donc, si n_k est suffisamment grand, on a, d'après (7),

$$\frac{2\Theta}{1+\Theta}\left(\frac{1-\Theta}{3-\Theta}\right)^{\varepsilon}\sqrt{\frac{1+\Theta}{2}}<\Theta$$

et enfin, d'après (10),

$$\frac{2\Theta}{1+\Theta}\sqrt{\frac{1+\Theta}{2}}\sqrt{\frac{1+\Theta}{2}}<\Theta,$$

ce qui donne l'inégalité fausse $\Theta<\Theta$.

Par suite, l'inégalité (2) ne peut pas avoir lieu, ce qui prouve que la thèse $M(z_0,r)\geqslant 1$ est vraie.

Remarque. *À chaque nombre $R\geqslant 1$ on peut faire correspondre une suite* (1) *telle que la quantité $M(z_0,r)$ soit égale à R.*

En effet, soient $\eta_0^{(n)},\eta_1^{(n)},\ldots,\eta_n^{(n)}$ les sommets du polygone régulier inscrit dans le cercle $|z|=1$. Alors

$$\Delta_n=\min_j\prod_{\substack{k=0\\k\neq j}}^{n}|\eta_j^{(n)}-\eta_k^{(n)}|=n+1\qquad\text{donc}\qquad\sqrt[n]{\Delta_n}\to 1.$$

D'autre part, soit $\{\delta_n\}$ une suite de nombres positifs tels qu'on ait

$$\delta_n\leqslant\Delta_n\qquad\text{et}\qquad\sqrt[n]{\delta_n}\to 1/R.$$

En changeant convenablement la position d'un seul des points $\eta_0^{(n)},\eta_1^{(n)},\eta_2^{(n)},\ldots,\eta_n^{(n)}$ sur la circonférence $|z|=1$, on peut être conduit au cas où

$$\delta_n=\min_j\prod_{\substack{k=0\\k\neq j}}^{n}|\eta_j^{(n)}-\eta_k^{(n)}|.$$

Posons $z_0=0$ et $\zeta_k^{(n)}=r_n\eta_k^{(n)}$, où $\{r_n\}$ est une suite de nombres positifs tendant vers zéro. Lorsque $r_n\leqslant r$, on a

$$M_n(z_0,r)=\max_j\left[\prod_{\substack{k=0\\k\neq j}}^{n}\frac{|\zeta_k^{(n)}|}{|\zeta_j^{(n)}-\zeta_k^{(n)}|}\right]=\frac{r_n^n}{\min\limits_j\prod\limits_{\substack{k=0\\k\neq j}}^{n}|\zeta_j^{(n)}-\zeta_k^{(n)}|}=\frac{1}{\delta_n},$$

donc

$$M(z_0,r)=\overline{\lim}\sqrt[n]{M_n(z_0,r)}=1/\sqrt[n]{\delta_n}=R.$$

Non-local problems in the calculus of variations (I)

by A. Krzywicki, J. Rzewuski, J. Zamorski and A. Zięba (Wrocław)

Introduction. Non-local variational problems occur in the developments of modern theoretical physics, especially in the theory of elementary particles. On the other hand they are intimately connected with the theory of integro-differential equations. In view of the above applications and of the fact that they constitute a new domain of the calculus of variations, we thought it worth while to carry out a systematic investigation of their mathematical structure.

A variational problem is called *non-local* if the unknown functions under the integral sign are taken at several different points of the domain considered. The integrals are multiple, each taken over the same domain of integration. Integrals of various multiplicity may occur.

In the present paper we shall confine ourselves to the study of non-local problems in which the unknown functions $q_1(t),q_2(t),\ldots,q_n(t)$ are functions of one independent variable t. We denote by $\dot q_i,\ddot q_i,\ldots$ the first, the second, etc. derivatives of these functions with respect to t. The fixed domain of integration is the closed interval $a\leqslant t\leqslant b$. We shall further restrict our investigations to problems in which higher order derivatives than the first do not occur under the integral sign and in which the multiplicity of integrals is at most two.

With all these restrictions we may finally write out the general form of the functional W to be investigated

$$W=\int_a^b L^{(1)}[t,q_i(t),\dot q_i(t)]\,dt+\lambda\int_a^b\!\!\int L^{(2)}[t,t',q_i(t),q_i(t'),\dot q_i(t),\dot q_i(t')]\,dt\,dt'. \tag{0.1}$$

We assume all the conventional conditions for $q_i(t),L^{(1)}$ and $L^{(2)}$ that are necessary for the existence of the integrals in (0.1); they will be listed in detail in due course. In the following investigations the parameter λ will play a similar role to that of the analogous parameter in the integral equations of Fredholm's type.

We must emphasize at this place that the restriction to first order derivatives and twofold integrals is not essential since problems with higher order derivatives of q_i and integrals of higher order multiplicity may be treated by trivial generalizations of the methods developed for (0.1).

On the contrary, the restriction to functions of one independent variable is essential and we intend in future to investigate also problems containing functions of more independent variables. The variational problem corresponding to the functional (0.1) consists in the investigation of conditions for an extremum of this functional.

In section 1 the necessary conditions for the existence of an extremum are derived. The investigation of the second variation and the construction of the non-local analogue of the Weierstrass function are postponed until a later paper[1]) since they demand certain theoretical means to be developed first.

The necessary conditions have the form of integro-differential equations and in section 2 the existence of their solution is investigated. These investigations bring to light an interesting equivalence between integro-differential equations and pure differential equations of the same order (of derivatives).

Section 3 is devoted to the explicit solution of the integro-differential equation in the linear case (linear in the q_i), and to the explicit construction of the equivalent differential equation.

In section 4 the problem of reduction of integro-differential equations to differential equations is illustrated on a simple example of a linear integro-differential equation with constant coefficients.

1. The first variation. Consider the following variation

$$\delta q_i(t) = \delta_0 q_i(t) + \dot{q}_i(t)\,\delta t \tag{1.1}$$

consisting of the variation $\delta_0 q_i(t)$ for constant t and of the variation $\dot{q}_i(t)\,\delta t$ derived from the variation of the independent variable t. (We assume that the functions $L^{(1)}, L^{(2)}, q_i, \delta_0 q_i, \delta t$ possess continuous second order derivatives with respect to each of the arguments.) The corresponding variation of the functional (0.1) is

$$\begin{aligned}\delta W = \int_a^b \Big\{ \sum_i \frac{\partial L^{(1)}}{\partial q_i(t)}\delta_0 q_i(t) + \sum_i \frac{\partial L^{(1)}}{\partial \dot{q}_i(t)}\delta_0 \dot{q}_i(t) + \frac{d(L^{(1)}\delta t)}{dt}\Big\} dt + \\ + \lambda \iint_a^b \Big\{ \sum_i \frac{\partial L^{(2)}}{\partial q_i(t)}\delta_0 q_i(t) + \sum_i \frac{\partial L^{(2)}}{\partial q_i(t')}\delta_0 q_i(t') + \sum_i \frac{\partial L^{(2)}}{\partial \dot{q}_i(t)}\delta_0 \dot{q}_i(t) + \\ + \sum_i \frac{\partial L^{(2)}}{\partial \dot{q}_i(t')}\delta_0 \dot{q}_i(t') + \frac{d(L^{(2)}\delta t)}{dt} + \frac{d(L^{(2)}\delta t')}{dt'}\Big\} dt\,dt'.\end{aligned} \tag{1.2}$$

[1]) A. Krzywicki, J. Rzewuski, J. Zamorski and A. Zięba, *Non-local problems in the calculus of variations (II)*, to appear.

