From the definition of $b_0^{\varepsilon,\eta}$ and $b_n^{\varepsilon,\eta}(a,F)$ and by lemma 2, we infer that in the case of $F\subset S_{n+1}$

$$b_0^{\varepsilon,\eta}=b_n^{\varepsilon,\eta}(a,F)+1. \tag{17}$$

From (17), (1), (2), (15) and (16) we obtain the following

THEOREM. *If $a\in F=\overline{F}\subset S_{n+1}$, then the number of components $b_0(a,S_{n+1}-F)$ in which the set F decomposes the $(n+1)$-dimensional sphere S_{n+1} at the point a is determined by the local cohomotopy number $b_n(a,F)$ of F at the point a by the formula*

$$b_0(a,S_{n+1}-F)=b_n(a,F)+1. \tag{18}$$

Since the number $b_n(a,F)$ is topologically invariant, we obtain the following

COROLLARY. *The number of components $b_0(a,S_{n+1}-F)$ in which a closed set $F\subset S_{n+1}$ decomposes the $(n+1)$-dimensional sphere S_{n+1} at the point $a\in F$ is topologically invariant.*

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Effectiveness of the representation theory for Boolean algebras [1])

by

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Stone calls Fundamental Existence Proposition of Ideal Arithmetic the lemma according to which:

(I) *In every Boolean algebra there is a prime ideal.*

This lemma plays a chief part in the demonstration that

(R) *any Boolean algebra is isomorphic with a field of sets*,

which is the most important result of Stone's representation theory.

According to Stone, (R) is effectively equivalent [2]) to (I). It was a long time ago noticed [3]), that (I) holds only with the help of transfinite methods. All the known proofs of it (S. Ulam [4]), A. Tarski [5]), M. H. Stone [6])) are based upon the principle of choice (or well-ordering theorem). A problem arises, whether the proposition (I) is really dependent on the principle of choice, and especially whether some particular cases of that principle [7]) are the consequences of the above-mentioned proposition.

A partial solution of this problem is given by W. Sierpiński [8]). It is known that the result of (I), without the use of transfinite methods, is that in the field of all subsets of an arbitrary infinite set E, there exists a two-valued measure which vanishes for one-point sets. Sierpiński has proved that the existence of such a measure in the set of

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[1]) Presented to the Polish Mathematical Society, Warsaw Section, on May 12, 1950.

[2]) Cf. [8], Fund. Exist. Prop., p. 78; Theorem 67, p. 106 and Theorem 70, p. 110.

[3]) Cf. [9], p. 812.

[4]) [11].

[5]) [10], Lemma 1, p. 43.

[6]) [8], Theorem 63, p. 100.

[7]) By particular cases of the principle of choice we mean those forms of this principle in which the family of sets $\{M_t\}_{t\in T}$ is subject to some restrictions *e. g.* that every M_t is finite, or that it is a bicompact space, etc.

[8]) [7].

integers allows us to construct a non measurable function in the sense of Lebesgue. It proves that (I) is at least of the same degree of ineffectivity, as the existence of such a function.

In this paper we give a full solution of the above mentioned problem. We demonstrate that (I) is effectively equivalent to other theorems, among others to the theorem of consistent choice [9]). From this immediately results the principle of choice from finite sets, and even the ordering principle, therefore the problem is solved.

The problem whether the choice principle is independent of (I) is not discussed here; it remains open. We suppose that it shall have a positive solution.

Definitions and lemmas. A *Boolean algebra* is a set $\boldsymbol{A}$ of elements (denoted by $a,b,\ldots$) with three operations: addition $(a+b)$, multiplication $(a\cdot b)$ and complementation (a'), satisfying the well-known axioms. The relation $a+b=b$, denoted by $a\subset b$, partly orders the set $\boldsymbol{A}$. This partial order has the least and the greatest element in $\boldsymbol{A}$; those elements are denoted by 0 and $|\boldsymbol{A}|$ respectively. If X is a subset of $\boldsymbol{A}$, then the smallest subalgebra of $\boldsymbol{A}$ containing X exists; it is denoted by $[X]_0$. The fields of sets are examples of Boolean algebras. An *s-ideal* [*d-ideal*] of the algebra $\boldsymbol{A}$ is a proper subset J of $\boldsymbol{A}$ satisfying the following condition:

$$a+b\in J \quad \text{if and only if} \quad a,b\in J$$
$$[a\cdot b\in J \quad \text{if and only if} \quad a,b\in J].$$

An s-ideal [d-ideal] J_p of $\boldsymbol{A}$ is called *prime* if, additionally, for every $a\in\boldsymbol{A}$ either $a\in J_p$ or $a'\in J_p$.

