To complete the proof we shall show that the projection π of $\mathcal{Y}$ onto $\mathcal{X}$ is open (clearly π is continuous). It suffices to prove that $\pi(\mathcal{Y}\cdot(G\times V))=G$ for every open $G\subset\mathcal{X}$ and for every open interval $V\neq 0$, $V\subset\mathcal{R}$.

Obviously $\pi(\mathcal{Y}\cdot(G\times V))\subset G$. If $x\in G$, then, by (1), there is an $r\in R_x\cdot V$. Consequently $x\in\pi(\mathcal{Y}\cdot(G\times V))$, which yields $G\subset\pi(\mathcal{Y}\cdot(G\times V))$, q. e. d.

The problem whether every Hausdorff space with an enumerable basis is an interior image of a separable metric space is unsolved [22]).

Notice that (H) implies easily (H_0).

References

[1] Alexandroff-Hopf, *Topologie* I.

[2] Anderson, R. D., *On monotone interior mappings in the plane*, Trans Amer. Math. Soc. 73 (1932), p. 211-222.

[3] Birkhoff, G., *Lattice Theory*, 1948 (second edition), Foreword on Algebra.

[4] Eilenberg, S., *Sur les transformations d'espaces métriques en circonférence*, Fund. Math. 24 (1935), p. 160-176.

[5] Каждан, Я., *Пример открытого отображения одномерного локальносвязного континуума на квадрат*, Доклады Ак. Наук СССР 56 (1947), стр. 339-342.

[6] Kolmogoroff, A., *Über offene Abbildungen*, Annals of Mathematics 38 (1937), p. 36-38.

[7] Kuratowski, C., *Topologie*, Vol. I, Warszawa 1952, troisième édition.

[8] Loomis, L. H., *On the representation of σ-complete Boolean algebras*, Bull. Am. Math. Soc. 53 (1947), p. 757-760.

[9] Sikorski, R., *Closure algebras*, Fund. Math. 36 (1949), p. 165-206.

[10] — *On the inducing of homomorphisms by mappings*, Fund. Math. 36 (1949), p. 7-22.

[11] — *On the representation of Boolean algebras as fields of sets*, Fund. Math. 35 (1948), p. 247-257.

[12] — *Products of abstract algebras*, Fund. Math. 39 (1952), p. 211-228.

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On existential theorems in non-classical functional calculi [1])

by

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Let $\mathcal{S}_\chi$ be the Heyting propositional calculus, and let $\mathcal{S}_\chi^*$ be the Heyting functional calculus. The individual variables of the system $\mathcal{S}_\chi^*$ will be denoted by $x_1, x_2, \ldots$, the quantifiers — by $\sum\limits_{x_p}$ and $\prod\limits_{x_p}$. The formulas from $\mathcal{S}_\chi^*$ will be denoted by the letters α, β. If α is a formula from $\mathcal{S}_\chi^*$, then $\alpha\binom{x_q}{x_p}$ denotes the formula obtained from α by replacing each free occurrence of x_p by x_q (each bound occurrence of x_q should be replaced earlier by x_l which does not appear in α, $l\neq q$).

Gödel [2]) formulated (without proof) the following theorem:

(χ_0) *Let σ, τ be two formulas from the Heyting propositional calculus $\mathcal{S}_\chi$. If the disjunction $\sigma+\tau$ is a theorem of $\mathcal{S}_\chi$, then either σ or τ is a theorem of $\mathcal{S}_\chi$.*

Theorem (χ_0) was later proved by McKinsey and Tarski [2] by an algebraical method. Another algebraical proof was given by Rieger [3]).

The purpose of this paper is to prove the following theorem (χ) which is an extension of (χ_0) over the Heyting functional calculus $\mathcal{S}_\chi^*$. The second part of Theorem (χ) shows that the Heyting functional calculus is the well formalization of Brouwer's ideas concerning existential theorems.

(χ) *If the formula $\alpha+\beta$ is provable in $\mathcal{S}_\chi^*$, then either α or β are provable in $\mathcal{S}_\chi^*$. If the formula $\sum\limits_{x_p}\alpha$ is provable in $\mathcal{S}_\chi^*$, then there is a positive integer q such that the formula $\alpha\binom{x_q}{x_p}$ is provable in $\mathcal{S}_\chi^*$.*

Clearly if the sequence $x_{i_1},\ldots,x_{i_n}$ contains all the free variables which appear in α, the integer q can be chosen among the numbers $i_1,\ldots,i_n$. If α contains no free variable, then q is an arbitrary integer, *e. g.* $q=p$.

