On summability of double sequences (I)

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The subject of this paper is a theorem on the consistency of methods of summability for bounded double sequences (theorem 3.3) which extends a theorem of Mazur and Orlicz ([7], theorem 2) concerning the summability of single sequences. Our theorem will be proved for completely permanent methods *i. e.* methods preserving not only the principal limit but also the row- and column-limits if such exist. The method of proof is simpler than the original proof of [7] and, applied for single sequences will appear in the paper [9] of Orlicz. J. D. Hill [4] has transferred for double sequences an earlier consistency theorem of Banach ([2], p. 95) which is less general than the theorem in [7]. Hence the results of this paper may be considered as a generalization of the theorems of Hill. The results of this paper may be extended to the case of transformations of sequences with an arbitrary degree of multiplicity, without any essential alterations of the proofs.

1. Definitions and preliminaries. By a *convergent sequence* will be meant any double sequence $x=\{x_{ik}\}$ converging in the sense of Pringsheim. The principal limit of such a sequence will be written $\lim\limits_{i,k\to\infty} x_{ik}$ or simply $x_{..}$. The sequence $\{x_{ik}\}$ is said to *converge regularly* if it is convergent and if the row- and column-limits

$$\lim_{k\to\infty} x_{ik}=x_{i.}, \qquad \lim_{i\to\infty} x_{ik}=x_{.k}$$

exist for every i and k respectively. Every regularly convergent sequence is bounded. The sequence x will be called *perfectly convergent* if it is regularly convergent and $x_{i.}=x_{.k}$ for $i,k=0,1,\ldots$

The most frequently used methods of summability for double sequences may be obtained by the following procedure. Given a four-dimensional matrix

$$A=(a_{ik\mu\nu}), \qquad i,k,\mu,\nu=0,1,\ldots$$

and a double sequence $x=\{x_{ik}\}$ let us consider the transforms

$$A_{ik}(x)=\sum_{\mu,\nu=0}^{\infty} a_{ik\mu\nu}x_{\mu\nu}.$$

If these transforms are defined for every i and k *i.e.* if the series on the right converge (in the sense of Pringsheim), we say that *the method A transforms the sequence x into the sequence $\{A_{ik}(x)\}$*; if the sequence $\{A_{ik}(x)\}$ converges to a, the sequence x will be said to be *A-summable* (or to be *summable by the method A*) to a; the number a will be denoted in this case by $A\text{-}\lim\limits_{i,k\to\infty} x_{ik}$ or simply $A_{..}(x)$.

The sequence x will be said to be *regularly A-summable to a* if the sequence $\{A_{ik}(x)\}$ converges regularly to a (we shall also say that the method A *transforms the sequence x regularly*) and we shall write

$$\lim_{k\to\infty} A_{ik}(x)=A_{i.}(x), \qquad \lim_{i\to\infty} A_{ik}(x)=A_{.k}(x).$$

The method A will be said to *fulfil the condition*

(r_0) *if it transforms every sequence x regularly convergent to 0 into a sequence regularly convergent in such a manner that*

(1) $$A_{i.}(x)=x_{i.}, \qquad A_{.k}(x)=x_{.k} \qquad \textit{for} \quad i,k=0,1,\ldots$$

in this case, of course, $A_{..}(x)=0$.

The method A will be called *completely permanent* if every regularly convergent sequence is regularly A-summable and the conditions (1) are satisfied.

The sequence x will be termed *perfectly A-summable* if it is regularly A-summable and

$$A_{i.}(x)=A_{.k}(x) \qquad \text{for} \qquad i,k=0,1,\ldots$$

i.e. if the sequence of the transforms converges perfectly.

The following Toeplitzian conditions are known[1]) —

(i) *The method A transforms regularly every sequence regularly convergent to 0 if and only if the following conditions are satisfied*:

(i_1) $$\sup_{i,k=0,1,\ldots} \sum_{\mu,\nu=0}^{\infty} |a_{ik\mu\nu}|<\infty,$$

(i_2) *there exist*

$$\lim_{i,k\to\infty} a_{ik\mu\nu}=a_{..\mu\nu}, \qquad \lim_{k\to\infty} a_{ik\mu\nu}=a_{i.\mu\nu}, \qquad \lim_{i\to\infty} a_{ik\mu\nu}=a_{.k\mu\nu},$$

[1]) These conditions are due in principle to Kojîma [5] (this paper was unavailable to the authors); for a complete discussion of conditions of this kind see Hamilton [3].