Without loss of generality we may assume $L^{(2)}$ to be a symetric function of t and t'

$$L^{(2)}[t,t',q_i(t),q_i(t'),\dot{q}_i(t),\dot{q}_i(t')] = L^{(2)}[t',t,q_i(t'),q_i(t),\dot{q}_i(t'),\dot{q}_i(t)]. \tag{1.3}$$

Indeed, if this is not the case we can always symmetrize the integrand of the double integral in (0.1), making use of the fact that the integration limits for both integrations are the same.

In (1.2) one may interchange the integration variables in those terms of the double integral in which the variations depend on t'. Then, making use of (1.3) and carrying out some conventional integrations by parts, we may write (1.2) in the form

$$\begin{aligned}\delta W = \int_a^b \sum_i \Big\{ \frac{\partial L^{(1)}}{\partial q_i(t)} - \frac{d}{dt}\frac{\partial L^{(1)}}{\partial \dot{q}_i(t)}\Big\}\delta_0 q_i(t)\,dt + \\ + \Big[\sum_i \frac{\partial L^{(1)}}{\partial \dot{q}_i(t)}\delta_0 q_i(t) + L^{(1)}\delta t\Big]\Big|_{t=a}^{t=b} + \\ + 2\lambda \iint_a^b \sum_i \Big\{ \frac{\partial L^{(2)}}{\partial q_i(t)} - \frac{d}{dt}\frac{\partial L^{(2)}}{\partial \dot{q}_i(t)}\Big\}\delta_0 q_i(t)\,dt\,dt' + \\ + 2\lambda \int_a^b \Big[\sum_i \frac{\partial L^{(2)}}{\partial \dot{q}_i(t)}\delta_0 q_i(t) + L^{(2)}\delta t\Big]\Big|_{t=a}^{t=b} dt.\end{aligned} \tag{1.4}$$

We may now interchange in (1.4) the differentiation connected with t and integration over t'. Introducing a functional

$$L = L^{(1)} + 2\lambda \int_a^b L^{(2)}\,dt', \tag{1.5}$$

which may be called the non-local Euler-Lagrange functional, we may finally write (1.4) in the simple form

$$\begin{aligned}\delta W = \int_a^b \sum_i \Big\{ \frac{\partial L}{\partial q_i(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i(t)}\Big\}\delta_0 q_i(t)\,dt + \\ + \Big[\sum_i \frac{\partial L}{\partial \dot{q}_i(t)}\delta_0 q_i(t) + L\,\delta t\Big]\Big|_{t=a}^{t=b}.\end{aligned} \tag{1.6}$$

The second term in (1.6), containing the variations $\delta_0 q_i(t)$ and δt taken at the points a and b, is important for the investigation of the transformation properties of non-local variational problems. We shall discuss this point in a forthcoming paper.

In the following we assume that the variations vanish at a and b

$$\delta q_i(a)=\delta q_i(b)=0, \qquad i=1,2,\ldots,n, \qquad \delta t(a)=\delta t(b)=0. \tag{1.7}$$

With these assumptions the variation of (0.1) becomes

$$\delta W=\int_a^b \sum_i \left\{\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right\}\delta_0 q_i\,dt. \tag{1.8}$$

On account of the theorem of Du Bois-Reymond the necessary (and of course sufficient) condition for vanishing of the first variation (1.8) is

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}=0, \qquad i=1,2,\ldots,n. \tag{1.9}$$

Just as in the conventional local case, we show that (1.9) (or $\delta W=0$) is a necessary condition for an extremum of the functional (0.1). The equations (1.9) have the same form as the usual Euler-Lagrange equations, known from the conventional variational problems. There exists, however, a profound difference due to the fact that L is now a functional (not a function as in the local case) of the form (1.5). A consequence of this fact is that now the equations (1.9) are integro-differential equations, in contrast to the differential equations occuring in the local case.

For further purposes we shall write down the explicit form of these equations using again the functions $L^{(1)}$ and $L^{(2)}$

$$\begin{aligned}\frac{\partial L^{(1)}}{\partial q_i(t)}&-\frac{\partial^2 L^{(1)}}{\partial t\,\partial \dot{q}_i(t)}-\sum_k\frac{\partial^2 L^{(1)}}{\partial q_k(t)\,\partial \dot{q}_i(t)}\dot{q}_k(t)-\sum_k\frac{\partial^2 L^{(1)}}{\partial \dot{q}_k(t)\,\partial \dot{q}_i(t)}\ddot{q}_k(t)+\\&+2\lambda\int_a^b\left\{\frac{\partial L^{(2)}}{\partial q_i(t)}-\frac{\partial^2 L^{(2)}}{\partial t\,\partial \dot{q}_i(t)}-\sum_k\frac{\partial^2 L^{(2)}}{\partial q_k(t)\,\partial \dot{q}_i(t)}\dot{q}_k(t)-\right.\\&\qquad\left.-\sum_k\frac{\partial^2 L^{(2)}}{\partial \dot{q}_k(t)\,\partial \dot{q}_i(t)}\ddot{q}_k(t)\right\}dt'=0.\end{aligned} \tag{1.10}$$

This is an integro-differential equation containing second order derivatives and simple integrals with constant integration limits. It may be noted that second order derivatives occur only outside the integral.

In the next section we shall investigate the problem of existence of solutions of (1.10) *i. e.* existence of extremals of the non-local variational problem.

The investigations of the second variation bring us (as will be demonstrated in part II of this paper) to the problem of discussion of the non-local analogue of Jacobi's conditions which in our case takes the form

of the homogeneous, linear integro-differential equations of the second order. It is, therefore, necessary to investigate first this type of equation (section 3) in some detail.

2. Existence theorems. Before starting any further considerations on non-local variational problems, we must investigate under what conditions there exist extremals of these problems. Therefore we shall prove in this section an existence theorem for the integro-differential equations (1.9) or (1.10).

We shall consider first equations with one unknown function $q(t)$. In a final remark we shall indicate how to generalize the procedure to the case of n unknown functions.

Denoting $q=q(t)$, $q'=q(t')$,

$$\varphi_1(t,q,\dot{q})=-\frac{\partial^2 L^{(1)}}{\partial\dot{q}\,\partial q}\dot{q}-\frac{\partial^2 L^{(1)}}{\partial\dot{q}\,\partial t}+\frac{\partial L^{(1)}}{\partial q}, \qquad \varphi_2(t,q,\dot{q})=\frac{\partial^2 L^{(1)}}{\partial\dot{q}^2},$$

$$\Phi_1(t,t',q,q',\dot{q},\dot{q}')=-2\left[\frac{\partial^2 L^{(2)}}{\partial\dot{q}\,\partial q}\dot{q}+\frac{\partial^2 L^{(2)}}{\partial\dot{q}\,\partial t}-\frac{\partial L^{(2)}}{\partial q}\right],$$

$$\Phi_2(t,t',q,q',\dot{q},\dot{q}')=2\frac{\partial^2 L^{(2)}}{\partial\dot{q}^2},$$

and solving (1.10) with respect to $\ddot{q}$ we get

$$\ddot{q}=\frac{\varphi_1(t,q,\dot{q})+\lambda\int_a^b\Phi_1(t,t',q,q',\dot{q},\dot{q}')\,dt'}{\varphi_2(t,q,\dot{q})+\lambda\int_a^b\Phi_2(t,t',q,q',\dot{q},\dot{q}')\,dt'}. \tag{2.1}$$

Introducing the new functions

$$q_1(t)=q(t), \qquad q_2(t)=\dot{q}(t)$$

we get, instead of (2.1), the system of two equations

$$\dot{q}_1=q_2, \qquad \dot{q}_2=\frac{\varphi_1(t,q_1,q_2)+\lambda\int_a^b\Phi_1(t,t',q_1,q_1',q_2,q_2')\,dt'}{\varphi_2(t,q_1,q_2)+\lambda\int_a^b\Phi_2(t,t',q_1,q_1',q_2,q_2')\,dt'}. \tag{2.2}$$