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[1]) Presented at the Seminar on Foundations of Mathematics in the Mathematical Institute of the Polish Academy of Sciences in November 1952.

[2]) See K. Gödel [1]. See also G. Gentzen [1].

[3]) See L. Rieger [1], p. 29.

We notice that

(χ^*) *Each formula from $\mathcal{S}_\chi^*$ without quantifiers is provable in $\mathcal{S}_\chi^*$ if and only if it is a substitution of a theorem of the Heyting sentential calculus $\mathcal{S}_\chi$.*

Since the calculus $\mathcal{S}_\chi$ is decidable [4]), we infer from (χ) and (χ^*) that

(χ') *Each formula β from $\mathcal{S}_\chi^*$ of the form*

$$\beta = \overset{(1)}{\underset{x_{p_1}}{\Xi}} \overset{(2)}{\underset{x_{p_2}}{\Xi}} \dots \overset{(n)}{\underset{x_{p_n}}{\Xi}} \alpha \tag{+}$$

where α contains no quantifier and $\overset{(i)}{\Xi}$ is either the sign $\sum$ or $\prod$ $(i=1,\dots,n)$, is decidable.

Theorem (χ) is a simple application of the results obtained in our paper on *Algebraic Treatment of the Notion of Satisfiability* [5]) cited hereafter as [AT]. The main idea of the proof (the extension of the space X to the space X_0 — see p. 24) is essentially due to McKinsey and Tarski [4].

The knowledge of [AT] is assumed in the sequel. Terminology and notation are the same as in [AT], therefore they will not be explained here [6]).

Theorems analogous to (χ) hold also for the other non-classical functional calculi examined in [AT]. More exactly, we shall prove the following theorems where α and β denote formulas from the functional calculus under consideration:

(π) *If the formula $\alpha+\beta$ is provable in the positive functional calculus [7]) $\mathcal{S}_\pi^*$, then either α or β is provable in $\mathcal{S}_\pi^*$. If the formula $\sum\limits_{x_p} \alpha$ is provable in $\mathcal{S}_\pi^*$, then there is an integer q such that $\alpha\binom{x_q}{x_p}$ is provable in $\mathcal{S}_\pi^*$.*

(μ) *If the formula $\alpha+\beta$ is provable in the minimal functional calculus [8]) $\mathcal{S}_\mu^*$, then either α or β is provable in $\mathcal{S}_\mu^*$. If the formula $\sum\limits_{x_p} \alpha$ is provable in $\mathcal{S}_\mu^*$, then there is an integer q such that $\alpha\binom{x_q}{x_p}$ is provable in $\mathcal{S}_\mu^*$.*

(ν) *If the formula $\alpha+\beta$ is provable in the functional calculus $\mathcal{S}_\nu^*$ [9]), then either α or β is provable in $\mathcal{S}_\nu^*$. If the formula $\sum\limits_{x_p} \alpha$ is provable in $\mathcal{S}_\nu^*$, then there is an integer q such that $\alpha\binom{x_q}{x_p}$ is provable in $\mathcal{S}_\nu^*$.*

[4]) See *e. g.* S. Jaśkowski [1], G. Gentzen [1], M. Wajsberg [1], J. C. C. McKinsey and A. Tarski [3], L. Rieger [1].

[5]) See References at the end of this paper. Theorem (χ) can be also deduced from a fundamental theorem of G. Gentzen [1].

[6]) The Heyting functional calculus is exactly described in [AT] § 11.

[7]) [AT] § 12.

[8]) [AT] § 13.

[9]) [AT] § 14.

(λ) *If the formula $\mathbf{I}\alpha+\mathbf{I}\beta$ is provable in the Lewis functional calculus [10]) $\mathcal{S}_\lambda^*$, then either α or β is provable in $\mathcal{S}_\lambda^*$. If the formula $\sum\limits_{x_p} \mathbf{I}\alpha$ is provable in $\mathcal{S}_\lambda^*$, then there is an integer q such that $\alpha\binom{x_q}{x_p}$ is provable in $\mathcal{S}_\lambda^*$.*

Notice that the formulation of Theorem (λ) is somewhat different from the formulation of Theorems (χ), (π), (μ), (ν).