(i_3) *there exist*

$$\lim_{i,k\to\infty}\sum_{\nu=0}^{\infty} a_{ik\mu\nu}=S_{..\mu.},\qquad \lim_{k\to\infty}\sum_{\mu=0}^{\infty} a_{ik\mu\nu}=S_{i..\nu},\qquad \lim_{i\to\infty}\sum_{\nu=0}^{\infty} a_{ik\mu\nu}=S_{.k\mu.},$$

$$\lim_{i,k\to\infty}\sum_{\mu=0}^{\infty} a_{ik\mu\nu}=S_{...\nu},\qquad \lim_{k\to\infty}\sum_{\nu=0}^{\infty} a_{ik\mu\nu}=S_{i.\mu.},\qquad \lim_{i\to\infty}\sum_{\mu=0}^{\infty} a_{ik\mu\nu}=S_{.k.\nu}.$$

If these conditions are satisfied, then for every sequence x regularly convergent to 0 *the following formulae hold:*

$$A_{i.}(x)=\sum_{\mu,\nu=0}^{\infty} a_{i.\mu\nu}x_{\mu\nu}+\sum_{\mu=0}^{\infty} x_{\mu.}\Big(S_{i.\mu.}-\sum_{\nu=0}^{\infty} a_{i.\mu\nu}\Big)+\sum_{\nu=0}^{\infty} x_{.\nu}\Big(S_{i..\nu}-\sum_{\mu=0}^{\infty} a_{i.\mu\nu}\Big),$$

$$A_{.k}(x)=\sum_{\mu,\nu=0}^{\infty} a_{.k\mu\nu}x_{\mu\nu}+\sum_{\mu=0}^{\infty} x_{\mu.}\Big(S_{.k\mu.}-\sum_{\nu=0}^{\infty} a_{.k\mu\nu}\Big)+\sum_{\nu=0}^{\infty} x_{.\nu}\Big(S_{.k.\nu}-\sum_{\mu=0}^{\infty} a_{.k\mu\nu}\Big),$$

$$A_{.\nu}(x)=\sum_{\mu,\nu=0}^{\infty} a_{..\mu\nu}x_{\mu\nu}+\sum_{\mu=0}^{\infty} x_{\mu.}\Big(S_{..\mu.}-\sum_{\nu=0}^{\infty} a_{..\mu\nu}\Big)+\sum_{\nu=0}^{\infty} x_{.\nu}\Big(S_{...\nu}-\sum_{\mu=0}^{\infty} a_{..\mu\nu}\Big);$$

moreover

$$\lim_{i\to\infty} a_{i.\mu\nu}=\lim_{k\to\infty} a_{.k\mu\nu}=a_{..\mu\nu},$$

$$\lim_{i\to\infty} S_{i.\mu.}=\lim_{k\to\infty} S_{.k\mu.}=S_{..\mu.},$$

$$\lim_{i\to\infty} S_{i..\nu}=\lim_{k\to\infty} S_{.k.\nu}=S_{...\nu}.$$

(ii) *The method A transforms regularly every sequence regularly convergent to* 0 *into a sequence convergent to* 0 *if and only if the conditions* (i_1), (i_2), *and* (i_3) *are satisfied and*

$$a_{..\mu\nu}=0,\qquad S_{..\mu.}=0,\qquad S_{...\nu}=0$$

for every μ,ν.

(iii) *The method A fulfils the condition* (r_0) *if and only if the conditions of* (i) *and* (ii) *are satisfied and*

$$a_{i.\mu\nu}=0,\qquad S_{i.\mu.}=S_{.i.\mu}=\delta_{i\mu},\qquad S_{i..\mu}=S_{.i\mu.}=0,$$

where $\delta_{i\mu}$ denotes the delta of Kronecker.

(iv) *The method A transforms regularly every sequence regularly convergent if and only if the conditions* (i_1), (i_2), (i_3), *and the following are satisfied:*

(iv_1) *there exist*

$$\lim_{i,k\to\infty}\sum_{\mu,\nu=0}^{\infty} a_{ik\mu\nu}=S_{....},\qquad \lim_{k\to\infty}\sum_{\mu,\nu=0}^{\infty} a_{ik\mu\nu}=S_{i...},\qquad \lim_{i\to\infty}\sum_{\mu,\nu=0}^{\infty} a_{ik\mu\nu}=S_{.k..}.$$

In this case

$$\lim_{i\to\infty} S_{i...}=\lim_{k\to\infty} S_{.k..}=S_{....},$$

$$A_{i.}(x)=x_{..}\Big(S_{i...}-\sum_{\mu=0}^{\infty} S_{i.\mu.}-\sum_{\nu=0}^{\infty} S_{i..\nu}\Big)+\sum_{\mu,\nu=0}^{\infty} a_{i.\mu\nu}x_{\mu\nu}+$$
$$+\sum_{\mu=0}^{\infty}\Big(S_{i.\mu.}-\sum_{\nu=0}^{\infty} a_{i.\mu\nu}\Big)x_{\mu.}+\sum_{\nu=0}^{\infty}\Big(S_{i..\nu}-\sum_{\mu=0}^{\infty} a_{i.\mu\nu}\Big)x_{.\nu},$$

and similar formulae hold for $A_{.k}(x)$ and $A_{..}(x)$.