If J_p is a prime s-ideal [d-ideal] of $\boldsymbol{A}$, then $\boldsymbol{A}-J_p$ is a prime d-ideal [s-ideal] of $\boldsymbol{A}$. The following lemma is obvious [10]).

LEMMA 1. *If* $X\subset\boldsymbol{A}$ *and* $a_1+a_2+\ldots+a_n\neq|\boldsymbol{A}|$ $[a_1\cdot a_2\cdot\ldots\cdot a_n\neq 0]$ *for each* $a_1,a_2,\ldots,a_n\in X$, *then there exists an s-ideal [d-ideal] J of* $\boldsymbol{A}$, *which includes* X.

By a *measure* in the Boolean algebra $\boldsymbol{A}$ we mean a non-negative real function μ on $\boldsymbol{A}$ which is additive (*i. e.* $\mu(a+b)=\mu(a)+\mu(b)$ for $a\cdot b=0$, $a,b\in\boldsymbol{A}$) and $\mu(|\boldsymbol{A}|)=1$. If $X\subset\boldsymbol{A}$, f is a function on X, μ a measure in $\boldsymbol{A}$ and $f(a)=\mu(a)$ for $a\in X$, then μ is called *extension* of f from X to $\boldsymbol{A}$.

The measure μ in $\boldsymbol{A}$ is *two-valued* if $\mu(a)=1$ or $\mu(a)=0$ for $a\in\boldsymbol{A}$.

The following lemmas give a connection between prime ideals and two-valued measures.

[9]) [6], Theorem 2, p. 235-236.

[10]) See *e. g.* [1], Theorem 1, p. 22.

LEMMA 2. *If* μ *is a two-valued measure in* $\boldsymbol{A}$, *then* $\underset{x}{E}[\mu(x)=0]$ *is a prime s-ideal and* $\underset{x}{E}[\mu(x)=1]$ *a prime d-ideal of* $\boldsymbol{A}$.

LEMMA 3. *If* J_p *is a prime s-ideal, then the function*

$$\mu(x)=\begin{cases}1 & \text{for } x\in\boldsymbol{A}-J_p\\ 0 & \text{for } x\in J_p\end{cases} \quad \textit{is a measure in } \boldsymbol{A}.$$

These lemmas allow us to translate some propositions from the language of Boolean algebra to the language of the measure theory.

In this paper by a *topological space* we always mean a Hausdorff space (*i. e.* a space in which for all pairs of different points p_1,p_2, two exclusive neighbourhoods G_1 and G_2 exist, so that $p_1\in G_1$ and $p_2\in G_2$). The notions of *compact* ($=$*bicompact*) space and *product space* always have the usual meaning [11]).

If $\mathfrak{M}=\{M_t\}_{t\in T}$ is a family of topological spaces, then the relation $\sigma(p_1,p_2)$ defined for $p_1,p_2\in\sum_{t\in T}M_t$ is called *a relation of consistency* for the class $\mathfrak{M}$ if it is symmetrical $\big(i.\ e.\ \sigma(p_1,p_2)=\sigma(p_2,p_1)\big)$ and the set $\underset{\langle p_1,p_2\rangle}{E}[\sigma(p_1,p_2),\, p_1\in M_{t_1},\, p_2\in M_{t_2}]$ is closed in every product space of different spaces $M_{t_1},M_{t_2}\in\mathfrak{M}$.

A set $X\subset\sum_{t\in T}M_t$ is called a *partial choice-set* from sets of $\mathfrak{M}$, if $\overline{\overline{X\cdot M_t}}\leqslant 1$ for every $t\in T$; if $\overline{\overline{X\cdot M_t}}=1$ always for $t\in T$, then the set X is called a *choice-set* from $\mathfrak{M}$. The choice-set (or partial choice-set) X is *σ-consistent*, if $\sigma(p_1,p_2)$ for $p_1,p_2\in X$ [12]).