Theorems (χ), (π), (μ), (ν) can also be formulated in a purely algebraical way. The first part of these theorems asserts that the class R of all non-provable formulas forms an ideal I in the corresponding Lindenbaum algebra [11]) L_i $(i=\chi,\pi,\mu,\nu)$. The second part asserts that the ideal I is enumerably additive in the following sense: if all components of an infinite sum corresponding to the quantifier $\sum\limits_{x_p}$ (*i. e.* of the sum [AT] 4.3 (*)) are in I, then the sum also belongs to I. Clearly I is the unique maximal ideal of L_i.

Clearly theorems analogous to (χ_0) are also true for the positive propositional calculus $\mathcal{S}_\pi$, the minimal propositional calculus $\mathcal{S}_\mu$, the propositional calculus $\mathcal{S}_\nu$, and for the Lewis propositional calculus $\mathcal{S}_\lambda$ [12]). In the case of $\mathcal{S}_\lambda$, $\sigma+\tau$ should be replaced by $\mathbf{I}\sigma+\mathbf{I}\tau$.

Theorem (χ^*) is a particular case of the following general theorem.

(*) *Let $\mathcal{S}$ be the propositional calculus described in* [AT] § 1, *and let $\mathcal{S}^*$ be the functional calculus determined by $\mathcal{S}$* ([AT] § 2). *Suppose that $\mathcal{S}$ has the extension property* (E) [13]). *A formula $\alpha \in S^*$ without quantifiers is provable in $\mathcal{S}^*$ if and only if α is a substitution of a theorem of the propositional calculus $\mathcal{S}$.*

In particular, Theorem (*) is true for all the systems examined in Part II of [AT]. The hypothesis that $\mathcal{S}$ has the property (E) seems to be inessential.

Since each formula from the Lewis propositional calculus $\mathcal{S}_\lambda$ is decidable [14]), we infer from (λ) and (*) that

[10]) [AT] § 10.

[11]) See [AT], p. 81, 85, 88, 90, 92.

[12]) In the case of $\mathcal{S}_\pi$ this theorem follows from the fact that a formula $\sigma \in \mathcal{S}_\pi$ is provable in $\mathcal{S}_\pi$ if and only if it is provable in $\mathcal{S}_\chi$ (see Hilbert-Bernays [1], p. 450). The proof of this theorem in the case of $\mathcal{S}_\mu$ and $\mathcal{S}_\nu$ is similar to that of $\mathcal{S}_\chi$. See McKinsey-Tarski [2] and [3]. It is based on the algebraic interpretation mentioned in [AT] §§ 3, 13, 14. For the proof of this theorem in the case of $\mathcal{S}_\lambda$, see McKinsey-Tarski [2].

[13]) See [AT], p. 69.

[14]) McKinsey [1].

(λ') *Each formula β from the Lewis functional calculus $\mathcal{S}_\chi^*$, such that*

$$\beta = \overset{(1)}{\underset{x_{p_1}}{\Xi}}\ \overset{(2)}{\underset{x_{p_2}}{\Xi}} \cdots \overset{(n)}{\underset{x_{p_n}}{\Xi}} \alpha$$

where α contains no quantifier and $\overset{(i)}{\Xi}$ is either the sign $\sum$ or $\prod$ $(i=1,\ldots,n)$, is decidable.

Since a formula α from the functional calculus $\mathcal{S}_\pi^*$ is provable in $\mathcal{S}_\pi^*$ if and only if α is provable in $\mathcal{S}_\chi^*$ [15]), and since each formula from the sentential calculus $\mathcal{S}_\pi$ is a theorem of $\mathcal{S}_\pi$ if and only if it is a theorem in $\mathcal{S}_\chi$ [16]), we infer that Theorem (χ') remains true if we replace $\mathcal{S}_\chi^*$ by $\mathcal{S}_\pi^*$.

