To this end, associate with each element j of I_1 the set u_j consisting of all elements x of A such that $j \in S(x)$, and let U be the class of all such sets u_j, $j \in I_1$. We assert that this class U satisfies the required conditions.

First let $x_1, \ldots, x_n$ be any finite number of elements from one of the sets u_j. Now $S(x_1) \circ_1 \ldots \circ_1 S(x_n) \neq 0_1$ since $j \in S(x_1) \circ_1 \ldots \circ_1 S(x_n)$. But S is an isomorphism, so $x_1 \ldots x_n \neq 0$. Thus condition (Ui) is satisfied.

Next consider any $x \in A$, $x \neq 0$. Since S is an isomorphism, $S(x) \neq 0_1$, and so there exists a j in $S(x)$. But then $x \in u_j$, so condition (Uii) is satisfied.

Finally, (Uiii) is an immediate consequence of our assumption on the cardinality of I_1, since the cardinality of U clearly does not exceed that of I_1.

This completes the proof of our theorem.

5. We do not know whether the theorem of part 4 can be proven from the Gödel-Malcev (propositional) theorem without using the axiom of choice. However, without the axiom of choice we can show by Stone's method that the possibility of representing a given boolean algebra $\boldsymbol{a}$ by a boolean algebra of sets $\boldsymbol{a}_1$ whose unit element has smaller cardinality than that of A, is equivalent to the existence of a non-empty class V satisfying the following conditions:

(Vi) Every element v of V is a maximal ideal of $\boldsymbol{a}$.

(Vii) The intersection of all the elements v of V is empty.

(Viii) The cardinality of V is less than that of A.

Using the axiom of choice, one can give a direct proof that the existence of a class U satisfying (Ui)-(Uiii) is equivalent to the existence of a class V satisfying (Vi)-(Viii).

References

[1] M. H. Stone, *The theory of representations for boolean algebras*, Transactions of the American Mathematical Society 40 (1936), p. 37-111.

[2] K. Gödel, *Die Vollständigkeit der Axiome der logischen Funktionenkalküle*, Monatshefte für Mathematik und Physik 37 (1930), p. 349-360.

[3] A. Malcev, *Untersuchungen aus dem Gebiete der mathematischen Logik*, Recueil Mathématique, n. s. 1 (1936), p. 323-336.

[4] L. Henkin, *The completeness of the first-order functional calculus*, Journal of Symbolic Logic 14 (1949), p. 159-166.

[5] — *Some interconnections between modern algebra and mathematical logic*, Transactions of the American Mathematical Society 74 (1953), p. 410-427.

[6] A. Robinson, *On the Metamathematics of Algebra*, 1951.

[7] H. Rasiowa and R. Sikorski, *A proof of the Skolem-Löwenheim theorem*, Fundamenta Mathematicae 38 (1951), p. 230-232.

[8] D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, erster Band. Berlin 1934.

Reçu par la Rédaction le 6. 7. 1953

On the existence of totally heterogeneous spaces

by

J. R. Büchi (Ann Arbor)

The main purpose of this note is to prove the existence of a set M of real numbers, which is heterogeneous in the sense that every Borel-function defined on a subset X of M into M is trivial. Some consequences and related facts are pointed out in notes at the end of the paper.

We first state the following fact:

(1) *Let f be a real valued measurable function defined on a measurable set X of real numbers. Then the set D of all y, for which $f^{-1}(y)$ is of positive measure, is at most of cardinality $\aleph_0$.*

Now we prove,

LEMMA 1. *Let F be a class of real valued measurable functions, defined on measurable sets of real numbers, and suppose the cardinality of F is $\aleph_1$. Then there exists a set M of real numbers, which is of cardinality $\aleph_1$, such that the sets $[f(x)|x \in M, f(x) \in M, f(x) \neq x]$ are at most of cardinality $\aleph_0$, for all members f of F.*

Proof. Let ω_1 be the first ordinal of cardinality $\aleph_1$. By hypothesis the class F can be arranged into a ω_1-series $[f_\xi | \xi < \omega_1]$. Let $D_\xi = [y | f_\xi^{-1}(y)$ of positive measure$]$ and define a ω_1-series of real numbers x_ξ by the following induction.