1. Method. The method of this paper is non axiomatic. It may easily be seen that all proofs could be formalized in every sufficiently large system of axiomatic set theory $\big($*e. g.* in the system Σ of Bernays-Gödel [13])$\big)$, without the axiom of choice.

2. Results. We deal here with six propositions, the Fundamental Existence Proposition of Ideal Arithmetic (I) and the following five:

(II) *If J is an s-ideal [d-ideal] of a field of sets* $\boldsymbol{A}$, *there is a prime s-ideal [d-ideal]* J_p *of* $\boldsymbol{A}$, *which contains J.*

(II*) *If* $\boldsymbol{A}_1$ *is a sub-field of a field of sets* $\boldsymbol{A}$, *and* μ_1 *a two-valued measure in* $\boldsymbol{A}_1$, *there exists a two-valued measure* μ *in* $\boldsymbol{A}$, *which is an extension of* μ_1.

(III) *The product space of non-empty compact spaces is non-empty and compact.*

[11]) Cf. [1], Definition 1, p. 59 and Definition. 1, p. 42, 43.

[12]) Those notions are introduced in [6], see (3,1), (3,2), (3,3).

[13]) Cf. [2].

(IV) *If* $\mathcal{M}=\{M_t\}_{t\in T}$ *is a family of compact spaces,* σ *a relation of consistency for the class* $\mathcal{M}$, *and if, moreover, for every finite set* $T_0\subset T$ *there exists a* σ*-consistent choice from the class* $\{M_t\}_{t\in T_0}$, *then there exists a* σ*-consistent choice from the whole class* $\mathcal{M}$.

(V) *If* $\boldsymbol{A}$ *is a Boolean algebra,* A *a subset of* $\boldsymbol{A}$, f *a real valued function on* A *and* W *a closed subset of the* $[0,1]$ *interval, and if, moreover, for any finite* $X\subset\boldsymbol{A}$ *there exists a measure* ν *in* $[X]_0$ *such that* $\nu(a)\in W$, *for* $a\in[X]_0$ *and* $\nu(a)=f(a)$, *for* $a\in A\cdot X$, *then there exists a measure* μ *in the whole algebra* $\boldsymbol{A}$, *which is an extension of* f, *and* $\mu(a)\in W$ *for every* $a\in\boldsymbol{A}_0$.

Evidently (II*) is a translation [14]) of (II) from the language of Boolean algebras into the language of the measure theory so as for instance the proposition

(I*) *in every Boolean algebra, there is a two-valued measure*

is a similar translation of (I). Therefore (I), (II) and (I*), (II*) are respectively equivalent.

The proposition (III) is the well known theorem of Tychonoff [15]) with the addition of the condition of non-emptiness of the product space. This condition is equivalent to the choice principle from compact spaces, which implies the choice principle from finite sets.

The proposition (IV) is the principle of consistent choice, it was discussed in our previous paper [16]).

Finally (V) is a theorem of extension of measure. It follows from the known theorems of A. Horn and A. Tarski [17]) and from one theorem of E. Marczewski [18]).

The main result of this paper is that all propositions (I)-(V) are effectively equivalent.

3. The implication (I)→(II). This implication is known. Let $\boldsymbol{A}$ be a field of sets and J an [s- or d-] ideal of $\boldsymbol{A}$. It results from (I) that in the quotient algebra $\mathfrak{A}=\boldsymbol{A}/J$ [19]) there exists a prime ideal $\mathfrak{J}$; by setting $J_p=\underset{x\in A}{E}[[x]\in\mathfrak{J}]$ we obtain a prime ideal J_p of $\boldsymbol{A}$.

4. Proof of the implication (II)→(III). One part of this implication is due to N. Bourbaki [20]). He has remarked that (II) has the following theorem as its consequence:

[14]) Cf. lemmas 2 and 3, p. 51.

[15]) Cf. [1], Theorem 2.

[16]) [6], Theorem 2, p. 235.

[17]) See [3], 1. Measure and partial measure, p. 469-480, especially Th. 1.22, p. 477.

[18]) [5], Theorem 1, p. 269 and (i), p. 270.

[19]) This is the algebra of the class of abstraction of J.

[20]) See [1], § 10, especially p. 59-63.