Proof of (χ). By [AT] 11.2 there is a topological space X such that the Heyting algebra [17]) $\boldsymbol{H}(X)$ of all open subsets of X is an $\mathcal{S}_\chi^*$-extension [18]) of the Lindenbaum algebra L_χ [19]), *i. e.* there exists an $\mathcal{S}_\chi^*$-isomorphism [20]) h of L_χ into $\boldsymbol{H}(X)$.

Let x_0 be a fixed element, x_0 non ϵ X, and let $X_0 = X + (x_0)$. We shall treat the set X_0 as a topological space with the following definition of topology: open subsets of X_0 are open subsets of X and the whole space X_0.

There is exactly one open subset of X_0 which contains the element x_0, viz. the whole space X_0. Consequently

(s) if $G_k \epsilon \boldsymbol{H}(X_0)$ and $X_0 = \sum_k G_k$, then there is a q such that $G_q = X_0$.

Since X is the open subset of X_0, the formula

$$g(A) = AX \quad \text{for} \quad A \epsilon \boldsymbol{H}(X_0)$$

defines an $\mathcal{S}_\chi$-homomorphism [21]) of the $\mathcal{S}_\chi^*$-algebra $\boldsymbol{H}(X_0)$ onto the $\mathcal{S}_\chi^*$-algebra $\boldsymbol{H}(X)$. The homomorphism g preserves all infinite sums and products.

Let $\varphi_m^k \epsilon \boldsymbol{F}^k\big(I_0, \boldsymbol{H}(X_0)\big)$ [22]) $(k, m = 1, 2, \ldots)$ be the mapping defined as follows:

$$\varphi_m^k(i_1, \ldots, i_k) = h\big(|F_m^k(x_{i_1}, \ldots, x_{i_k})|\big) \quad \text{for} \quad i_1, \ldots, i_k \epsilon I_0.$$

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[15]) [AT], Theorem 15.5.
[16]) See Hilbert-Bernays [1], p. 450.
[17]) [AT], p. 85.
[18]) [AT], p. 71.
[19]) [AT], p. 85.
[20]) [AT], p. 70.
[21]) [AT], p. 68.
[22]) See [AT], p. 71. I_0 always denotes the set of all positive integers (see [AT], p. 65).

Suppose that the formula $\alpha + \beta \epsilon S_\chi^*$ is provable in $\mathcal{S}_\chi^*$. By [AT] 5.1 and 5.3 [23])

$$\big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_\alpha(\{i\}, \{\varphi_m^k\}) + \big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_\beta(\{i\}, \{\varphi_m^k\})$$
$$= \big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_{\alpha+\beta}(\{i\}, \{\varphi_m^k\}) = X_0 = \text{the unit of } \boldsymbol{H}(X_0).$$

By (s) either

(a) $$\big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_\alpha(\{i\}, \{\varphi_m^k\}) = X_0,$$
or
(b) $$\big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_\beta(\{i\}, \{\varphi_m^k\}) = X_0.$$

Suppose that, for instance, the equation (a) is true. Since $g\varphi_m^k = \varphi_m^k$ we have by [AT] 5.5 and 5.2

$$X = g(X_0) = g\Big(\big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_\alpha(\{i\}, \{\varphi_m^k\})\Big) = \big(I_0, \boldsymbol{H}(X)\big)\, \Phi_\alpha(\{i\}, \{\varphi_m^k\}) = h(|\alpha|).$$

Since h is an isomorphism, we infer that $|\alpha|$ is the unit element of L_χ, *i. e.* that α is provable in $\mathcal{S}_\chi^*$ (see [AT] 4.2).

The first part of (χ) is proved. The proof of the second part is similar.

Suppose that $\beta = \sum_{x_p} \alpha \epsilon S_\chi^*$ is provable in $\mathcal{S}_\chi^*$. By [AT] 5.1 and 5.3

$$\sum_{x_p \epsilon I_0} \big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_\alpha(\{i\}', \{\varphi_m^k\}) = \big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_\beta(\{i\}, \{\varphi_m^k\})$$
$$= X_0 = \text{the unit of } \boldsymbol{H}(X_0),$$

where $\{i\}'$ denotes the sequence $\{i\}$ where the p-th term is replaced by x_p.

