\nu 1} \cos\psi x + \boldsymbol{q}^{\mu\nu 1} \sin\psi x),$$

where $\varrho = \mathrm{mod}(a_1 + ia_2)$, $\psi = \arg(a_1 + ia_2)$ and $\boldsymbol{q}^{\mu\nu 1}, \boldsymbol{q}^{\mu\nu 2}$ are vectors whose components are real and imaginary parts of components of the vector $\boldsymbol{g}^{\mu\nu}$ [6]).

This result we can express by the following:

Theorem II. *If the independent variable in* (10) *is real and the coefficients a_{ij} are real numbers, then a real fundamental system of solutions can be obtained, and solutions of this fundamental system are of the form* (14) *or consist of pairs* (15).

References

[1] E. Weyr, *O theorii forem bilinearných*, Spisy poctěné jubilejní cenou král. české spol. nauk, Praha 1889; reprinted under the title *Zur Theorie der bilinearen Formen*, Mh. Math. Physik 1 (1890), p. 163-236.

[2] L. Stickelberger, *Zur Theorie der linearen Differentialgleichungen*, Leipzig 1881.

[3] O. Perron, *Zur Theorie der Matrizes*, Math. Annalen 64 (1907), p. 248.

[4] J. Kaucký, *Quelques remarques sur les chaînes de Markoff*, Publ. Fac. sc. Univ. Masaryk 131 (1930).

[5] M. Kumorovitz, *Une solution du système linéaire homogène d'équations différentielles du premier ordre à coefficients constants*, Annales Soc. Pol. Math. 23 (1950), p. 190-200.

[6] P. Funk, *Die linearen Differenzengleichungen*, Berlin 1920, p. 22.

Sur certaines fractions continues finies

par J. Mikusiński (Wrocław)

1. Développons chacune des $n-1$ fractions

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$$

en une fraction continue, et désignons par $K(n)$ le plus grand nombre de termes dans les développements obtenus.

Par exemple, on a

$$\frac{1}{7} = (7), \quad \frac{2}{7} = (3,2), \quad \frac{3}{7} = (2,3), \quad \frac{4}{7} = (1,1,3), \quad \frac{5}{7} = (1,2,2), \quad \frac{6}{7} = (1,6).$$

Les plus longues des fractions continues précédentes contiennent 3 termes, on a donc

$$K(7) = 3.$$

Le procédé ci-dessus détermine une fonction[1]) qui fait correspondre un nombre naturel $K(n)$ à tout entier $n \geqslant 2$. Le but de cette note est de démontrer les inégalités

$$\frac{1}{2a} < \frac{K(n)}{\log n} < \frac{1}{a} \qquad (n = 2, 3, \ldots), \tag{1}$$

où $a = \log \dfrac{1+\sqrt{5}}{2}$.

2. Supposons que

$$(a_1, \ldots, a_k) \qquad (k = K(n), \quad a_k \geqslant 2) \tag{2}$$

soit le plus long des développements de $1/n, \ldots, (n-1)/n$ en fractions continues. Considérons encore la fraction continue

$$(1, \ldots, 1, 2) \tag{3}$$

[1]) M. W. Urbański m'a fait remarquer cette fonction.