(4,1) *Any topological space* M *is compact if and only if for every two-valued measure* μ, *defined for all subsets of* M, *there exists precisely one point* $p\in M$ *such that* $\mu(G)=1$ *for every neighbourhood* G *of* p,

and that from (4,1) and (II) it follows that

(4,2) *the product space of compact spaces is compact.*

Therefore it is sufficient to show that from (II) it results that:

(4,3) *The product of non-empty compact spaces is non empty.*

Let $\mathcal{M}=\{M_t\}_{t\in T}$ be a family of compact spaces; we set

(4,3,1) $$\mathcal{X}=\underset{X}{E}[X\subset\sum_{t\in T}M_t \text{ and } \overline{\overline{X\cdot M_t}}\leqslant 1 \text{ for every } t\in T],$$

(4,3,2) $$\mathcal{C}_t=\underset{X\in\mathcal{X}}{E}[\overline{\overline{X\cdot M_t}}=1].$$

$\mathcal{X}$ is evidently the family of partial choice-sets from the sets of $\mathcal{M}$, and $\mathcal{C}_t$ the family of such partial choice-sets which have one element in common with M_t. Clearly we have

(4,3,3) $$\prod_{i=1}^{n}\mathcal{C}_{t_i}\neq 0 \quad \text{for every } n \text{ and } t_1,t_2,\ldots,t_n\in T$$

and in view of lemma 1 there exists an d-ideal J of the field $\mathcal{A}$ of all subsets of $\mathcal{X}$, such that $\mathcal{C}_t\in J$ for all $t\in T$. But in consequence of (II) we conclude that there exists a prime ideal J_p which includes J. The function

(4,3,4) $$\mu(E)=\begin{cases}0 & \text{if } E\in\mathcal{A}-J_p,\\ 1 & \text{if } E\in J_p\end{cases}$$

is a two-valued measure in $\mathcal{A}$ [21]) such that

(4,3,5) $$\mu(\mathcal{C}_t)=1 \quad \text{for every } t\in T$$

because $\mathcal{C}_t\in J\subset J_p$, for all $t\in T$.

Denoting by $\mathcal{O}(p)$ the set $\underset{X\in\mathcal{X}}{E}[p\in X]$ for $p\in M_t$, we find that

(4,3,6) $$\text{if } p,q\in M_t \text{ and } p\neq q, \text{ then } \mathcal{O}(p)\cdot\mathcal{O}(q)=0,$$

(4,3,7) $$\sum_{p\in M_t}\mathcal{O}(p)=\mathcal{C}_t.$$

It follows therefore that the function

(4,3,8) $$m_t(E)=\mu\big(\sum_{p\in E}\mathcal{O}(p)\big)$$

[21]) See Lemma 3, p. 51.

defined for all $E\subset M_t$, is a two-valued measure. But in view of (4,1) every such measure distinguishes in M_t one and only one point whose every neighbourhood has the measure 1.

The set of all those points is evidently a choice-set from the class $\mathcal{M}$.

5. The implication (III)→(IV). The proof of this implication was given in our previous paper [22].

As the proposition (IV) may be interesting, we also give other formulations of it:

(IV⁺) *If $\mathcal{P}=\{(a_s,b_s)\}_{s\in S}$ is a family of pairs and $\tau(x_1,x_2,x_3)$ a totally symmetrical relation (i. e. such that $\tau(x_1,x_2,x_3)$ implies $\tau(x_{a_1},x_{a_2},x_{a_3})$ for every permutation a_1,a_2,a_3 of 1,2,3) defined for $x_1,x_2,x_3\in\sum_{s\in S}(a_s,b_s)$, so that for every finite set $S_0\subset S$ there is a choice-set X from $\{(a_s,b_s)\}_{s\in S_0}$ such that $\tau(x_1,x_2,x_3)$ for all $x_1,x_2,x_3\in X$, then there exists a choice-set from the whole $\mathcal{P}$, with the same property.*

The proposition (I) follows from (IV⁺) by substituting $P=\{(x,x')\}_{x\in\mathfrak{A}}$ (where $\mathfrak{A}$ is a Boolean algebra) and $\tau(x_1,x_2,x_3)=(x_1,x_2,x_3\neq 0)$.

Since every finite set can be looked upon as a bicompact space, therefore (IV) holds true if every set M_t of $\mathcal{M}$ is finite. That particular case of (IV) in which every set M_t is precisely of the power m (where m is a finite cardinal number) we denote by $(\mathrm{IV}^{(m)})$.