(v) *The method A is completely permanent if and only if the conditions* (i_1), (i_2), (i_3), *and* (iv_1) *are satisfied and*

$$a_{..\mu\nu}=0,\qquad S_{..\mu.}=S_{...\nu}=0,$$
$$a_{i.\mu\nu}=a_{.k\mu\nu}=0,\qquad S_{i.\mu.}=S_{.i.\mu}=\delta_{i\mu},$$
$$S_{i..\mu}=S_{.i\mu.}=0,\qquad S_{....}=S_{i...}=S_{.k..}=1.$$

The proof of these theorems may easily be carried out by considering the space $\boldsymbol{R}$ of regularly convergent sequences and its subspace $\boldsymbol{R}_0$ composed of the sequences convergent to 0; both are Banach spaces under the norm $\|x\|=\sup\limits_{i,k=0,1,\ldots}|x_{ik}|$. Denoting by $e_{i.}$ the sequence having 1 in the ith row and 0 elsewhere, by $e_{.k}$ an analogous sequence with 1 in the kth column, by e_{ik} the sequence with 1 at the intersection point of the ith row and the kth column, one can easily prove that the linear combinations of these elements compose a dense set in $\boldsymbol{R}_0$ and that every linear functional ξ in $\boldsymbol{R}_0$ is of the form

$$\xi(x)=\sum_{\mu,\nu=0}^{\infty} x_{\mu\nu}\xi(e_{\mu\nu})+\sum_{\mu=0}^{\infty} x_{\mu.}\Big[\xi(e_{\mu.})-\sum_{\nu=0}^{\infty}\xi(e_{\mu\nu})\Big]+\sum_{\nu=0}^{\infty} x_{.\nu}\Big[\xi(e_{.\nu})-\sum_{\mu=0}^{\infty}\xi(e_{\mu\nu})\Big]$$

(see Hill [4]). This enables us to get the representations of $A_{i.}(x)$, $A_{.k}(x)$, and $A_{..}(x)$ in (i).

2. A linear space. Let us denote by $\boldsymbol{P}_0$ the space of all sequences $x=\{x_{ik}\}$ perfectly convergent to 0; it is a linear subspace of the space $\boldsymbol{R}_0$. Given a Banach space X we shall denote by $T_0\{X\}$ the space of the sequences $\boldsymbol{x}=\{x_n\}$ with $x_n\in X$ and such that $\|x_n\|\to 0$; this linear space is a Banach space under the norm $\|\boldsymbol{x}\|=\sup\limits_{n=0,1,\ldots}\|x_n\|$. It may easily be proved that the general form of linear functionals in $T_0\{X\}$ is

$$\xi(\boldsymbol{x})=\sum_{n=0}^{\infty}\xi_n(x_n)$$

where ξ_i is a linear functional in X and, moreover,

$$\|\xi\|=\sum_{n=0}^{\infty}\|\xi_n\|.$$

In the case of X being the space of reals we shall write simply T_0 for $T_0\{X\}$.

2.1. Lemma. *The space $\boldsymbol{P}_0$ is equivalent to the space $T_0\{T_0\}$.*

Proof. For every $x=\{x_{ik}\}\,\epsilon\,\boldsymbol{P}_0$ write

$$x^{n\cdot}=\{x_{nn},x_{n,n+1},\ldots\},\qquad x^{\cdot n}=\{x_{n+1,n},x_{n+2,n},\ldots\}:$$

these are elements of T_0. It is easily shown that the correspondence

$$x\rightleftarrows\{x^{0\cdot},x^{\cdot 0},x^{1\cdot},x^{\cdot 1},\ldots\}$$

definies the equivalence between the two spaces under consideration.