Choose any real number as x_1. If the x_η are already defined for all $\eta < \xi$, then choose x_ξ such that the following conditions are satisfied:

$$(\alpha) \qquad x_\xi \neq x_\eta \quad \text{for all} \quad \eta < \xi,$$

$$(\beta) \qquad x_\xi \neq f_\nu(x_\eta) \quad \text{for all} \quad \eta < \xi \quad \text{and} \quad \nu < \xi,$$

$$(\gamma) \qquad f_\nu(x_\xi) \neq x_\eta \quad \text{or} \quad f_\nu(x_\xi) \in D_\nu \quad \text{for all} \quad \eta < \xi \quad \text{and} \quad \nu < \xi.$$

That such an element x_ξ exists one shows as follows. To realize (α) and (β) one has to avoid a set of cardinality less than $\aleph_1$ only. As for the realization of (γ) note first that in case $x_\eta \in D_\nu$ the condition (γ) is void. In the alternative case the pair (η, ν) is such that $x_\eta \notin D_\nu$. Then, by definition of D_ν, $f_\nu^{-1}(x_\eta)$ is of measure 0. Therefore, for any pair (η, ν), one can satisfy (γ) by avoiding a set of measure 0 only. But $\xi<\omega_1$, and therefore the conditions (α), (β), (γ) can be realized simultaneously by avoiding a set of measure 0. Thus, the ω_1-series $[x_\xi|\xi<\omega_1]$ is well-defined.

Now by (β), $x_\eta=f_\nu(x_\xi)$ implies $\eta\leqslant\nu$ or $\eta=\xi$ or $\eta<\xi$. By (γ), $x_\eta=f_\nu(x_\xi)$ and $\eta<\xi$ implies $f_\nu(x_\xi)\in D_\nu$ or $\eta\leqslant\nu$. We conclude that $x_\eta=f_\nu(x_\xi)$ implies $\eta\leqslant\nu$ or $x_\eta=x_\xi$ or $f_\nu(x_\xi)\in D_\nu$. Or, if we now define $M=[x_\xi|\xi<\omega_1]$, $x\in M$ and $f_\nu(x)\in M$ and $f_\nu(x)\neq x$ implies $f_\nu(x)\in[x_\eta|\eta<\nu]$ or $f_\nu(x)\in D_\nu$. But both sets D_ν and $[x_\eta|\eta<\nu]$ are at most of cardinality $\aleph_0$, as it follows from (1) and $\nu<\omega_1$. Thus the set M clearly satisfies the conditions in lemma 1.

THEOREM 1. *If $2^{\aleph_0}=\aleph_1$, there exists a set M of real numbers, such that M is of cardinality $2^{\aleph_0}$, and such that the sets $[f(x)|f(x)\neq x]$ are at most of cardinality $\aleph_0$, for all Borel-measurable functions f: $X\to M$, defined on arbitrary subsets X of M.*

Proof. Let F be the class of all real valued Borel-measurable functions, defined on Borel-sets of real numbers. The cardinality of F is $2^{\aleph_0}$. Thus, by $2^{\aleph_0}=\aleph_1$ and lemma 1, there exists a set M of cardinality $2^{\aleph_0}$, such that $[g(x)|x\in M, g(x)\in M, g(x)\neq x]$ is at most of cardinality $\aleph_0$, for all members g of F. Now suppose X is a subset of M and f is any Borel-measurable function of X into M. It is known, (see [4] and [8]), that such an f can be extended to a function g, which is a member of F. Since $[f(x)|f(x)\neq x]$ is a subset of $[g(x)|x\in M, g(x)\in M, g(x)\neq x]$, it follows that the cardinality of $[f(x)|f(x)\neq x]$ is at most $\aleph_0$. This proves theorem 1.

Some notes and further results

1. Theorem (1) strengthens a result of B. Dushik and E. W. Miller [1], who proved it with *Borel-measurable functions* replaced by *strictly monotonic functions*. A similar result can be proved without assuming the continuum-hypothesis. (See theorem 2).

2. It does not seem to be easy to eliminate the assumption $2^{\aleph_0}=\aleph_1$ from theorem 1. We do not know how to do this, even if we restrict our attention to continuous functions. However, by an obvious variation of the proof of lemma 1 we can get, without assuming $2^{\aleph_0}=\aleph_1$.