We assume that for $a\leqslant t\leqslant b$, $-\infty<q_1,q_2,q_1^*,q_2^*<\infty$

$$\begin{aligned}&|\varphi_2(t,q_1,q_2)|\geqslant\sigma>0,\\&|\varphi_i(t,q_1^*,q_2^*)-\varphi_i(t,q_1,q_2)|\leqslant K(|q_1^*-q_1|+|q_2^*-q_2|), \qquad i=1,2,\end{aligned} \tag{2.3}$$

$\varphi_i(t, q_1, q_2)$ — continuous functions of t. With these assumptions the system

$$\dot{\bar q}_1 = \bar q_2, \qquad \dot{\bar q}_2 = \frac{\varphi_1(t, \bar q_1, \bar q_2)}{\varphi_2(t, \bar q_1, \bar q_2)} \tag{2.4}$$

has, in the interval (a, b), a unique and continuous solution $\bar q_1(t), \bar q_2(t)$ passing through the fixed point (t_0, q_{01}, q_{02}).

Now let us consider the full system (2.2). On account of (2.3) there exists a positive number Q such that

$$|\bar q_1(t) - q_{01}| \leqslant Q, \qquad |\bar q_2(t) - q_{02}| \leqslant Q.$$

THEOREM. *With the assumptions* (2.3) *and the additional assumptions*

$$\begin{aligned} &|\Phi_i(t, t', q_1^*, q_1^{*\prime}, q_2^*, q_2^{*\prime}) - \Phi_i(t, t', q_1, q_1', q_2, q_2')| \\ &\qquad \leqslant K(|q_1^* - q_1| + |q_1^{*\prime} - q_1'| + |q_2^* - q_2| + |q_2^{*\prime} - q_2'|), \qquad i = 1, 2 \end{aligned} \tag{2.5}$$

(Φ_i — *continuous functions of* t, t') *to be satisfied in the domain* D:

$$a \leqslant t, t' \leqslant b,$$

$$\begin{aligned} q_{01} - 2Q \leqslant q_1, q_1^*, q_1', q_1^{*\prime} \leqslant q_{01} + 2Q, \\ q_{02} - 2Q \leqslant q_2, q_2^*, q_2', q_2^{*\prime} \leqslant q_{02} + 2Q, \end{aligned} \tag{2.6}$$

there exists, for sufficiently small values of λ, *a unique solution of the system* (2.2) (*and, therefore, of equation* (2.1)) *passing through the point* (t_0, q_{01}, q_{02}).

The proof is based on a simple application of Banach's fixed point theorem.

We note first that from the assumptions it follows that φ_i and Φ_i are bounded in D

$$|\varphi_i| \leqslant M, \qquad |\Phi_i| \leqslant M. \tag{2.7}$$

Let us rewrite (2.2) in the equivalent form

$$\dot q_1 = q_2, \qquad \dot q_2 = \frac{\varphi_1}{\varphi_2} + \lambda \frac{\varphi_2 \int_a^b \Phi_1\, dt' - \varphi_1 \int_a^b \Phi_2\, dt'}{\varphi_2 [\varphi_2 + \lambda \int_a^b \Phi_2\, dt']}. \tag{2.8}$$

Integrating over the interval (t_0, t) we get

$$q_1 = q_{01} + \int_{t_0}^t q_2\, dt, \qquad q_2 = q_{02} + \int_{t_0}^t \frac{\varphi_1}{\varphi_2}\, dt + \lambda \int_{t_0}^t \frac{\varphi \int_a^b \Phi_1\, dt' - \varphi_1 \int_a^b \Phi_2\, dt'}{\varphi_2 [\varphi_2 + \lambda \int_a^b \Phi_2\, dt']}\, dt.$$

The space of pairs of continuous functions (q_1, q_2) which are contained in D with a metric defined by

$$\varrho[(q_1^*, q_2^*), (q_1, q_2)] = \max_{a \leqslant t \leqslant b} (|q_1^* - q_1| + |q_2^* - q_2|)$$

is complete. We define a transformation $U(q_1, q_2) = (q_1^*, q_2^*)$ of the *point* (q_1, q_2) into the *point* (q_1^*, q_2^*) as follows

$$\begin{aligned} q_1^* &= q_{01} + \int_{t_0}^t q_2^*\, dt, \\ q_2^* &= q_{02} + \int_{t_0}^t \frac{\varphi_1(q^*)}{\varphi_2(q^*)}\, dt + \lambda \int_{t_0}^t \frac{\varphi_2(q) \int_a^b \Phi_1(q)\, dt' - \varphi_1(q) \int_a^b \Phi_2(q)\, dt'}{\varphi_2(q) [\varphi_2(q) + \lambda \int_a^b \Phi_2(q)\, dt']}\, dt, \end{aligned} \tag{2.9}$$

where $\varphi_i(q)$ is an abbreviation of $\varphi_i(t, q_1, q_2), \varphi_i(q^*)$ — of $\varphi_i(t, q_1^*, q_2^*)$, $\Phi_i(q)$ — of $\Phi_i(t, t', q_1, q_1', q_2, q_2')$ etc.

In the formulas (2.9) the functions q_1^*, q_2^* appear implicitly. It may be shown that for sufficiently small λ the two conditions are satisfied: (i) the transformated functions q_1^*, q_2^* fall in the domain D if the primary functions q_1, q_2 had this property, (ii) the distance of two *points* (q_1, q_2) and (p_1, p_2) is transformed in the following way:

$$\varrho([U(q_1, q_2), U(p_1, p_2)] \leqslant \Theta \cdot \varrho[(q_1, q_2), (p_1, p_2)], \qquad 0 < \Theta < 1. \tag{2.10}$$

If (i) and (ii) are satisfied, our theorem follows from Banach's theorem mentioned above.

Since $|\Phi_2| \leqslant M$ in D, we have $|\lambda \int_a^b \Phi_2\, dt'| \leqslant |\lambda| M(b-a)$ and for

$$|\lambda| < \frac{\sigma}{2M(b-a)} \tag{2.11}$$

we shall have

$$|\varphi_2 + \lambda \int_a^b \Phi_2\, dt'| > \frac{\sigma}{2}. \tag{2.12}$$

Integrating (2.4) over the interval (t_0, t) and subtracting the result from (2.9) we get

$$q_1^* - \bar q_1 = \int_{t_0}^t (q_2^* - \bar q_2)\, dt,$$

$$q_2^* - \bar q_2 = \int_{t_0}^t \left[\frac{\varphi_1(q^*)}{\varphi_2(q^*)} - \frac{\varphi_1(\bar q)}{\varphi_2(\bar q)}\right] dt + \lambda \int_{t_0}^t \frac{\varphi_2(q) \int_a^b \Phi_1(q)\, dt' - \varphi_1(q) \int_a^b \Phi_2(q)\, dt'}{\varphi_2(q) [\varphi_2(q) + \lambda \int_a^b \Phi_2(q)\, dt']}\, dt.$$

Hence and in virtue of (2.3), (2.7), (2.12) we have further

$$|q_1^* - \bar{q}_1| \leqslant \int_{t_0}^{t} |\bar{q}_2^* - \bar{q}_2|\, dt,$$

$$|q_2^* - \bar{q}_2| \leqslant \int_{t_0}^{t} \left| \frac{\varphi_1(q^*)\varphi_2(\bar{q}) - \varphi_2(q^*)\varphi_1(\bar{q})}{\varphi_2(q^*)\varphi_2(\bar{q})} \right| dt + |\lambda| \frac{4M^2(b-a)^2}{\sigma^2}.$$