Taking into account the general form of the linear functionals in the space T_0 we obtain as the general form of linear functionals in $\boldsymbol{P}_0$:

$$\xi(x)=\sum_{i,k=0}^{\infty}a_{ik}x_{ik}$$

with $\sum_{i,k=0}^{\infty}|a_{ik}|=\|\xi\|$. This easily gives

2.2. Lemma. *If the sequences $x^n=\{x_{ik}^n\}$ and $x=\{x_{ik}\}$ are in $\boldsymbol{P}_0$ and satisfy the conditions*

$$\sup_{i,k,n=0,1,\ldots}|x_{ik}^n|<\infty,\qquad \lim_{n\to\infty}x_{ik}^n=x_{ik}\quad \textit{for}\quad i,k=0,1,\ldots,$$

then x^n converges weakly to x in $\boldsymbol{P}_0$.

Now we shall prove a lemma which, for single sequences, was stated in its original form by Banach ([2], p. 93) and in the final one by Mazur and Orlicz ([7], theorem 2.2).

2.3. Lemma. *Let the method A fulfil the condition* (r_0). *If the sequence $x=\{x_{ik}\}$ is bounded and perfectly A-summable to 0, then for every $\varepsilon>0$ and n there exists a positive integer $p>n$ and a sequence $z=\{z_{ik}\}$ such that*

$$z_{ik}=\begin{cases}x_{ik} & \textit{for}\quad i,k\leqslant n,\\ 0 & \textit{for}\quad \max(i,k)>p,\end{cases}\tag{2}$$

$$\begin{gathered}|z_{ik}|\leqslant|x_{ik}|\quad \textit{for}\quad i,k=0,1,\ldots,\\ |A_{ik}(x)-A_{ik}(z)|<\varepsilon\quad \textit{for}\quad i,k=0,1,\ldots\end{gathered}\tag{3}$$

Proof. Write

$$A_{ik}^n(x)=\sum_{\mu,\nu=0}^{n}a_{ik\mu\nu}x_{\mu\nu},\qquad y^n=\{A_{ik}^n(x)\},\qquad y=\{A_{ik}(x)\}.$$

By the condition (r_0) the elements y^n belong to $\boldsymbol{P}_0$, and the element y belongs there by hypothesis, moreover,

$$\sup_{n=0,1,\ldots}\ \sup_{i,k=0,1,\ldots}|A_{ik}^n(x)|\leqslant\sup_{i,k=0,1,\ldots}\sum_{\mu,\nu=0}^{\infty}|a_{ik\mu\nu}|\sup_{\mu,\nu=0,1,\ldots}|x_{\mu\nu}|<\infty,$$

$\lim_{n\to\infty}A_{ik}^n(x)=A_{ik}(x)$, whence by Lemma 2.2 the sequence y^n converges weakly to y in $\boldsymbol{P}_0$. By a theorem of Mazur ([6], p. 81) there exist non-negative numbers $\lambda_n,\ldots,\lambda_p$ such that $\lambda_n+\ldots+\lambda_p=1$ and $\|y-(\lambda_n y^n+\ldots+\lambda_p y^p)\|<\varepsilon$. Writing $x^m=\{x_{ik}^m\}$ where

$$x_{ik}^m=\begin{cases}x_{ik} & \text{for}\quad i,k\leqslant m,\\ 0 & \text{elsewhere},\end{cases}$$

we see that $\lambda_n y^n+\ldots+\lambda_p y^p=\{A_{ik}(\lambda_n x^n+\ldots+\lambda_p x^p)\}$, whence it easily follows that the element $\lambda_n x^n+\ldots+\lambda_p x^p=z=\{z_{ik}\}$ satisfies the conditions (2) and (3) and that

$$|A_{ik}(x)-A_{ik}(z)|<\varepsilon\quad \text{for}\quad i,k=0,1,\ldots$$

Suppose that the method A has the property (r_0) and denote by $\boldsymbol{P}_b$ the space of all bounded sequences $x=\{x_{ik}\}$ which are perfectly A-summable to 0. Let us introduce in $\boldsymbol{P}_b$ two norms

$$\|x\|=\sup_{i,k=0,1,\ldots}|x_{ik}|,$$

$$\|x\|^*=\sum_{i,k=0}^{\infty}2^{-(i+k)}|x_{ik}|+\sup_{i,k=0,1,\ldots}|A_{ik}(x)|.$$

A sequence $\{x_n\}$ of elements of $\boldsymbol{P}_b$ will be called γ-convergent to x_0 (in symbols $x_n\xrightarrow{\gamma}x_0$) if

$$\sup_{n=0,1,\ldots}\|x_n\|<\infty\quad\text{and}\quad\lim_{n\to\infty}\|x_n-x_0\|^*=0.\tag{4}$$

The convergence γ satisfies the following conditions:

(n_1) *If* $\sup_{n=0,1,\ldots}\|x_n\|<\infty$ *and* $\lim_{p,q\to\infty}\|x_p-x_q\|^*=0$, *then there exists an element x_0 such that $x_n\xrightarrow{\gamma}x_0$.*

(n_2) *If* $x_n\xrightarrow{\gamma}x_0$, *then* $\|x_0\|\leqslant\varliminf_{n\to\infty}\|x_n\|$.

