On Dirichlet series with complex exponents

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1. We shall consider the series

$$a_1 \exp(-\beta_1 z) + a_2 \exp(-\beta_2 z) + \ldots \tag{1}$$

where the coefficients a_n, the exponents β_n and the variable z are complex. We shall assume throughout this paper that

$$\lim(\beta'_n/\log n) = \infty, \tag{2}$$

denoting by β'_n the real part of β_n.

2. Consider first the particular series

$$\exp(-\beta_1 x) + \exp(-\beta_2 x) + \ldots \tag{3}$$

where x is real. We put $k_n = \beta'_n/\log n$, then the series of absolute values $|\exp(-\beta_1 x)| + |\exp(-\beta_2 x)| + \ldots$ can be written in the form

$$|\exp(-\beta_1 x)| + 2^{-k_2 x} + 3^{-k_3 x} + \ldots$$

which makes it evident that (3) converges absolutely for $x>0$.

3. This allows us to prove that *if*

$$x_0 = \overline{\lim}\,(\log|a_n|/\beta'_n), \tag{4}$$

the series (1) *converges absolutely for real* $z=x>x_0$.

Let $x_0<x_1<x_2\leqslant x$. By (2) and (4) there is a number M such that $|a_n|\exp(-\beta'_n x) < M$. We have $|a_n \exp(-\beta_n x)| \leqslant M \exp[-\beta'_n(x_2-x_1)]$ and the majorant

$$M(\exp[-\beta'_1(x_2-x_1)] + \exp[-\beta'_2(x_2-x_1)] + \ldots)$$

is convergent. Thus we have proved even more, namely that the convergence is uniform in any interval $x_2 \leqslant x < \infty$, where $x_2 > x_0$.

4. It is also easy to see that (1) diverges for all real $z=x<x_0$. In fact, in this case it follows from (4) that there is an increasing sequence of positive integers $n_1, n_2, \ldots$ such that $|a_n \exp(-\beta_n x)| = |a_n|\exp(-\beta'_n x) > 1$ for $n=n_1, n_2, \ldots$, which implies the divergence.

5. Now suppose that

$$\delta_1 = \underline{\lim}\, \arg\beta_n > -\pi/2 \quad \text{and} \quad \delta_2 = \overline{\lim}\, \arg\beta_n < \pi/2.$$

It is easy to see that in this case assumption (2) is equivalent to

$$\lim(|\beta_n|/\log n) = \infty.$$

We restrict ourselves to the case

$$-\delta_1 = \delta_2 = \delta < \pi/2,$$

which gives the symmetry with respect to the real axis and simplifies further considerations. There is no loss of generality, because the general case can be reduced to this by a proper rotation of the plane of z.

We are going to prove that if (1) converges at a point z_0, it does so absolutely at any point $z=z_0+\varrho \exp i\Theta$, where $\varrho>0$ and $|\Theta|<\pi/2-\delta$.

In fact, for these values of z the series (1) takes the form

$$\bar{a}_1 \exp(-\bar{\beta}_1 \varrho) + \bar{a}_2 \exp(-\bar{\beta}_2 \varrho) + \ldots, \tag{5}$$

where $\bar{a}_n = a_n \exp(-\beta_n z_0)$ and $\bar{\beta}_n = \beta_n \exp i\Theta$.

From the convergence of (1) at z it follows that $|a_n \exp(-\beta_n z_0)| < M$. On the other hand, we have

$$\underline{\lim}\, \arg\bar{\beta}_n = -\delta + \Theta > -\pi/2 \quad \text{and} \quad \overline{\lim}\, \arg\bar{\beta}_n = \delta + \Theta < \pi/2$$

and $\lim(|\bar{\beta}_n|/\log n) = \infty$; thus also, for $\bar{\beta}'_n = R(\bar{\beta}_n)$, $\lim(\bar{\beta}'_n/\log n) = \infty$.

Hence the series

$$M(\exp(-\bar{\beta}'_1 \varrho) + \exp(-\bar{\beta}'_2 \varrho) + \ldots)$$

is convergent and (5) is absolutely convergent.

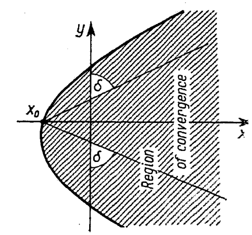

Fig. 1

6. From the last property it follows that the region of convergence D of series (1) is connected and that the convergence is absolute at any interior point of D. Moreover, if $x_0=-\infty$, D is the whole plane of the variable z. If $-\infty<x_0<\infty$, D contains at least the angular region

$$z = x_0 + \varrho \exp i\Theta \qquad (\varrho>0,\ |\Theta|<\pi/2-\delta); \tag{6}$$

the point $z=x_0$ lies on the boundary of D. Finally, if $x_0=\infty$, D is empty and the series diverges at any point z.