Making use of the identity

$$\frac{\varphi_1(q^*)\varphi_2(\bar{q}) - \varphi_2(q^*)\varphi_1(\bar{q})}{\varphi_2(q^*)\varphi_2(\bar{q})} = \frac{\varphi_2(\bar{q})[\varphi_1(q^*) - \varphi_1(\bar{q})] + \varphi_1(\bar{q})[\varphi_2(\bar{q}) - \varphi_2(q^*)]}{\varphi_2(q^*)\, \varphi_2(\bar{q})}$$

and of the formulas (2.3), (2.7), we finally get

$$|q_1^* - \bar{q}_1| \leqslant \int_{t_0}^{t} |q_2^* - \bar{q}_2|\, dt,$$

(2.13)

$$|q_2^* - \bar{q}_2| \leqslant \frac{2KM}{\sigma^2} \int_{t_0}^{t} (|q_1^* - \bar{q}_1| + |q_2^* - \bar{q}_2|)\, dt + |\lambda| \frac{4M^2(b-a)^2}{\sigma^2}.$$

In order to simplify the notations we assume that $\lambda > 0$, $t \geqslant t_0$. For further applications we shall need the following lemma:

If a continuous function $f(t)$ satisfies the inequality

$$|f(t)| \leqslant A \int_{t_0}^{t} |f(t)|\, dt + B, \tag{2.14}$$

then

$$|f(t)| \leqslant B e^{A(t-t_0)}. \tag{2.15}$$

Now let us multiply the second equation (2.13) by $\mu > 0$ and add the result to the first equation. When μ satisfies the condition

$$\frac{1}{\mu} = \frac{2KM\sigma^{-2}\mu}{1 + 2KM\sigma^{-2}\mu} \quad \text{or} \quad \mu = \frac{1}{2}\left(1 + (1 + 2\sigma^2 K^{-1} M^{-1})^{1/2}\right), \tag{2.16}$$

we get

$$|q_1^* - \bar{q}_1| + \mu |q_2^* - \bar{q}_2| \leqslant \frac{2KM}{\sigma^2} \mu \int_{t_0}^{t} (|q_1^* - \bar{q}_1| + \mu |q_2^* - \bar{q}_2|)\, dt + \lambda \frac{4M^2(b-a)^2}{\sigma^2} \mu.$$

In virtue of the lemma (2.14), (2.15) and the inequality $|q_1^* - q_1| + \mu |q_2^* - q_2| \geqslant |q_1^* - q_1| + |q_2^* - q_2|$ (since $\mu > 1$) we have

$$|q_1^* - \bar{q}_1| + |q_2^* - \bar{q}_2| \leqslant \lambda \frac{4M^2(b-a)^2\mu}{\sigma^2} \exp \frac{2KM(b-a)\mu}{\sigma^2} = \lambda \bar{M}; \tag{2.17}$$

putting upon λ the condition

$$\lambda \leqslant \frac{Q}{\bar{M}} \tag{2.18}$$

we find that q_1^*, q_2^* fall in the domain D. Thus (i) is proved.

Let us proceed now to the demonstration of (ii). Writing out the formulas (2.9) for (q_1, q_2) and (p_1, p_2) respectively and subtracting, we get

$$q_1^* - p_1^* = \int_{t_0}^{t} (q_2^* - p_2^*)\, dt,$$

$$q_2^* - p_2^* = \int_{t_0}^{t} \left[\frac{\varphi_1(q^*)}{\varphi_2(q^*)} - \frac{\varphi_1(p^*)}{\varphi_2(p^*)}\right] dt + \lambda \int_{t_0}^{t} \left[\frac{\varphi_2(q) \int_a^b \Phi_1(q)\, dt' - \varphi_1(q) \int_a^b \Phi_2(q)\, dt'}{\varphi_2(q)[\varphi_2(q) + \lambda \int_a^b \Phi_2(q)\, dt']} - \frac{\varphi_2(p) \int_a^b \Phi_1(p)\, dt' - \varphi_1(p) \int_a^b \Phi_2(p)\, dt'}{\varphi_2(p)[\varphi_2(p) + \lambda \int_a^b \Phi_2(p)\, dt']} \right] dt.$$

On account of (2.3), (2.5), (2.7), (2.12) the last term in the second equation is smaller than

$$\lambda \frac{8M^3K(b-a)}{\sigma^4} [4 + 3(b-a)\lambda] \max_{(a,b)} (|q_1 - p_1| + |q_2 - p_2|).$$

Hence we get, in the same way as above (cf. (2.13)),

$$|q_1^* - p_1^*| \leqslant \int_{t_0}^{t} |q_2^* - p_2^*|\, dt,$$

$$|q_2^* - p_2^*| \leqslant \frac{2KM}{\sigma^2} \int_{t_0}^{t} (|q_1^* - p_1^*| + |q_2^* - p_2^*|)\, dt + \lambda \frac{8M^3K(b-a)}{\sigma^4} [4 - 3(b-a)\lambda] \max_{(a,b)} [|q_1 - p_1| + |q_2 - p_2|].$$

Multiplying the second equation by μ (cf. (2.16)), subtracting it from the first and repeating the procedure which brought us to (2.17) we finally get

$$\max_{(a,b)} [|q_1^* - p_1^*| + |q_2^* - p_2^*|] \leqslant \lambda \frac{8M^3K(b-a)}{\sigma^4} [4 + 3(b-a)\lambda] \exp \frac{2KM(b-a)\mu}{\sigma^2} \max_{(a,b)} [|q_1 - p_1| + |q_2 - p_2|].$$

Condition (2.10) will be satisfied if λ is smaller than the positive root λ^+ of the equation

$$\frac{8M^3K(b-a)}{\sigma^4}\exp\frac{2KM(b-a)\mu}{\sigma^2}\cdot[4\lambda+3(b-a)\lambda^2]=1.$$

Thus the proof is complete, the condition for λ being ((2.11), (2.18))

$$\lambda<\min\left\{\frac{\sigma}{2M(b-a)},\ \frac{Q\sigma^2}{4M^2(b-a)^2\mu}\exp\left(-\frac{2KM(b-a)\mu}{\sigma^2}\right),\lambda^+\right\}.$$

The above considerations, without essential changes, may be applied to the case of n unknown functions $q_1(t),\ldots,q_n(t)$. To solve the system (1.10) with respect to $\ddot q_1,\ldots,\ddot q_n$ we must assume that the determinant

$$F(\lambda)=|a_{ij}|,\qquad a_{ik}=\frac{\partial^2 L^{(1)}}{\partial\dot q_i\,\partial\dot q_k}+2\lambda\int_a^b\frac{\partial^2 L^{(2)}}{\partial\dot q_i\,\partial\dot q_k}\,dt'$$

satisfies the condition

$$|F(0)|>\sigma\quad\text{for}\quad a\leqslant t\leqslant b,\quad -\infty<q_i,\dot q_i<+\infty,\quad i=1,2,\ldots,n.$$

After reducing the system of n second order equations to a set of $2n$ first order equations with assumptions analogous to (2.3), (2.5) we get the generalization of the above theorem for n unknown functions.

An interesting consequence of the above theorem is the equivalence of integro-differential equations and pure differential equations. Indeed, it has been shown that through every system of n line-elements of a certain domain passes one and only one solution of the system of n integro-differential equations (1.10). There exists, therefore, a system of n pure differential equations which is satisfied by the solutions of (1.10). In regular cases, when the corresponding system of differential equations satisfies the condition of uniqueness, this system is equivalent to (1.10) in the sense that both of them possess the same family of solutions. In further considerations we shall frequently make use of this result.