The space $\boldsymbol{P}_b$ is a particular case of a space X in which a two norm convergence [1] is defined, viz. if in a linear space x there are defined two norms $\|x\|$ and $\|x\|^*$ such that $\|x_n\|\to 0$ implies $\|x_n\|^*\to 0$, we may define in this space a convergence γ by the condition (4). Let us call a functional ξ in X γ-linear if it is additive and $x_n\xrightarrow{\gamma}x_0$ implies $\xi(x_n)\to\xi(x_0)$. We may ask under what conditions the space X is such that the limit of a convergent sequence of γ-linear functionals is γ-linear. It may easily be shown that the conditions (n_1) and (n_2) alone do not suffice; this property, however, is fulfilled if the space X satisfies the following condition in which S stands for the set $\underset{x}{E}\{\|x\|\leqslant 1\}$ ([1], theorem 4.1):

(n_3) *Given any $x_0 \epsilon S$ and $\varepsilon>0$, there is a $\delta>0$ such that $\|x\|^*<\delta, x\epsilon S$ implies $x=x'-x''$ with $x', x'' \epsilon S$, $\|x_0-x'\|<\varepsilon$, $\|x_0-x''\|<\varepsilon$.*

2.4. THEOREM. *The space $\boldsymbol{P}_b$ has the property* (n_3)[2].

Proof. Let us write $x_0=\{x_{ik}^0\}$, $M=\sup\limits_{i,k=0,1,\ldots}\sum\limits_{\mu,\nu=0}^{\infty}|a_{ik\mu\nu}|$, $\eta=\varepsilon/(2M+5)$. Choose n so that

$$\sum_{\max(\mu,\nu)>n} 2^{-(\mu+\nu)}<\eta;$$

by Lemma 2.3 there exists an element $z=\{z_{ik}\}$ such that $|z_{ik}|\leqslant|x_{ik}|\leqslant 1$ and

$$z_{ik}=\begin{cases} x_{ik}^0 & \text{for } i,k\leqslant n,\\ 0 & \text{for } \max(i,k)>n+p,\end{cases}$$

$$|A_{ik}(x_0)-A_{ik}(z)|<\eta \quad \text{for } i,k=0,1,\ldots$$

Then $\|x_0-z\|^*<3\eta$. Now let $\|x\|\leqslant 1$, $\|x\|^*<2^{-(n+p)}\eta$, $x=\{x_{ik}\}$. Then $|x_{ik}|\leqslant 2^{i+k}\|x\|^*<\eta<1$ for $i,k\leqslant n+p$. Since $\min(|x_{ik}^0+x_{ik}|, |x_{ik}^0-x_{ik}|)\leqslant 1$, there exist $\varepsilon_{ik}=\pm 1$ such that $|x_{ik}^0+\varepsilon_{ik}x_{ik}|\leqslant 1$. Now set

$$x'_{ik}=\begin{cases} x_{ik}+\varepsilon_{ik}x_{ik} & \text{if } i,k\leqslant n+p, \quad \varepsilon_{ik}=1,\\ z_{ik} & \text{if } i,k\leqslant n+p, \quad \varepsilon_{ik}=-1,\\ x_{ik} & \text{elsewhere},\end{cases}$$

$$x''_{ik}=\begin{cases} z_{ik} & \text{if } i,k\leqslant n+p, \quad \varepsilon_{ik}=1,\\ z_{ik}+\varepsilon_{ik}x_{ik} & \text{if } i,k\leqslant n+p, \quad \varepsilon_{ik}=-1,\\ 0 & \text{elsewhere},\end{cases}$$

$$x'=\{x'_{ik}\}, \qquad x''=\{x''_{ik}\}.$$

Then $\|x'_{ik}\|\leqslant 1$, $\|x''_{ik}\|\leqslant 1$, $x=x'-x''$ and

$$|A_{ik}(x')-A_{ik}(z)|\leqslant\sum_{\min(\mu,\nu)\leqslant n+p}|a_{ik\mu\nu}x_{\mu\nu}|+\Big|\sum_{\max(\mu,\nu)>n+p}a_{ik\mu\nu}x_{\mu\nu}\Big|$$

$$\leqslant M\eta+\Big|\sum_{\mu,\nu=0}^{\infty}a_{ik\mu\nu}x_{\mu\nu}\Big|+\sum_{\min(\mu,\nu)\leqslant n+p}|a_{ik\mu\nu}x_{\mu\nu}|$$

$$<2M\eta+|A_{ik}(x)|\leqslant 2M\eta+\|x\|^*,$$

[2]) For the case of single sequences this was proved in [8], p. 248.