We obtain (IV⁺) from $(\mathrm{IV}^{(4)})$ by substituting $T=S^2$ (the set of all ordered pairs $\langle s_1,s_2\rangle$, where $s_1,s_2\in S$),

$$M_{\langle s_1,s_2\rangle}=\big(\langle a_{s_1},a_{s_2}\rangle,\langle a_{s_1},b_{s_2}\rangle,\langle b_{s_1},a_{s_2}\rangle,\langle b_{s_1},b_{s_2}\rangle\big)$$

and $\sigma\big(\langle x_{s_1},x_{s_2}\rangle,\langle x_{s_3},x_{s_4}\rangle\big)$ if and only if $\tau(x_{s_k},x_{s_l},x_{s_n})$ for $k<l<n\leqslant 4$.

It is easy to see that the implication $(\mathrm{IV}^{(m)})\to(\mathrm{IV}^{(k)})$ holds true if $m\geqslant k$, and so the propositions (I)-(IV), (IV⁺) and $(\mathrm{IV}^{(m)})$, $m=4,5,\ldots$ are effectively equivalent.

6. Proof of the implication (IV)→(V). Let $\boldsymbol{A}$, A, f and W be respectively: a Boolean algebra, a subset of $\boldsymbol{A}$, a function on A and a closed subset of the [0,1] interval. Suppose that $\boldsymbol{A}$, A, f and W fulfil the conditions of (V). Let $\mathfrak{X}$ be the family of all finite subsets of $\boldsymbol{A}$ and let M_X for $X\in\mathfrak{X}$ be the set of all real functions on X with values in W which are extensions of f from $X\cdot A$ to X. Evidently $\mathcal{M}=\{M_X\}_{X\in\mathfrak{X}}$ is a family of compact spaces. For $X_1,X_2\in\mathfrak{X}$ and $\varphi_1\in M_{X_1}$, $\varphi_2\in M_{X_2}$ — $\sigma(\varphi_1,\varphi_2)$ denotes that the functions φ_1 and φ_2 are additive respectively in X_1,X_2 and $\varphi_1(x)=\varphi_2(x)$ for $x\in X_1\cdot X_2$. σ appears to be a relation of consistency for the class $\mathcal{M}$ and, in consequence of (IV), there exists

[22]) [6], see the proof of Theorem 2, p. 235.

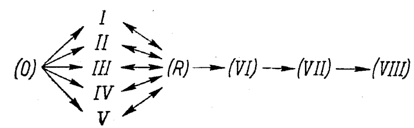
a σ-consistent choice-set Φ from $\mathcal{M}$. From σ-consistency of Φ it follows that for every $a\in A$ and $\varphi_1,\varphi_2\in\Phi$, $\varphi_1(a)=\varphi_2(a)$ if only φ_1 and φ_2 are defined for a (i. e. if $\varphi_1\in\Phi\cdot M_{X_1}$, $\varphi_2\in\Phi\cdot M_{X_2}$ and $a\in X_1\cdot X_2$). Therefore we can set $\mu(a)=\varphi(a)$, where $\varphi\in\Phi M_{(a)}$. μ is a measure in $\mathfrak{A}$ with the required properties.

7. Proof of the implication (V)→(I*). If in (V) A is the empty set, f the empty function and W the set which consists of two numbers 0 and 1, then (V) may be expressed in the following form

(V⁰) *If in every finite subalgebra $\boldsymbol{A}_0$ of the Boolean algebra $\boldsymbol{A}$ there exists a two-valued measure, then also in the whole algebra $\boldsymbol{A}$ there exists such a measure.*

The antecedens of (V⁰) holds for every algebra $\boldsymbol{A}$ and therefore so does the succedens, which is (I*).

8. Table of results. We have proved that all the propositions (I)-(V) are effectively equivalent. Let us denote by (VI) the theorem of extending partial order to order [23]), by (VII) the ordering principle, by (VIII) the principle of choice from finite sets and finally by (0) the axiom of choice (without restrictions). We demonstrated previously the implication (III)→(VI) [24]), therefore we obtain the following table



Nevertheless we do not know, whether any implication outside those resulting from this table is valid for any pair of the propositions (0)-(VIII).