3. The linear integro-differential equation and the corresponding differential equation. In this section we shall consider the linear integro-differential equation of the general type[2])

$$p_2(t)\ddot q(t)+p_1(t)\dot q(t)+p_0(t)q(t)=f(t)+\lambda\int_a^b K(t,t')q(t')dt'. \tag{3.1}$$

[2]) Similar equations have been considered independently by В. Н. Николенко (*Задача Коши для интегро-дифференциального уравнения типа Фредгольма*, Усп. мат. наук, том VII, 5 (51), 1952, p. 225-228) and В. В. Васильев (*Решение линейных обобщенных интегро-дифференциальных уравнений*, Прикл. мат. и мех., том XV, 5, 1951, p. 609-614). The methods developed in those papers differ essentialy from ours and are less convenient for our purposes.

THEOREM. *Equation (3.1) has a two-parameter set of solutions if λ is not a characteristic value to be determined below and if the following assumptions are satisfied:*

(i) *the functions p_0,p_1,p_2,f are continuous and $p_2\neq 0$ in the interval $\langle a,b\rangle$,*

(ii) *in the square $a\leqslant t,t'\leqslant b$ there exists a bounded first order derivative*

$$\left|\frac{\partial K(t,t')}{\partial t}\right|\leqslant C, \tag{3.2}$$

(iii) *$K(t,t')$ is a continuous function of t'.*

The proof is given by an effective method of constructing the solution of (3.1).

We shall first replace (3.1) by the equivalent pure integral equation. For this purpose we write the kernel $K(t,t')$ as the sum of three functions

$$K(t,t')=K_0(t,t')+K_1(t,t')+K_2(t,t'). \tag{3.3}$$

The last term in (3.1), on account of (3.3) and after partial integrations, takes the form

$$\begin{aligned}\int_a^b K(t,t')q(t')dt' &= \int_a^b\Big[q(t')K_0(t,t')-\dot q(t')\int^{t'}K_1(t,\tau)d\tau+\\ &+\ddot q(t')\iint^{t'}K_2(t,\tau)d\tau d\tau\Big]dt'+\Big[\int^{t'}K_1(t,\tau)d\tau+\int^{t'}K_2(t,\tau)d\tau\Big]q(t')\Big|_{t'=a}^{t'=b}-\\ &-\iint^{t'}K_2(t,\tau)d\tau d\tau\,\dot q(t')\Big|_{t'=a}^{t'=b}\end{aligned} \tag{3.4}$$

where the symbols $\int^{t'}d\tau$ and $\iint^{t'}d\tau d\tau$ denote indefinite integrals. They are uniquely determined as will become clear from the following consideration.

Now we determine the functions $K_i(t,t')$ to satisfy the following equations:

$$\begin{aligned}K_0(t,t')&=N(t,t')p_0(t'),\\ -\int^{t'}K_1(t,\tau)d\tau&=N(t,t')p_1(t'),\\ \iint^{t'}K_2(t,\tau)d\tau d\tau&=N(t,t')p_2(t').\end{aligned} \tag{3.5}$$

The equation (3.3) takes the form

$$\frac{d^2}{dt'^2}[N(t,t')p_2(t')]-\frac{d}{dt'}[N(t,t')p_1(t')]+N(t,t')p_0(t')=K(t,t'). \tag{3.6}$$

This is an equation for the function $N(t,t')$ where t plays the role of a parameter, so that (3.6) is an ordinary second order differential equation. If $p_1=p_2^{\cdot}$, which may be assumed without loss of generality [3]), then equation (3.6) is selfadjoint and, therefore, may be written in the form

$$p_2(t')\frac{d^2}{dt'^2}N(t,t')+p_1(t')\frac{d}{dt'}N(t,t')+p_0(t')N(t,t')=K(t,t'). \tag{3.7}$$

Denoting by $q_1(t)$ and $q_2(t)$ two linearly independent solutions of the equation

$$L[q(t)]\equiv p_2(t)q^{\cdot\cdot}(t)+p_1(t)q^{\cdot}(t)+p_0(t)q(t)=0$$

we may write the partial solution of the non-homogeneous equation (3.7) in the form

$$N(t,t';t_0')=\int\limits_{t_0'}^{t'}\frac{q_1(\tau)q_2(t')-q_2(\tau)q_1(t')}{W(\tau)}K(t,\tau)\,d\tau, \tag{3.8}$$

where $W(\tau)$ denotes Wroński's determinant of q_1,q_2. In the notation for $N(t,t';t_0')$ we have expressed the dependence of N on t_0', since it affects characteristic values of N. Formula (3.4), on account of (3.5), now takes the form

$$\int\limits_a^b K(t,t')q(t')\,dt'=\int\limits_a^b N(t,t';t_0')L[q(t')]\,dt'-$$
$$-\left\{N(t,t';t_0')p_1(t')-\frac{\partial}{\partial t'}[N(t,t';t_0')p_2(t')]\right\}q(t')\Big|_{t'=a}^{t'=b}-N(t,t';t_0')p_2(t')q^{\cdot}(t')\Big|_{t'=a}^{t'=b}.$$

Denoting

$$-\left\{N(t,t';t_0')p_1(t')-\frac{\partial}{\partial t'}[N(t,t';t_0')p_2(t')]\right\}q(t')\Big|_{t'=a}^{t'=b}- \tag{3.9}$$
$$-N(t,t';t_0')p_2(t')q^{\cdot}(t')\Big|_{t'=a}^{t'=b}=\Phi(t),$$

$$f(t)+\lambda\Phi(t)=\psi(t), \tag{3.10}$$

$$L[q(t)]=F(t) \tag{3.11}$$

we may write equation (3.1) in the final form

$$F(t)=\psi(t)+\lambda\int\limits_a^b N(t,t';t_0')F(t')\,dt'. \tag{3.12}$$

[3]) (3.1) may always be transformed into a form in which $p_1=p_2^{\cdot}$ by the multiplication by a suitable function not vanishing in $\langle a,b\rangle$.

The solution of (3.1) is, therefore, equivalent to the successive solution of the equations (3.12) and (3.11). If $\psi(t)\not\equiv 0$ in the interval $\langle a,b\rangle$ and λ is not a characteristic value of the kernel $N(t,t';t_0')$, then (3.12) has a unique solution

$$F(t)=\psi(t)+\lambda\int\limits_a^b R(t,t';t_0')\psi(t')\,dt' \tag{3.13}$$

where $R(t,t';t_0)$ is the resolving kernel of equation (3.12). We note that (3.11) has not the usual form of a non-homogeneous linear differential equation of the second order since $F(t)$ depends linearly on the values of $q(t)$ and $q^{\cdot}(t)$ at the points a and b (cf. (3.9) and (3.10)). In spite of this fact we shall show that (3.11) has a two-parameter set of solutions.

Lemma. *Equation*

$$p_2q^{\cdot\cdot}+p_1q^{\cdot}+p_0q=h_0+\lambda[h_1q(a)+h_2q^{\cdot}(a)+h_3q(b)+h_4q^{\cdot}(b)] \tag{3.14}$$

has, for all values of λ with the exception of at most two, a two-parameter set of solutions if

(i) *the functions $h_i=h_i(t)$ are continuous in $\langle a,b\rangle$,*

(ii) *the functions p_i satisfy the assumptions of our theorem.*

Proof. The formal solution of equation (3.14) may be written in the form

$$q(t)=C_1q_1(t)+C_2q_2(t)+H_0(t)+ \tag{3.15}$$
$$+\lambda[H_1q(a)+H_2q^{\cdot}(a)+H_3q(b)+H_4q^{\cdot}(b)]$$

where

$$H_i=H_i(t)=\int\limits_a^t\frac{q_1(\tau)q_2(t)-q_2(\tau)q_1(t)}{W(\tau)}h_i(\tau)\,d\tau+a_{1i}q_1(t)+a_{2i}q_2(t)$$
$$(i=0,1,2,3,4),$$

q_1,q_2,W have the same meaning as in (3.8) and a_{ki} are arbitrary constants.