$$\|z-x'\|^*=\sum_{\mu,\nu=0}^{\infty}2^{-(\mu+\nu)}|z_{\mu\nu}-x'_{\mu\nu}|+\sup_{i,k=0,1,\ldots}|A_{ik}(x')-A_{ik}(z)|$$

$$<\sum_{\mu,\nu=0}^{\infty}2^{-(\mu+\nu)}|x_{\mu\nu}|+2M\eta+\|x\|^*<2\|x\|^*+2M\eta<2(M+1)\eta,$$

$$\|x_0-x'\|^*\leqslant\|x_0-z\|^*+\|z-x'\|^*<3\eta+2(M+1)\eta<\varepsilon.$$

Similarly $\|x_0-x''\|<(2M+4)\eta<\varepsilon$.

3. Consistency theorems.

3.1. THEOREM. *Let the method A fulfil the condition (r_0) and let every sequence regularly convergent to 0 be B-summable to 0. If every bounded sequence perfectly A-summable to 0 is B-summable, then $B\text{-}\lim\limits_{i,k\to\infty}x_{ik}=0$.*

Proof. Let the method B correspond to the matrix $B=(b_{ik\mu\nu})$. The functionals $B_{ik\mu\nu}(x)=\sum\limits_{\sigma,\tau=0}^{\mu,\nu}b_{ik\sigma\tau}x_{\sigma\tau}$ are linear in the space $\boldsymbol{P}_b$, whence the functionals

$$B_{ik}(x)=\sum_{\sigma,\tau=0}^{\infty}b_{ik\sigma\tau}x_{\sigma\tau} \quad \text{and} \quad B_{..}(x)=\lim_{n\to\infty}B_{nn}(x)$$

are also linear. By hypothesis, $B_{..}(x)=0$ for every sequence regularly convergent to 0, and since these sequences form a dense set in $\boldsymbol{P}_b$ (lemma 2.3), $B_{..}(x)=0$ in $\boldsymbol{P}_b$.

3.2. THEOREM. *Let the methods A_n fulfil the condition (r_0) for $n=0,1,\ldots$, and let any sequence regularly convergent to 0 be B-summable to 0. If every bounded sequence perfectly summable to 0 by all the methods A_n is B-summable, then $B\text{-}\lim\limits_{i,k\to\infty}x_{ik}=0$.*

Proof. The theorem may be proved by a similar method to that used in Theorem 3.1. We must merely replace the space $\boldsymbol{P}_b$ by the space $\boldsymbol{P}_b^\omega$ of all bounded sequences $x=\{x_{ik}\}$ which are simultaneously perfectly A_n-summable to 0. The second norm must be defined now as

$$\|x\|^*=\sum_{i,k=0}^{\infty}2^{-(i+k)}|x_{ik}|+\sum_{i=0}^{\infty}2^{-i}\sup_{\mu,\nu=0,1,\ldots}|A_{\mu\nu}^{(i)}(x)|[1+\sup_{\mu,\nu=1,2,\ldots}|A_{\mu\nu}^{(i)}(x)|]^{-1},$$

$A_{\mu\nu}^{(i)}(x)$ denoting the (μ,ν)th transform of x obtained by the method A_i. Then a lemma analogous to 2.3 holds, which shows that the sequences perfectly convergent to 0 lie dense in $\boldsymbol{P}_b^\omega$. It is not difficult to prove the condition (n_3) also for the space $\boldsymbol{P}_b^\omega$.

3.3. THEOREM[3]). *Let the method A be completely permanent, and let every regularly convergent sequence be B-summable to its limit.*

[3]) This theorem was proved by Hill [4] under the additional hypothesis that the method A is reversible.