By (s) there is an integer q such that the q-th component of the sum on the left side is equal to X_0, *i. e.*

$$\big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_\alpha(\{\bar{j}_i\}, \{\varphi_m^k\}) = X_0$$

where $\bar{j}_i = i$ for $i \neq p$ and $\bar{j}_p = q$.

Let $\delta = \alpha\binom{x_q}{x_p}$. By [AT] 5.4

$$\big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_\delta(\{i\}, \{\varphi_m^k\}) = \big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_\alpha(\{\bar{j}_i\}, \{\varphi_m^k\}) = X_0.$$

Since $g\varphi_m^k = \varphi_m^k$ we obtain from [AT] 5.5 and 5.2 that

$$X = g(X_0) = g\Big(\big(I_0, \boldsymbol{H}(X_0)\big)\, \Phi_\delta(\{i\}, \{\varphi_m^k\})\Big) = \big(I_0, \boldsymbol{H}(X)\big)\, \Phi_\delta(\{i\}, \{\varphi_m^k\}) = h(|\delta|).$$

Since h is an isomorphism, we infer that $|\delta|$ is the unit of L_χ, *i. e.* that $\delta = \alpha\binom{x_q}{x_p}$ is provable in $\mathcal{S}_\chi^*$ (see [AT] 4.2).

Proof of (π) is completely analogous to that of (χ).

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[23]) $\{i\}$ is the sequence of all positive integers.

Proof of (μ) is analogous to that of (χ). Only the following supplement is needed.

Let $B = \sim X$ in the $\mathcal{S}_\mu^*$-algebra $\boldsymbol{H}(X)$. We define the operation $\sim$ in the $\mathcal{S}_\mu^*$-algebra $\boldsymbol{H}(X_0)$ by the formula

$$\sim A = A \to B \quad \text{for} \quad A \in \boldsymbol{H}(X_0)$$

where $\to$ on the right side is taken in $\boldsymbol{H}(X_0)$ $\big($not in $\boldsymbol{H}(X)\big)$. Clearly g is an $\mathcal{S}_\mu$-homomorphism of the $\mathcal{S}_\mu^*$-algebra $\boldsymbol{H}(X_0)$ onto the $\mathcal{S}_\mu^*$-algebra $\boldsymbol{H}(X)$.

Proof of (ν) is analogous to that of (χ). Only the following supplement is needed.

We define the operation $\dot{-}$ in $\boldsymbol{H}(X_0)$ as follows: if $A \in \boldsymbol{H}(X_0)$, then $\dot{-}A$ $\big($in the $\mathcal{S}_\nu^*$-algebra $\boldsymbol{H}(X_0)\big)$ is the set $\dot{-}(AX)$ where $\dot{-}$ is taken in the $\mathcal{S}_\nu^*$-algebra $\boldsymbol{H}(X)$. Clearly g is an $\mathcal{S}_\nu$-homomorphism of the $\mathcal{S}_\nu^*$-algebra $\boldsymbol{H}(X_0)$ onto $\boldsymbol{H}(X)$.

Proof of (λ) is also analogous to that of (χ). Instead of $\boldsymbol{H}(X)$ and $\boldsymbol{H}(X_0)$ we should write everywhere $\boldsymbol{C}(X)$ and $\boldsymbol{C}(X_0)$ respectively. The mapping g is an $\mathcal{S}_\lambda$-homomorphism of $\boldsymbol{C}(X_0)$ onto $\boldsymbol{C}(X)$ since X is open in X_0 [24]).

Remarks. Theorem (π) can be directly deduced from Theorem (χ) and [AT] 15.5. Theorem (χ) follows directly from (λ) and [AT] 15.2.

In the proof of (χ) we can replace $\boldsymbol{H}(X)$ by any other $\mathcal{S}_\chi^*$-extension H of L_χ, which need not be the Heyting algebra of all open subsets of a topological space. The Heyting algebra corresponding to $\boldsymbol{H}(X_0)$ must then be somewhat differently defined [25]).

Proof of (*). Let $\mathcal{L}$ be Lindenbaum algebra constructed for the system $\mathcal{S}$. The exact definition of $\mathcal{L}$ is completely analogous to the Lindenbaum algebra $L(R)$ described in [AT] § 4 (where R= the empty set). Elements of $\mathcal{L}$ are classes $\|\sigma\|$ ($\sigma \in S$) of equivalent formulas, *i. e.* $\tau \in \|\sigma\|$ if and only if the formula $\tau \equiv \sigma$ is a theorem of $\mathcal{S}$ ($\tau, \sigma \in S$). The definition of algebraic operations is the same as in [AT] § 4.