To prove the lemma it is sufficient to show that the quantities $q(a)$, $q(b),q^{\cdot}(a),q^{\cdot}(b)$ may be eliminated from (3.15). In fact, that is the case if the set of four linear equations

$$\begin{aligned}
q(a)&=C_1q_1(a)+C_2q_2(a)+H_0(a)+\lambda[H_1(a)q(a)+\ldots+H_4(a)q^{\cdot}(b)],\\
q^{\cdot}(a)&=C_1q_1^{\cdot}(a)+C_2q_2^{\cdot}(a)+H_0^{\cdot}(a)+\lambda[H_1^{\cdot}(a)q(a)+\ldots+H_4^{\cdot}(a)q^{\cdot}(b)],\\
q(b)&=C_1q_1(b)+C_2q_2(b)+H_0(b)+\lambda[H_1(b)q(a)+\ldots+H_4(b)q^{\cdot}(b)],\\
q^{\cdot}(b)&=C_1q_1^{\cdot}(b)+C_2q_2^{\cdot}(b)+H_0^{\cdot}(b)+\lambda[H_1^{\cdot}(b)q(a)+\ldots+H_4^{\cdot}(b)q^{\cdot}(b)]
\end{aligned} \tag{3.16}$$

can be solved with respect to these quantities. Equations (3.16) are non-homogeneous on account of $W(t)\neq 0$ in $\langle a,b\rangle$. It is sufficient to demand, therefore, that

$$\Delta=\begin{vmatrix} \lambda J_{11}-1 & \lambda J_{12} & \lambda J_{13} & \lambda J_{14} \\ \lambda J_{21} & \lambda J_{22}-1 & \lambda J_{23} & \lambda J_{24} \\ \lambda J_{31} & \lambda J_{32} & \lambda J_{33}-1 & \lambda J_{34} \\ \lambda J_{41} & \lambda J_{42} & \lambda J_{43} & \lambda J_{44}-1 \end{vmatrix}\neq 0$$

where

$$J_{1i}=a_{1i}q_1(a)+a_{2i}q_2(a),$$

$$J_{2i}=a_{1i}\dot{q}_1(a)+a_{2i}\dot{q}_2(a),$$

$$J_{3i}=\left(-\int_a^b \frac{q_2(\tau)}{W(\tau)}h_i(\tau)d\tau+a_{1i}\right)q_1(b)+\left(\int_a^b \frac{q_1(\tau)}{W(\tau)}h_i(\tau)d\tau+a_{2i}\right)q_2(b),$$

$$J_{4i}=\left(-\int_a^b \frac{q_2(\tau)}{W(\tau)}h_i(\tau)d\tau+a_{1i}\right)\dot{q}_1(b)+\left(\int_a^b \frac{q_1(\tau)}{W(\tau)}h_i(\tau)d\tau+a_{2i}\right)\dot{q}_2(b)$$

for $i=1,2,3,4$.

Taking $a_{ij}=-H_{ij}$ where

$$H_{1j}=-\int_a^b \frac{q_2(\tau)}{W(\tau)}h_j(\tau)d\tau, \qquad H_{2j}=\int_a^b \frac{q_1(\tau)}{W(\tau)}h_j(\tau)d\tau$$

we get

$$\Delta=\begin{vmatrix} \lambda\big(H_{11}q_1(a)+H_{21}q_2(a)\big)+1 & \lambda\big(H_{12}q_1(a)+H_{22}q_2(a)\big) \\ \lambda\big(H_{11}\dot{q}_1(a)+H_{21}\dot{q}_2(a)\big) & \lambda\big(H_{12}\dot{q}_1(a)+H_{22}\dot{q}_2(a)\big)+1 \end{vmatrix}.$$

From the last formula it can be seen that $\Delta=0$ for at most two values of λ. It may be shown that this number cannot in general be reduced.

From (3.9) and (3.10) it can easily be seen that equation (3.11) has the form (3.14). This remark completes the proof of our theorem, the characteristic values of λ being the two zeros of Δ and the characteristic values of the kernel $N(t,t';t_0')$. The existence of $dF(t)/dt$, which was necessary for the derivation of equation (3.15), follows from the assumption (3.2) about $K(t,t')$ since it guarantees the existence of $\partial[R(t,t';t_0')]/\partial t$. This follows from simple considerations concerning the convergence of $\partial R/\partial t$ analogous to those carried out by Fredholm with respect to R.

It may be of some interest to discuss the particular case of a homogeneous equation (3.1) ($f(t)\equiv 0$). In this case it can easily be seen from

(3.9) that for the condition $\varphi(t)\not\equiv 0$ in $\langle a,b\rangle$ to be satisfied it is necessary and sufficient that at least one of expressions

$$N(t,\xi;t_0') \qquad N(t,\xi;t_0')p_1(\xi)-\frac{\partial}{\partial t'}[N(t,t';t_0')p_2(t')]_{t'=\xi} \qquad (\xi=a,b)$$

does not vanish identically in $\langle a,b\rangle$.

We may now proceed to the final conclusion of this section *i. e.* to the explicit construction of the differential equation which is equivalent to the original equation (3.1). For this purpose we rewrite the equations (3.16) with the functions $H_0,H_i,i=1,2,3,4$ determined from equations (3.11), (3.13), and (3.10). Together with equation (3.15) and its first and second derivative

$$-q(t)+C_1q_1(t)+C_2q_2(t)+H_0(t)+\lambda[H_1(t)q(a)+\ldots+H_4(t)\dot{q}(b)]=0,$$

$$-\dot{q}(t)+C_1\dot{q}_1(t)+C_2\dot{q}_2(t)+\dot{H}_0(t)+\lambda[\dot{H}_1(t)q(a)+\ldots+\dot{H}_4(t)\dot{q}(b)]=0,$$

$$-\ddot{q}(t)+C_1\ddot{q}_1(t)+C_2\ddot{q}_2(t)+\ddot{H}_0(t)+\lambda[\ddot{H}_1(t)q(a)+\ldots+\ddot{H}_4(t)\dot{q}(b)]=0$$

they form a set of seven equations for the six arbitrary parameters C_1, $C_2,q(a),\dot{q}(a),q(b),\dot{q}(b)$. The result of elimination of these parameters

$$\text{(3.17)}\qquad \begin{vmatrix} -q(t)+H_0(t) & q_1(t) & q_2(t) & \lambda H_1(t) & \ldots & \lambda H_4(t) \\ -\dot{q}(t)+\dot{H}_0(t) & \dot{q}_1(t) & \dot{q}_2(t) & \lambda\dot{H}_1(t) & \ldots & \lambda\dot{H}_4(t) \\ -\ddot{q}(t)+\ddot{H}_0(t) & \ddot{q}_1(t) & \ddot{q}_2(t) & \lambda\ddot{H}_1(t) & \ldots & \lambda\ddot{H}_4(t) \\ H_0(a) & q_1(a) & q_2(a) & \lambda H_1(a)-1 & \ldots & \lambda H_4(a) \\ \dot{H}_0(a) & \dot{q}_1(a) & \dot{q}_2(a) & \lambda\dot{H}_1(a) & \ldots & \lambda\dot{H}_4(a) \\ H_0(b) & q_1(b) & q_2(b) & \lambda H_1(b) & \ldots & \lambda H_4(b) \\ \dot{H}_0(b) & \dot{q}_1(b) & \dot{q}_2(b) & \lambda\dot{H}_1(b) & \ldots & \lambda\dot{H}_4(b)-1 \end{vmatrix}=0$$

is just the required differential equation. This equation is equivalent to equation (3.1) in the sense that each solution of (3.1) satisfies (3.17) and vice versa.