If every bounded sequence $x=\{x_{ik}\}$ regularly A-summable is B-summable, then

$$A\text{-}\lim_{i,k\to\infty} x_{ik}=B\text{-}\lim_{i,k\to\infty} x_{ik}.$$

Proof. Consider the sequences $z=\{z_{ik}\}$ and $y=\{y_{ik}\}$ with the terms

$$z_{ik}=\begin{cases}A_{i.}(x) & \text{for } k\geqslant i,\\ A_{.k}(x) & \text{for } i>k,\end{cases}$$

$$y_{ik}=x_{ik}-z_{ik}.$$

The sequence z converges regularly to $A_{..}(x)$, and $A_{i.}(z)=A_{i.}(x)$, $A_{.k}(z)=A_{.k}(x)$ by the complete permanency of the method A, whence the sequence y is perfectly A-summable to 0. By Theorem 3.1 $B\text{-}\lim\limits_{i,k\to\infty} y_{ik}=0$; by hypothesis $B\text{-}\lim\limits_{i,k\to\infty} z_{ik}=A_{..}(x)$; it follows that $B\text{-}\lim\limits_{i,k\to\infty} x_{ik}$ exists and is equal to

$$B\text{-}\lim_{i,k\to\infty} y_{ik}+B\text{-}\lim_{i,k\to\infty} z_{ik}=A_{..}(x).$$

Theorem 3.2 leads in a similar manner to

3.4. THEOREM. *Let the methods A_n be completely permanent for $n=0,1,\ldots$ and let every regularly convergent sequence be B-summable to its limit. If every bounded sequence x regularly summable by all the methods A_n to the same value is B-summable, then $A_n\text{-}\lim\limits_{i,k\to\infty} x_{ik}=B\text{-}\lim\limits_{i,k\to\infty} x_{ik}$.*

4. Transformations reducing the degree of multiplicity of the sequence. Now let us consider double-to-single sequence transformations defined by a three-dimensional matrix

$$A=(a_{i\mu\nu}),\qquad i,\mu,\nu=0,1,\ldots$$

with the transforms $A_i(x)=\sum\limits_{\mu,\nu=0}^{\infty} a_{i\mu\nu}x_{\mu\nu}$, such methods map double sequences upon single ones. We define A-summability as usual, the generalized limit will be written $A\text{-}\lim\limits_{i,k\to\infty} x_{ik}$ or simply $A_{.}(x)$. The method A will be said to *fulfil the condition*

(p_0) *if it transforms every sequence perfectly convergent to 0 into a sequence convergent to 0.*

For the methods under consideration the following statements analogous to the conditions of section 1 are valid:

(i) *The method A transforms every sequence perfectly convergent to 0 into a convergent one if and only if the following conditions are satisfied:*

(i_1) $$\sup_{i=0,1,\ldots}\sum_{\mu,\nu=0}^{\infty}|a_{i\mu\nu}|<\infty,$$

(i_2) *there exists* $\lim\limits_{i\to\infty} a_{i\mu\nu}=a_{.\mu\nu}$ *for* $\mu,\nu=0,1,\ldots$

(ii) *The method A satisfies the condition* (p_0) *if and only if the conditions* (i_1) *and* (i_2) *are satisfied and* $a_{.\mu\nu}=0$.

(iii) *The method A transforms every sequence regularly convergent into a convergent one if and only if the conditions* (i_1), (i_2), *and the following are satisfied*

(iii_1) *there exist*

$$\lim_{i\to\infty}\sum_{\mu=0}^{\infty}a_{i\mu\nu}=S_{..\nu},\qquad \lim_{i\to\infty}\sum_{\nu=0}^{\infty}a_{i\mu\nu}=S_{.\mu.},\qquad \lim_{i\to\infty}\sum_{\mu,\nu=0}^{\infty}a_{i\mu\nu}=S_{...}.$$

(iv) *The method A transforms every regularly convergent sequence into a sequence convergent to the same limit if and only if the conditions* (i_1), (i_2), (iii_1) *are satisfied and*

$$a_{.\mu\nu}=S_{..\nu}=S_{.\mu.}=0,\qquad S_{...}=1.$$

If the method A satisfies the conditions of (iii), then

$$A_{.}(x)=\sum_{\mu,\nu=0}^{\infty}a_{.\mu\nu}x_{\mu\nu}+\sum_{\mu=0}^{\infty}\Big(s_{.\mu.}-\sum_{\nu=0}^{\infty}a_{.\mu\nu}\Big)x_{\mu.}+\sum_{\nu=0}^{\infty}\Big(S_{..\nu}-\sum_{\mu=0}^{\infty}a_{.\mu\nu}\Big)x_{.\nu},$$