Clearly $\mathcal{L}$ is an $\mathcal{S}$-algebra, and $\|\sigma\|$ is the unit element of $\mathcal{L}$ if and only if σ is a theorem of $\mathcal{S}$.

Let $\mathcal{L}^*$ be an $\mathcal{S}^*$-algebra containing $\mathcal{L}$ as a subalgebra [26]).

Let ψ be a one-to-one transformation of the set of all primitive formulas $F_m^k(x_{i_1}, \ldots, x_{i_k})$ onto the set $(a_1, a_2, \ldots)$ of all sentential variables (see [AT] §§ 1-2). If $\alpha \in S^*$ contains no quantifier, we shall denote by σ_α the formula obtained from α by replacing formulas $F_m^k(x_{i_1}, \ldots, x_{i_k})$ by the sentential variables $\psi(F_m^k(x_{i_1}, \ldots, x_{i_k}))$ respectively. Clearly $\sigma_\alpha \in S$ and α is a substitution of σ_α.

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[24]) Sikorski [1].

[25]) See McKinsey-Tarski [3].

[26]) The unit of $\mathcal{L}^*$ is the unit of $\mathcal{L}$.

Now let $\varphi_m^k \in \boldsymbol{F}^k(I_0, \mathcal{L}^*)$ ($k, m = 1, 2, \ldots$) be defined as follows:

$$\varphi_m^k(i_1, \ldots, i_k) = \|\psi(F_m^k(x_{i_1}, \ldots, x_{i_k}))\| \quad \text{for} \quad i_1, \ldots, i_k \in I_0.$$

It is easy to prove by induction with respect to the length of α that

$$(I_0, \mathcal{L}^*)\,\Phi_\alpha(\{i\}, \{\varphi_m^k\}) = \|\sigma_\alpha\|.$$

Consequently, if α is provable in $\mathcal{S}^*$, then $\|\sigma_\alpha\|$ is the unit of $\mathcal{L}$ ([AT] 5.3), *i. e.* σ_α is a theorem of $\mathcal{S}$.

Proof of (χ'). The expression $\alpha\binom{x_p}{x_k}$ described in [AT] § 4 (p. 69-70) was not uniquely determined. However, it will be uniquely determined if, for instance, we require the integer l (see the definition in [AT], p. 70) to be the least positive integer such that $l \neq p$ and α contains neither x_l nor $\sum\limits_{x_l}$ nor $\prod\limits_{x_l}$. In the sequel we assume the definition of $\alpha\binom{x_p}{x_k}$ from [AT] § 4 with the above correction. Hence $\alpha\binom{x_p}{x_k}$ is uniquely determined.

If $\beta = \sum\limits_{x_p} \alpha \in S_\chi^*$, then we shall denote by $Z(\beta)$ the set of all formulas $\alpha\binom{x_q}{x_p}$ where either $q = p$ or α contains at least one occurrence of x_q.

If $\beta = \prod\limits_{x_p} \alpha \in S_\chi^*$, then $Z(\beta)$ is the set containing only one element: the formula α.

More generally, if $R \subset S_\chi^*$ is a set of formulas β of the form

$$\beta = \sum_{x_p} \alpha \qquad \text{or} \qquad \beta = \prod_{x_p} \alpha,$$

then $Z(R)$ is the union of all sets $Z(\beta)$ where $\beta \in R$. Clearly $Z(R)$ is finite if R is finite, and the number of elements of $Z(R)$ can easily be estimated from the above.

Suppose now that $\beta \in S_\chi^*$ is a formula of the form (+). Let $R_1 = Z(\beta)$ and, by induction, $R_k = Z(R_{k-1})$ ($k = 2, \ldots, n$). It follows from (χ) that β is provable if and only if R_1 contains at least one provable formula. By an easy induction with respect to k, we find that β is provable if and only if R_k contains at least one provable formula. Consequently β is provable if and only if R_n contains at least one provable formula. However, all formulas in R_n contain no quantifier. Hence, by (χ^*), we can decide by a finite method whether there is a provable formulas in R_n.