LEMMA 2. *Let F be a class of real valued measurable functions, defined on measurable sets of real numbers. Suppose the cardinality of F is $2^{\aleph_0}$ and, for every f in F and every real number y, $f^{-1}(y)$ is either of positive measure or at most of cardinality $\aleph_0$. Then there exists a set M of cardinality $2^{\aleph_0}$, such that the sets $[f(x)|x\in M, f(x)\in M, f(x)\neq x]$ are of cardinality less than $2^{\aleph_0}$, for every member f of F.*

An example of a class F which satisfies the conditions in lemma 2 consists of all real valued weakly monotonic functions, defined for all real numbers. Furthermore, every weakly monotonic function defined on any set of reals can be extended to a member of this class F. We obtain at once

THEOREM 2. *Without assuming $2^{\aleph_0}=\aleph_1$, it is possible to prove the existence of a set M of real numbers, such that the cardinality of M is $2^{\aleph_0}$, but the cardinality of $[f(x)|f(x)\neq x]$ is less than $2^{\aleph_0}$, for every weakly monotonic function f, which maps a subset of M into M.*

Another class F which satisfies all conditions of lemma 2 is the set of all generalized homeomorphisms (one-to-one mappings, which are Borel-measurable in both ways) between Borel-sets of real numbers. According to a result of C. Kuratowski [5], every generalized homeomorphism between any two sets of real numbers can be extended to a member of this class F. It follows that

(2) *Without assuming $\aleph_1=2^{\aleph_0}$ one can show the existence of a set M of real numbers, such that the cardinality of M is $2^{\aleph_0}$, and such that the set $[x|f(x)\neq x]$ is of cardinality less than $2^{\aleph_0}$, for every generalized homeomorphism f between two subsets of M.*

This theorem has been proved by W. Sierpiński [9], with the word *generalized* removed. By taking two disjoined subsets of M, both of cardinality $2^{\aleph_0}$, it follows that

(3) *Without assuming $\aleph_1=2^{\aleph_0}$ one can show the existence of two sets N_1 and N_2 of real numbers, both of cardinality $2^{\aleph_0}$, such that there is no set Z of cardinality $2^{\aleph_0}$ which can be mapped into N_1 and N_2 by generalized homeomorphisms.*

3. The set M of theorem 1 has in particular the property that no two exclusive subsets of it are homeomorphic, except, when they are of power less than $2^{\aleph_0}$. This suggests

Definition. *A set M of real numbers which has cardinality $2^{\aleph_0}$ is totally heterogeneous, if for every Borel-function f of a subset $X\subseteq M$ into M the set $[f(x)|f(x)\neq x]$ is of cardinality less than $2^{\aleph_0}$.*

Now, a subset of cardinality $2^{\aleph_0}$ of a heterogeneous set is clearly heterogeneous. Thus by theorem 1 we have

THEOREM 3. *If $2^{\aleph_0}=\aleph_1$ there exist $2^{\aleph_1}$ totally heterogeneous sets of real numbers.*

Next we note,

(4) *Every perfect set contains an order-isomorphic image of the set of all reals. Every Borel-set of cardinality greater than $\aleph_0$ contains a perfect set* (see Hausdorff [2]).

Now, Cantor's sets if a perfect set of measure zero and of first category. Together with (4) this yields another improvement of theorem 1.

THEOREM 4. *If $2^{\aleph_0}=\aleph_1$, there exists a totally heterogeneous set in every Borel-set of power greater than $\aleph_0$ (in every perfects set). Furthermore, there exist totally heterogeneous sets which are of measure zero and of first category.*

But (4) implies the following negative results, also.

THEOREM 5. *A totally heterogeneous set cannot be a Borel-set and is always of inner measure zero.*

Nevertheless in the sense of outer measure, a heterogeneous set may be thick.

THEOREM 6. *If $2^{\aleph_0}=\aleph_1$, there exists a totally heterogeneous set M of real numbers which has the outer measure ∞, and even stronger, the outer measure of $M\cap E$ is equal to the measure of E, for every measurable set E. Such a set M is automatically of second category.*

Proof. Since $2^{\aleph_0}=\aleph_1$, we can arrange the Borel-sets of real numbers which have a positive measure into a ω_1-series $[B_\xi|\xi<\omega_1]$. Now we refine the proof of lemma 1 by choosing x_ξ in B_ξ, which can be done, because B_ξ is of positive measure and the conditions (α), (β), (γ) eliminate a set of measure 0, only. The heterogeneous set $M=[x_\xi|\xi<\omega_1]$ then clearly intersects every set A of positive measure. Now let E be any measurable set. If $M\cap E\subseteq X\subseteq E$, then $(E-X)\cap M=0$. Therefore, if X is measurable, we conclude by the property of M that the measure of $E-X$ must be 0, and therefore the measure of X is equal to the measure of E. It follows that the outer measure of $M\cap E$ is equal to the measure of E.

To prove or disprove the existence of totally heterogeneous sets of second category appears to be difficult.