The reduction of an integro-differential equation to a pure differential equation of the same order (in derivatives) is of practical as well as theoretical importance since it enables us to apply to integro-differential equations the powerful methods of the theory of differential equations. One of the immediate and most important consequences of the above considerations is *e.g.* the oscillation theorem of Sturm for (3.1).

The above considerations may be generalized to higher order linear equations containing derivatives also under the integral sign:

$$p_n(t)\,q^{(n)}(t)+\ldots+p_0(t)\,q(t)$$
$$=f(t)+\lambda\int_a^b [K_0(t,t')\,q(t')+\ldots+K_m(t,t')\,q^{(m)}(t')]\,dt'.$$

The results will be published elsewhere.

We shall make use of the results of this section in the investigations of the second variation (Part II). Applications to the quantum theory of non local systems (systems described by integro-differential equations) are discussed elsewhere [4]).

4. Example. The calculation of the coefficients of the differential equation (3.17) corresponding to the integro-differential equation (3.1) is rather complicated apart from the fact that it requires knowledge of the resolving kernel $R(t,t';t_0')$. Therefore we shall consider in this section a simple example of an integro-differential equation with constant coefficients. For constant coefficients the method of section 3 may be simplified to a great extent and new methods may be developed which are of importance for the generalization of the results to partial integro-differential equations.

Consider the integro-differential equation

$$\ddot q(t)+\varkappa^2 q(t)=\lambda\int_a^b K(t,t')\,q(t')\,dt'. \tag{4.1}$$

This equation may immediately be converted into an integral equation of Fredholm's type by means of the function

$$\Delta^r(t)=\eta^r(t)\,\frac{1}{\varkappa}\sin\varkappa t,\qquad \eta^r(t)=\begin{cases}1 & \text{for}\quad t\geqslant 0,\\ 0 & \text{for}\quad t<0.\end{cases}$$

Indeed, multiplying (4.1) by $\Delta^r(\tau-t)$ and integrating with respect to t over the interval $\langle a,b\rangle$ we obtain (after some partial integrations)

$$q(\tau)=q^0(\tau)+\lambda\int_a^b N(\tau,t')\,q(t')\,dt' \tag{4.2}$$

where

$$q^0(\tau)=\dot q(a)\,\frac{1}{\varkappa}\sin\varkappa(\tau-a)+q(a)\cos\varkappa(\tau-a),$$

$$N(\tau,t')=\int_a^b \Delta^r(\tau-t)\,K(t,t')\,dt.$$

[4]) J. Rzewuski, *Differential structure of non-local theories*, I, Acta Phys. Pol. XIII (1954), p. 135-144; *On differential structure of non-local field theories*, Bull. Acad. Pol. Sc., Cl. III, 2 (1954), p. 429-433.

If $K(t,t')$ is bounded in the domain $a\leqslant t\leqslant b$, $a\leqslant t'\leqslant b$, then $N(\tau,t')$ is also bounded together with its first and second derivative with respect to τ. Further, if λ is not a characteristic value of the kernel $N(\tau,t')$, equation (4.2) has a unique solution. Since $q^0(\tau)$ contains two arbitrary parameters, $\dot q(a)$ and $q(a)$, (4.2) represents a two-parameter set of solutions. Equations (4.1) and (4.2) are equivalent in the sense that each solution of (4.1) satisfies (4.2) (with a particular choice of parameters), and vice versa: each solution of (4.2) (with arbitrary choice of parameters) satisfies (4.1). Therefore the solutions of (4.1) form (under the above assumptions about λ and $K(t,t')$) also a two-parameter set. They may be explicitly written down by means of the resolving kernel $R(\tau,t)$ of equation (4.2)

$$q(\tau)=q^0(\tau)+\lambda\int_a^b R(\tau,t')\,q^0(t')\,dt'. \tag{4.3}$$

Elimination of the arbitrary parameters $\dot q(a)$ and $q(a)$ from (4.3) yields immediately the corresponding differential equation in the form

$$p_2(t)\,\ddot q(t)-\dot p_2(t)\,\dot q(t)+p_0(t)\,q(t)=0 \tag{4.4}$$

where

$$p_2=\begin{vmatrix} r_1 & r_2\\ \dot r_1 & \dot r_2\end{vmatrix},\qquad p_0=\begin{vmatrix} \dot r_1 & \dot r_2\\ \ddot r_1 & \ddot r_2\end{vmatrix}$$

with

$$r_1(t)=\frac{1}{\varkappa}\sin\varkappa(t-a)+\lambda\int_a^b R(t,t')\,\frac{1}{\varkappa}\sin\varkappa(t'-a)\,dt',$$

$$r_2(t)=\cos\varkappa(t-a)+\lambda\int_a^b R(t,t')\cos\varkappa(t'-a)\,dt'.$$

The explicit calculation of the coefficients yields

$$p_2=-(1+c_2),\qquad p_0=-\varkappa^2(1+c_0) \tag{4.5}$$

with

$$\begin{aligned} c_0 &= -\lambda\int_a^b \frac{1}{\varkappa}\dot R(t,t')\sin\varkappa(t-t')\,dt'-\lambda\int_a^b\frac{1}{\varkappa^2}\ddot R(t,t')\cos\varkappa(t-t')\,dt'+\\ &\quad+\lambda^2\int_a^b\!\!\int\frac{1}{\varkappa^3}\ddot R(t,t')\sin\varkappa(t'-t'')\,\dot R(t,t'')\,dt'\,dt'',\\ c_2 &= -\lambda\int_a^b\frac{1}{\varkappa}\dot R(t,t')\sin\varkappa(t-t')\,dt'+\lambda\int_a^b R(t,t')\cos\varkappa(t-t')\,dt'-\\ &\quad-\lambda^2\int_a^b\!\!\int R(t,t')\sin\varkappa(t'-t'')\,\frac{1}{\varkappa}\dot R(t,t'')\,dt'\,dt''\end{aligned} \tag{4.6}$$

(dots always denote differentiation with respect to the first argument).

With the notation (4.5) equation (4.4) takes the form

$$(1+c_2)\ddot{q}-\dot{c}_2\dot{q}+\varkappa^2(1+c_0)q=0. \tag{4.7}$$

It can be seen from (4.6) that, for $\lambda=0$, $c_2=c_0=0$ in accordance with (4.1).

Taking λ sufficiently small we can always obtain

$$1+c_2\neq 0 \quad \text{in} \quad \langle a,b\rangle;$$

in this case (4.7) may be rewritten in the form

$$\ddot{q}+\varkappa^2 q=\frac{\dot{c}_2\dot{q}+\varkappa^2(c_2-c_0)q}{1+c_2}.$$

Comparison with the original equation (4.1) shows that in the space of solutions of (4.1) the following identity is satisfied:

$$\frac{\dot{c}_2\dot{q}+\varkappa^2(c_2-c_0)q}{1+c_2}=\lambda\int_a^b K(t,t')q(t')\,dt'. \tag{4.8}$$

The existence of an identity of the type (4.8) indicates a new possibility of converting integro-differential equations into differential equations, which puts the whole problem into a new light: indeed, given an equation of the type (4.1), we may seek two functions $A(t)$ and $B(t)$ such that the equation

$$A(t)\dot{q}(t)+B(t)q(t)=\lambda\int_a^b K(t,t')q(t')\,dt' \tag{4.9}$$

is satisfied in the space of solutions of (4.1). The construction of these functions may be carried out by the expansion of A, B, and q in power series of λ and equating coefficients on both sides of (4.9).