Proof of (λ') is completely analogous to that of (χ').

References

G. Gentzen [1], *Untersuchungen über das logische Schließen*, Math. Zeitschrift 39 (1934), p. 176-210, 405-431.

K. Gödel [1], *Eine Interpretation des intuitionistischen Aussagenkalküls*, Erg. Koll. Wien, 4 (1933), p. 39-40.

D. Hilbert und P. Bernays [1], *Grundlagen der Mathematik*, Band II, Berlin 1939.

S. Jaśkowski [1], *Recherches sur le système de la logique intuitioniste*, Actes du Congrès International de Philosophie Scientifique, VI Philosophie des mathématiques, Actualités scientifiques et industrielles 393 (1936), p. 58-61.

J. C. C. McKinsey [1], *A Solution of the Decision Problem for the Lewis Systems S_2 and S_4, with an Application to Topology*, The Journal of Symbolic Logic 6 (1941), p. 117-134.

J. C. C. Mc Kinsey and A. Tarski [2], *Some theorems about the sentential calculi of Lewis and Heyting*, The Journal of Symbolic Logic 13 (1948), p. 1-15.

J. C. C. Mc Kinsey and Tarski [3], *On closed elements in closure algebras*, Annals of Math. 47 (1946), p. 122-162.

J. C. C. Mc Kinsey and Tarski [4], *The Algebra of Topology*, Annals of Math. 45 (1944), p. 141-191.

H. Rasiowa and R. Sikorski [AT], *Algebraic Treatment of the Notion of Satisfiability*, Fund. Math. 40 (1953), p. 62-95.

L. Rieger [1], *On the lattice theory of Brouwerian propositional logic*, Spisy vyd. prirod. fak. Univ. Karlovy 189 (1949), p. 1-40.

R. Sikorski [1], *Closure homomorphisms and interior mappings*, this volume, p. 12-20.

M. Wajsberg [1], *Untersuchungen über den Aussagenkalkül von A. Heyting*, Wiadomości Matematyczne 46 (1938), p. 45-101.

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Sur le phénomène de convergence de M. Sierpiński

par

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M. Sierpiński a démontré en 1928 le Théorème suivant:

(T) *Si* $\aleph_1 = 2^{\aleph_0}$, *il existe une suite convergente de fonctions* $f_n(x)$ $(n=1,2,\ldots)$, *définies pour* $0 \leqslant x \leqslant 1$, *qui converge non uniformément sur tout ensemble non dénombrable* [1]).

Nous appelons *phénomène de convergence de Sierpiński* la singularité qui se présente dans la thèse du Théorème (T), à savoir la convergence non uniforme dans *tout ensemble indénombrable* d'un certain champ de convergence.

Si l'on abandonne l'hypothèse du continu, la question de l'existence des singularités de ce genre reste ouverte; la méthode de M. Sierpiński, essentiellement liée à l'hypothèse du continu, ne donne aucun renseignement sur ce sujet.

L'étude approfondie du Théorème (T) et des problèmes qui s'y rattachent m'a conduit à mettre en évidence une singularité connexe dont l'existence dans les espaces d'une certaine puissance indénombrable peut être démontrée sans prémisses hypothétiques.

Dans la Note présente, je m'occupe de cette démonstration, puis je donne certaines applications du résultat acquis à la théorie de la mesure abstraite.

1. Préliminaires

Étant données deux suites infinies d'entiers positifs $a=(a_1,a_2,\ldots)$ et $b=(b_1,b_2,\ldots)$, convenons d'écrire

(ρ_1) $a \leqslant b$ si $a_i \leqslant b_i$ pour $i=1,2,\ldots$;

(ρ_2) $a \precsim b$ si l'on a $a_i \leqslant b_i$ à partir d'un certain indice $i=i_0$.

Nous dirons qu'une famille $\mathcal{F}$ de suites infinies d'entiers positifs est *bornée*, resp. *finalement bornée*, lorsqu'il existe une suite fixe d'entiers positifs qui majore toutes les suites de $\mathcal{F}$ au sens de la relation (ρ_1), resp. (ρ_2).

[1]) W. Sierpiński [1] et [2], p. 52, Proposition C_9.