9. Remark. In a previous volume of this Journal J. L. Kelley [25]) showed that the theorem of Tychonoff for Kuratowski's closure spaces (*i. e.* the spaces in which the closure of every set is defined, fulfilling the known axioms of Kuratowski; those spaces need not be Hausdorff's spaces) implies the axiom of choice (0).

This result is not astonishing, because the theorem (4,1) of Bourbaki is valid only for Hausdorff's spaces, and this compels us, when proving the theorem of Tychonoff for Kuratowski's spaces, to use once more,

[23]) See *e. g.* [6], Theorem (2.1), p. 234.

[24]) [6], it is easy to see that in the proof of (2.1) only the theorem of Tychonoff is used, besides, of course, that of Theorem 1 of [6], which is effective.

[25]) Kelley [4].

additionaly, the principle of choice. But the method of Kelley permits us to demonstrate that the theorem of Tychonoff in the form (4,2) (for Hausdorff's spaces) implies the principle of choice for compact (Hausdorff's) spaces (4,3).

Let $\mathcal{M}=\{M_t\}_{t\in T}$ be a class of compact spaces, and let p_t non $\in \sum_{t\in T} M_t$. We set $M_t^*=M_t+(p_t)$ and $\mathcal{M}^*=\{M_t^*\}_{t\in T}$. If we consider the point p_t as isolated in M_t^*, then every M_t^* is a compact space, and M_t is closed in M_t^*.

Each set $F_{t_0}=\underset{X}{E}[\overline{\overline{X\cdot M_t^*}}=1$ for $t\in T$ and $X\cdot M_{t_0}^*\subset M_t]$ is a closed subset of the product space of $\mathcal{M}^*$ and $F_{t_1}\ldots F_{t_n}\neq 0$ for every finite number of $t_i\in T$. Therefore $\prod_{t\in T} F_t$ is non empty, but it is the product of $\mathcal{M}$. We have demonstrated that (4,2) is equivalent to each of the propositions (I)-(V).

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Intersections of prescribed power, type, or measure

by

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In 1914, Mazurkiewicz [5] showed that there exists a set of points in the plane, which intersects every straight line in the plane in precisely two points. Recently, Bagemihl [1] proved a general intersection theorem in the theory of sets, which, when applied to the plane, yields the following generalization of Mazurkiewicz's result: With every straight line s, associate a cardinal number $\mathfrak{q}_s\geqslant 2$ so that the sum of fewer than $2^{\aleph_0}$ of the numbers $\mathfrak{q}_s$ is always less than $2^{\aleph_0}$. Then there exists a set of points which intersects every straight line s in exactly $\mathfrak{q}_s$ points.

In the present paper, after extending the general intersection theorem alluded to above, we obtain several theorems dealing with plane point sets which intersect every straight line in a set of prescribed power, order type, or measure. In particular, we show that the aforementioned $\mathfrak{q}_s$ may be chosen arbitrarily in the range $2\leqslant\mathfrak{q}_s\leqslant 2^{\aleph_0}$. Free use is made of the well-ordering theorem.

THEOREM 1. *Let a be an arbitrary, fixed ordinal number, and S be a set with*

(1) $$\overline{\overline{S}}\leqslant\aleph_a.$$

To every $s\in S$ let there correspond a set L_s such that, for every $S'\subseteq S-\{s\}$ with $\overline{\overline{S'}}<\aleph_a$,

(2) $$\overline{\overline{L_s-\sum_{s'\in S'}L_{s'}}}\geqslant\aleph_a,$$

and put $P=\sum_{s\in S}L_s$.

Suppose that for every $s\in S$ there exists a cardinal number $\mathfrak{l}_s$, with $1\leqslant\mathfrak{l}_s\leqslant\aleph_a$, such that the following holds: If $D\subset P$, $\overline{\overline{D}}<\aleph_a$, and S_D is the set of elements $s'\in S$ for which $\mathfrak{l}_{s'}<\aleph_a$ and $\overline{\overline{L_{s'}D}}\geqslant\mathfrak{l}_{s'}$, then

(3) $$\overline{\overline{S_D}}<\aleph_a.$$

With every $s\in S$ let there be associated in an arbitrary manner a cardinal number $\mathfrak{q}_s$ satisfying

(4) $$\mathfrak{l}_s\leqslant\mathfrak{q}_s\leqslant\aleph_a.$$