Let

$$A=\sum_{n=0}^{\infty}\lambda^n A^{(n)}, \qquad B=\sum_{n=0}^{\infty}\lambda^n B^{(n)}. \tag{4.10}$$

From (4.3), remembering that $R(t,t')$ is the resolving kernel of (4.2), we obtain

$$q(t)=q^0(t)+\lambda\int_a^b\sum_{n=0}^{\infty}\lambda^n\{N^{n+1}\}(t,t')q^0(t')\,dt', \tag{4.11}$$

where

$$\begin{aligned}\{N^{n+1}\}(t,t')&\equiv\int_a^b\cdots\int_a^b N(t,\tau^1)N(\tau^1,\tau^2)\cdots N(\tau^n,t')\,d\tau^1\cdots d\tau^n,\\ \{N^1\}(t,t')&=N(t,t').\end{aligned} \tag{4.12}$$

In (4.11) we may express $q^0(t')$ by its initial value and the initial value of its first derivative at an arbitrary point $t=t^0$

$$q^0(t')=\dot{q}^0(t^0)\frac{1}{\varkappa}\sin\varkappa(t'-t^0)+q^0(t^0)\cos\varkappa(t'-t^0). \tag{4.13}$$

Introducing (4.13) and (4.11) into the right-hand side of (4.9) and denoting for convenience

$$\Delta(t)=\frac{1}{\varkappa}\sin\varkappa t, \qquad \dot{\Delta}(t)=\cos\varkappa t,$$

we obtain

$$\int_a^b K(t,t')q(t')\,dt'=\sum_{n=0}^{\infty}\lambda^n\big[\{KN^n\Delta\}(t,t^0)\dot{q}^0(t^0)+\{KN^n\dot{\Delta}\}(t,t^0)q^0(t^0)\big], \tag{4.14}$$

where

$$\{KN^n\Delta\}(t,t^0)=\iint_a^b K(t,\tau^1)\{N^n\}(\tau^1,\tau^2)\Delta(\tau^2-t^0)\,d\tau^1 d\tau^2, \qquad \{N^0\}=1,$$

and similarly for $\{KN^n\dot{\Delta}\}$, in an obvious generalization of the notation (4.12).

The shape of (4.14) already indicates the general form of the left-hand side of (4.9), which is not at all obvious a priori (*e.g.* one could introduce also $\ddot{q}$ on the left-hand side of (4.9)). Treating in exactly the same manner the left-hand side of (4.9) by use of (4.10), (4.11) and (4.13) we obtain

$$\begin{aligned}&\Big(A\frac{d}{dt}+B\Big)q\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\lambda^{n+m}\Big[A^{(n)}\frac{d}{dt}+B^{(n)}\Big]\big[\{N^m\Delta\}(t,t^0)\dot{q}^0(t^0)+\{N^m\dot{\Delta}\}(t,t^0)q^0(t^0)\big].\end{aligned} \tag{4.15}$$

Equating coefficients of (4.14) and (4.15) we obtain the infinite set of equations

$$\begin{aligned}&\sum_{m=0}^{n}\Big(A^{(m)}\frac{d}{dt}+B^{(m)}\Big)\big[\{N^{n-m}\Delta\}(t,t^0)\dot{q}^0(t^0)+\{N^{n-m}\dot{\Delta}\}(t,t^0)q^0(t^0)\big]\\ &\quad=\{KN^n\Delta\}(t,t^0)\dot{q}^0(t^0)+\{KN^n\dot{\Delta}\}(t,t^0)q^0(t^0), \qquad n=1,2,\ldots\end{aligned} \tag{4.16}$$

(4.16) are identities in the space of solutions of (4.1) and, therefore, they must be satisfied for an arbitrary choice of the parameters $\dot{q}^0(t^0)$ and $q^0(t^0)$. Therefore the equations (4.16) split into

$$\left.\begin{aligned}&\sum_{m=0}^{n}\Big(A^{(m)}\frac{d}{dt}+B^{(m)}\Big)\{N^{n-m}\Delta\}(t,t^0)=\{KN^n\Delta\}(t,t^0),\\ &\sum_{m=0}^{n}\Big(A^{(m)}\frac{d}{dt}+B^{(m)}\Big)\{N^{n-m}\dot{\Delta}\}(t,t^0)=\{KN^n\dot{\Delta}\}(t,t^0).\end{aligned}\right\} \quad n=1,2,\ldots \tag{4.17}$$

Taking $t^0=t$, we obtain from (4.17) simple recurrence formulas for the functions $A^{(n)}(t)$ and $B^{(n)}(t)$, $n=1,2,\ldots$

The consistency of the two methods described in this section may be verified by the expansion of $c_2^{\cdot}/(1+c_2)$ and $\varkappa^2(c_2-c_0)/(1+c_2)$ in power series of λ and a comparison with A and B respectively.

In contract to the first method described in this section the second method allows a generalization to partial differential equations. We shall return to these questions at another place.

Finally it may be noted that the considerations of this section apply equally well to general types of linear integro-differential equations with constant coefficients and higher order derivatives outside the integral.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Eingliedrige Gruppen der homogenen kanonischen Transformationen und Finslersche Räume

von K. Maurin (Warszawa)*

1. Einführung. Bisher wurden die kanonischen Transformationen

$$\tilde{x}^i=\varphi^i(x^1,\ldots,x^n;p_1,\ldots,p_n),\quad \tilde{p}_i=\psi_i(x^1,\ldots,x^n;p_1,\ldots,p_n)\quad (i=1,2,\ldots,n)$$

als Abbildungen des $2n$-dimensionalen linearen Raumes (des sog. Phasenraumes) interpretiert. In dieser Abhandlung werden $x^1,\ldots,x^n$ als Koordinaten des Punktes P einer (nicht notwendig linearen) Mannigfaltigkeit M_n der Klasse C^1, dagegen $p_1,\ldots,p_n$ als Koordinaten des Punktes $p\in\overline{T}_n(x)$ gedeutet. $\overline{T}_n(x)$ ist der duale Raum des tangentialen Raumes der M_n im Punkte P.

Auf diese Weise wird die geometrische Interpretation der infinitesimalen kanonischen homogenen Transformationen (k. h.) und ihr enger Zusammenhang mit den Indicatrizen Eichflächen der (kovarianten) Finsler-Metrik evident.

Damit die Zusammenhänge klarer vor Augen treten, werden die für die Anwendungen wichtigsten k. h. Transformationen ausgesondert und als *sternartige Transformationen* bezeichnet. Es wird bewiesen der folgende

Hauptsatz. *Jede sternartige 1-gliedrige Gruppe G_1 der k. h. Transformationen induziert in M_n eine Finsler-Metrik und umgekehrt: der Finslersche Raum bestimmt eine sternartige G_1, wobei die geodätischen Linien des F_n Trajektorien der G_1 sind.*

Die hier entwickelte Theorie zeigt, daß die Elementarwellen von Huyghens-Vessiot die Indikatrizen des von G_1 erzeugten F_n-Raumes sind; G_1 beschreibt die Ausbreitung der Störungen im permanenten Medium.

Als Anwendung gebe ich einen einfachen Beweis des Haupsatzes der Theorie der Normalkongruenzen geodätischer Linien im Finslerschen Raume.

In der Arbeit wurde alles auf koordinatenfreie Weise entwickelt; dadurch wurde der geometrische Inhalt der Theorie in den Vorder-

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