for every regularly convergent sequence. The number

$$c(A)=S_{...}-\sum_{\mu,\nu=0}^{\infty}a_{.\mu\nu}$$

will be termed *the characteristic of the method A*. Let $A'=(a'_{i\mu\nu})$ be another method satisfying the conditions of (iii); for this method denote the numbers defined by the conditions (i_2) and (iii_1) by $a'_{.\mu\nu}, S'_{..\nu}, S'_{.\mu.}, S'_{...}$ respectively. The methods A and A' are called *consistent for regularly convergent sequences* if $A_{.}(x)=A'_{.}(x)$ for every regularly convergent sequence. It may easily be shown that the methods A and A' are consistent for regularly convergent sequences if and only if $a_{.\mu\nu}=a'_{.\mu\nu}$, $S_{..\nu}=S'_{..\nu}$, $S_{.\mu.}=S'_{.\mu.}$, $S_{...}=S'_{...}$ for $\mu,\nu=0,1,\ldots$ In this case $c(A)=c(A')$.

Supposing that the method A has the property (p_0), we introduce in the space of bounded sequences A-summable to 0, which will be denoted by $\boldsymbol{P}'_b$, two norms

$$\|x\|=\sup_{i,k=0,1,\ldots}|x_{ik}|,\qquad \|x\|^*=\sum_{i,k=0}^{\infty}2^{-(i+k)}|x_{ik}|+\sup_{i=0,1,\ldots}|A_i(x)|.$$

Then by a method similar to that applied in section 2 we may prove the following

4.1. LEMMA. *Let the method A fulfil the condition* (p_0). *If the sequence $x=\{x_{ik}\}$ belongs to $\boldsymbol{P}'_b$, then for every $\varepsilon>0$ and n there exists a p and* *a sequence* $z = \{z_{ik}\}$ *satisfying the conditions* (1) *and* (2) *of* 2.3 *and such that*

$$|A_i(x) - A_i(z)| < \varepsilon \quad \textit{for} \quad i = 0, 1, \ldots$$

This lemma states that the sequences with a finite number of terms different from 0 form a dense set in the space P'_b with the two norm convergence; moreover, by the same proof as in section 2, we can show that the space P'_b fulfils the condition (n_3). Hence follows, as in section 3,

4.2. THEOREM. *Let the methods A and B satisfy the condition* (p_0). *If every bounded sequence x, A-summable to* 0 *is B-summable, then it is B-summable to* 0.

The following theorem requires a separate proof.

4.3. THEOREM. *Let $c(A) \neq 0$, and let the methods A and B be consistent for regularly convergent sequences. If every bounded A-summable sequence is B-summable, then* $A\text{-}\lim\limits_{i,k\to\infty} x_{ik} = B\text{-}\lim\limits_{i,k\to\infty} x_{ik}$.

Proof. The methods A' and B', defined by the matrices

$$A' = (a_{i\mu\nu} - a_{.\mu\nu}), \qquad B' = (b_{i\mu\nu} - b_{.\mu\nu})$$

transform every sequence perfectly convergent to 0 into a sequence convergent to 0, and every bounded A'-summable sequence is B'-summable. Write

$$r = \frac{1}{c(A)}\Big[A_.(x) - \sum_{\mu,\nu=0}^{\infty} a_{.\mu\nu} x_{\mu\nu}\Big].$$

The sequence $x'_{ik} = \{x_{ik} - r\}$ is A'-summable to 0, for

$$A_.(x') = A_.(x) - \sum_{\mu,\nu=0}^{\infty} a_{.\mu\nu} x_{\mu\nu} - rc(A).$$

By theorem 4.2 $B'_.(x') = 0$; since $B(x) = A(x)$ for regularly convergent sequences, $c(A) = c(B)$ and $a_{.\mu\nu} = b_{.\mu\nu}$, whence

$$\begin{aligned} 0 = B'_.(x') &= B_.(x) - \sum_{\mu,\nu=0}^{\infty} b_{.\mu\nu} x_{\mu\nu} - rc(B) \\ &= B_.(x) - \sum_{\mu,\nu=0}^{\infty} a_{.\mu\nu} x_{\mu\nu} - rc(A) = B_.(x) - A_.(x). \end{aligned}$$

By the method indicated in section 3 we can prove the following generalization of Theorem 4.1

4.4. THEOREM. *Let $c(A_0) \neq 0$ and let every regularly convergent sequence $x = \{x_{ik}\}$ be summable to the same value by all the methods $B, A_0, A_1, \ldots$ If every bounded sequence A_n-summable to the same value for $n = 1, 2, \ldots$ is B-summable, then* $A_n\text{-}\lim\limits_{i,k\to\infty} x_{ik} = B\text{-}\lim\limits_{i,k\to\infty} x_{ik}$ *for* $n = 0, 1, \ldots$

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