4. We say that the set V is a Borel-image of the set U, if there is a Borel-measurable function f defined on U, such that $f(U)=V$. Now, every Borel-measurable function $f:X\to Y$ can be extended to a Borel-measurable function $f':X'\to Y'$ where X' and Y' are Borel-sets of reals. Furthermore, if Y is of cardinality greater than $\aleph_0$, then Y' is automatically of cardinality $2^{\aleph_0}$ (see Hausdorff [2]). By a theorem of C. Kuratowski [7], and theorem 1, we conclude:

THEOREM 7. *Without assuming $2^{\aleph_0}=\aleph_1$, we can show the existence of $2^{2^{\aleph_0}}$ sets M_ν of real numbers, such that no Borel-image $f(M_\nu)$ of cardinality greater than $\aleph_0$ of any M_ν can be contained in a different M_ν.*

We define a Borel-invariant to be a class I of sets, which together with any set contains all its Borel-images. As a corollary to theorem 7 we then get the following improvement of a theorem by C. Kuratowski [6].

COROLLARY. *Without assuming $2^{\aleph_0}=\aleph_1$ one can show the existence of $2^{2^{2^{\aleph_0}}}$ Borel-invariants of the space of real numbers.*

As another obvious corollary to theorem 7 we get the following improvement of a theorem by W. Sierpiński [11].

COROLLARY. *Without assuming $2^{\aleph_0}=\aleph_1$, one can show the existence of $2^{2^{\aleph_0}}$ sets M_ν of real numbers, such that none of the M_ν is generalized homeomorphic to any subset of any different one of the M_ν's.*

To get W. Sierpiński's result, replace "$2^{2^{\aleph_0}}$" by "more than $2^{\aleph_0}$" and replace "generalized homeomorphic" by "order-isomorphic".

If we now assume the continuum-hypothesis, we can improve theorem 7 to

THEOREM 8. *If $2^{\aleph_0}=\aleph_1$ and $2^{\aleph_1}=\aleph_2$, then there exist $\aleph_2$ sets of real numbers M_ν, such that no subset of cardinality $\aleph_1$ of M_{ν_1} is a Borel-image of any subset of M_{ν_2}, where $M_{\nu_1}\neq M_{\nu_2}$.*

Proof. By a theorem of W. Sierpiński [10] (see A. Tarski [13]), if M is a set of cardinality m, then there exists a class K of subsets M_ν of M, such that K is of cardinality greater than m and such that $M_{\nu_1}\cap M_{\nu_2}$ is of cardinality less than m, whenever M_{ν_1} and M_{ν_2} are different members of K. Thus, under the hypotheses $2^{\aleph_0}=\aleph_1$ and $2^{\aleph_1}=\aleph_2$, theorem 8 clearly follows from theorem 1.

Note that similarly we can prove from theorem 2 and (2), without assuming $2^{\aleph_0}=\aleph_1$ or $2^{\aleph_1}=\aleph_2$, that there exist more than $2^{\aleph_0}$ sets of real numbers M_ν, such that no subset of M_{ν_1} of cardinality $2^{\aleph_0}$ is a monotonic image of (generalized homeomorphic to) any subset of M_{ν_2}.

5. In the proof of lemma 1 we can choose all numbers x_ξ from a given set of positive outer measure. Thus, in every set of real numbers X which has positive outer measure, there exists a totally heterogeneous subset M.

It is clear that all our results can be proved if we replace the set of real numbers by a complete separable metric space, for which there is a σ-measure on the Borel-sets, which is not identically 0.

6. The following fact has been proved by R. Sikorski [12]. Let B_1 and B_2 be the σ-complete fields of all Borel-sets of separable metric spaces X_1 and X_2 and let h be a σ-homomorphism of B_2 into B_1; then there exists a mapping f of X_1 into X_2, such that $h(U)=f^{-1}(U)$ for every member U of B_2. In other words, every σ-homomorphism h of B_2 into B_1 is generated by a Borel-measurable function f of X_1 into X_2. From theorem 1 we obtain at once

THEOREM 9. *If $2^{\aleph_0}=\aleph_1$, there exists a σ-complete Boolean algebra with $\aleph_1$ atoms, which is heterogeneous in the following sense. If x and y are* *any elements of B, such that $x \cap y = 0$ and y contains $\aleph_1$ atoms, then there is no σ-homomorphism defined on the Boolean algebra $[u | u \epsilon B, u \subseteq y]$ onto the Boolean algebra $[u | u \epsilon B, u \subseteq x]$.*

It is possible that the quotient-algebra Q of B in theorem 9, modulo the σ-ideal of all elements $x \epsilon B$, which are the union of at most $\aleph_0$ atoms, does not admit any σ-homomorphisms. (This would follow from a result of R. Sikorski [12] if the heterogeneous set M which generates B were a Borel-set of real numbers. But, by theorem 4 there is no such M). In this connection note the ingenious construction of B. Jónsson [3] of a Boolean algebra which admits no automorphism except the identity. His algebra is of very high cardinality.