We shall prove that the interior of D is a convex set. It suffices to show that if z_1 and z_2 are interior points of D, the series (1) converges on the segment $z=tz_1+(1-t)z_2$ $(0<t<1)$. We have $R(\beta_n z) = tR(\beta_n z_1)+$

$+(1-t)R(\beta_n z_2)$. Hence the modulus $|\exp(-\beta_n z)|$ lies between the moduli $|\exp(-\beta_n z_1)|$, $|\exp(-\beta_n z_2)|$, and the (absolute) convergence of (1) follows from the absolute convergence of the two series

$$a_1\exp(-\beta_1 z_1)+a_2\exp(-\beta_2 z_1)+\ldots$$

and

$$a_1\exp(-\beta_1 z_2)+a_2\exp(-\beta_2 z_2)+\ldots$$

7. We can find an explicit formula for the boundary of D. If $z=x+iy$ series (1) can be written in the form

$$\gamma_1\exp(-\beta_1 x)+\gamma_2\exp(-\beta_2 x)+\ldots,$$

where $\gamma_n=a_n\exp(-i\beta_n y)$. Since $\log|\gamma_n|=\log|a_n|+\beta_n'' y$ (β_n'' imaginary part of β_n), we find, by use of (4), the following equation for the boundary line of D:

$$x=\overline{\lim}\left(\frac{\log|a_n|}{\beta_n'}+\frac{\beta_n''}{\beta_n'}y\right).$$

From this formula it may also easily be deduced that D is a convex region containing tha angular region (6).

8. By refining the argument of section 5, we can prove that series (1) converges uniformly in each closed and bounded set in the interior of D. In fact, if $|\Theta|\leqslant\Theta_0<\pi/2-\delta$, we have

$$\underline{\lim}\arg\bar\beta_n>-\delta-\Theta_0>-\pi/2 \quad\text{and}\quad \overline{\lim}\arg\bar\beta_n<\delta+\Theta_0<\pi/2;$$

hence, for great values of n,

$$R(\bar\beta_n)>|\beta_n|\cos(\arg\beta_n)>|\beta_n|\varepsilon,$$

where $\varepsilon=[\cos(\delta+\Theta_0)]/2$. Thus, for great values of n, the terms of (5) are less (in absolute value) than the terms of the series

$$M(\exp(-\varepsilon|\beta_1|\varrho)+\exp(-\varepsilon|\beta_2|\varrho)+\ldots)$$

which converges for positive ϱ. This proves that (1) converges uniformly in any region

$$z=z_0+\varrho\exp i\Theta \qquad (0<\varrho_0<\varrho,\ |\Theta|<\Theta_0).$$

The uniform convergence in arbitrary closed and bounded sets in the interior of D follows by the Borel-Lebesgue theorem.

This implies that the sum of (1) is an analytic function inside D.

Problème aux limites de Poincaré généralisé

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1. Introduction. Le problème aux limites de Poincaré ([3], Chapitre X) consiste dans la recherche d'une fonction $u(x,y)$, harmonique à l'intérieur d'un domaine D, limité par une courbe fermée L, qui sur cette courbe satisfait à une relation linéaire

$$du/dn+a(s)u+b(s)du/ds=f(s) \tag{1}$$

entre les valeurs limites de la dérivée suivant la normale du/dn, de la dérivée tangentielle du/ds, et de la fonction u elle même; $a(s)$, $b(s)$, $f(s)$ sont les fonctions données de la longueur d'arc de la courbe L qui détermine la position du point sur la courbe L.

Le problème cité fut posé et résolu par Poincaré dans le cas particulier $a=0$ et sous la supposition que les fonctions données, $b(s)$, $f(s)$ et la ligne L, sont *analytiques*. L'auteur de ce travail [2] à résolu le problème pour le cas $a\neq 0$ mais sous la même supposition d'analyticité. Le problème a été résolu complétement, sous les suppositions plus générales que les fonctions $a(s)$, $b(s)$, $f(s)$ satisfont à la condition d'Hölder, par le mathématicien soviétique Hvedélidzé [1].

Dans ce travail nous nous proposons de résoudre le problème de la recherche d'une fonction harmonique $u(x,y)$ à l'intérieur du domaine D, qui en tout point (s) au bord L de ce domaine vérifie la relation généralisée suivante:

$$du/dn+a(s)u=\lambda F(s,u,du/ds), \tag{2}$$

où $F(s,u,v)$ est une fonction des trois variables, définie dans une certaine région et λ est un paramètre. On admet les suppositions suivantes:

I. La ligne fermée de Jordan L a la tangente *continue* en tout point et l'angle que fait cette tangente avec une direction fixe satisfait à la condition d'Hölder, c'est-à-dire qu'on a

$$|\delta_{ss_1}|\leqslant c_1|s-s_1|^{\gamma} \qquad (0<\gamma\leqslant 1), \tag{3}$$

où δ_{ss_1} désigne l'angle que font les tangentes aux deux points arbitraires de la courbe L aux coordonnées curvilignes s et s_1.