7. The rather ingenious use of well-orderings, employed to prove the fundamental lemma 1, has often been used to derive pseudo-antinomious results about the continuum. It seems to originate with G. Hamel, who devised it to show the existence of a base for the reals.

References

[1] B. Dushnik and E. W. Miller, *Partially ordered sets*, Am. Jour. of Math. 63 (1941), p. 600-610.

[2] F. Hausdorff, *Die Mächtigkeit der Borelschen Mengen*, Math. Ann. 77 (1916), p. 430-437.

[3] B. Jónsson, *A Boolean algebra without proper automorphisms*, Proc. Am. Math. Soc. 2 (1951), p. 766-770.

[4] C. Kuratowski, *Sur les théorèmes topologiques de la théorie des fonctions de variables réelles*, Comptes Rendus 197 (1933), p. 19-20.

[5] — *Sur le prolongement de l'homéomorphie*, Comptes Rendus 197 (1933) p. 1090-1091.

[6] — *Sur la puissance de l'ensemble des nombres de dimension au sens de M. Fréchet*, Fund. Math. 8 (1926), p. 201-208.

[7] — *Sur l'extension de deux théorèmes topologiques à la théorie de ensembles*, Fund. Math. 34 (1947), p. 34-38.

[8] W. Sierpiński, *Sur l'extension des fonctions de Baire définies sur les ensembles linéaires quelconques*, Fund. Math. 16 (1930), p. 81-89.

[9] — *Sur un problème concernant les types de dimension*, Fund. Math. 19 (1932), p. 65-71.

[10] — *Sur une décomposition d'ensembles*, Monatsheft für Math. und Phys. 35 (1928), p. 239-242.

[11] — *Sur les types d'ordre des ensembles linéaires*, Fund. Math. 37 (1950), p. 253-264.

[12] R. Sikorski, *On the inducing of homomorphisms by mappings*, Fund. Math. 36 (1949), p. 7-22.

[13] A. Tarski, *Sur la décomposition des ensembles en sous-ensembles presque disjoints*, Fund. Math. 12 (1928), p. 188-205.

Reçu par la Rédaction le 8. 7. 1953

On a problem concerning completely regular sets

by

J. Novák (Praha)

Słowikowski and Zawadowski have raised the following problem:

A topological space R has the property α if every function defined and continuous on R is bounded. Does the property α always imply the compacticity of any completely regular space R?

We are going to prove that the answer to this question is *negative*. Let $\beta(N)$ be the Čech bicompactification of an infinite isolated point-set N — for instance the set of all naturals. Let $N = \bigcup_{k=1}^{\infty} N_k$ where N_k are infinite subsets of N disjoint from one another. Let us identify in the space

$$\beta(N) - \beta[\bigcup_{k=1}^{\infty} \beta(N_k) - N] \bigcup_{k=1}^{\infty} \beta(N_k)$$

every set $\beta(N_k) - N$ with a new element $a_k \equiv \beta(N_k) - N$, the symbol β indicating the closure in the space $\beta(N)$. In such a way we get a new topological space R. The closure of the set A in R will be denoted by $\overline{A}$.

Some remarkable properties of the space R.

Clearly, the set N is isolated and dense in R.

Further, there is an open basis of R consisting of neighbourhoods which are ambiguous, *i. e.* open and closed in R. We have to prove that in every neighbourhood $O(x)$ of any point $x \epsilon R$ there is an ambiguous neighbourhood $U(x) = \overline{U(x)} \subset O(x)$. As a matter of fact, for $x \epsilon N$ we can put $U(x) = (x)$ and for $x = a_k$ we can choose $U(x) = O(x) \cap [N_k \cup (a_k)]$. Now, let $x \epsilon [N \cup \bigcup_{k=1}^{\infty} (a_k)]$. Then

$$x \epsilon \left(R - [N \cup \bigcup_{k=1}^{\infty} (a_k)]\right) \cap \left(\beta(N) - \beta[\bigcup_{k=1}^{\infty} \beta(N_k) - N]\right).$$

Since $\beta(N)$ is a normal space there is a set G open in $\beta(N)$ such that $x \epsilon \beta(G) \subset O(x)$ and such that

$$\beta[\bigcup_{k=1}^{\infty} \beta(N_k) - N] \subset \beta(N) - \beta(G).$$