On some metrizations of the hyperspace of compact sets

by

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1. Set-theoretical metric ϱ_s. Let M be a metric space and let ϱ denote the distance-function defined in M. By 2^M we denote the class of all non-empty compacta [1]) lying in M. It is known [2]) that, setting

$$(1)\qquad \varrho_s(X,Y)=\operatorname{Max}\,[\operatorname*{Sup}_{x\in X}\varrho(x,Y),\ \operatorname*{Sup}_{x\in Y}\varrho(y,X)]\qquad \text{for every}\quad X,Y\in 2^M,$$

we obtain a function ϱ_s which can be considered as a distance-function in 2^M (called in this note *set-theoretic metric*). The metric space obtained from 2^M in this manner will be denoted by 2^M_s. This space constitutes a useful tool in the investigation of the compact sets lying in M. If M is a complete space [3]), then also the space 2^M_s is complete [4]). This is the ground for the application of 2^M_s to some existential proofs [5]), based on the well known theorem of Baire on the category.

The set-theoretical distance $\varrho_s(X,Y)$ constitutes a measure of the difference between X and Y only from the set-theoretical and metric point of view. But it does not measure the difference between the topological structures of them. It is clear that $\varrho_s(X,Y)$ can be arbitrarily small, though the topological structures of X and Y are completely different.

It is the purpose of this note to construct other two metrics for compacta: one called *metric of continuity* ϱ_c and the other *metric of homotopy* ϱ_h. The first of them has a rather auxiliary character. Its definition is simple and intuitive and it sets off the topological differences between compacta—but unfortunately it is not complete. The second metric ϱ_h has a clear topological sense only for the ANR-sets (= absolute neigh-

[1]) By a *compactum* we understand a compact metric space.

[2]) See **Hausdorff** [9], p. 293.

[3]) A metric space is *complete* if every Cauchy sequence of its points has a limit.

[4]) See **Kuratowski** [12], p. 314.

[5]) Cf. **Mazurkiewicz** [15].

bourhood retracts) [6]) lying in M. By the metric ϱ_h the class of all ANR-sets lying in a compactum M of a finite dimension is complete, and for those sets the metric ϱ_h seems to be an adequate measure for the difference of their topological structures.

2. Metric of continuity ϱ_c. For every $X,Y\in 2^M$ we define $\varrho_c(X,Y)$ as the lower bound of the numbers $t\geqslant 0$ such that there exists a continuous mapping φ of X into Y and a continuous mapping ψ of Y into X satisfying the conditions

$$(2)\qquad \begin{aligned} &\varrho\big(x,\varphi(x)\big)\leqslant t &&\text{for every}\quad x\in X,\\ &\varrho\big(y,\psi(y)\big)\leqslant t &&\text{for every}\quad y\in Y.\end{aligned}$$

One sees at once that the function $\varrho_c(X,Y)$ is non-negative and that it satisfies the conditions

$$\begin{aligned} &\varrho_c(X,Y)=0 \quad\text{if and only if}\quad X=Y,\\ &\varrho_c(X,Y)=\varrho_c(Y,X),\\ &\varrho_c(X,Y)+\varrho_c(Y,Z)\geqslant\varrho_c(X,Z)\end{aligned}$$

for every $X,Y,Z\in 2^M$. Hence $\varrho_c(X,Y)$ constitutes a distance in the set 2^M. The metric space obtained in this manner from 2^M will be denoted by 2^M_c.

Let us observe that for every number t satisfying the inequalities (2) we have $\varrho_s(X,Y)\leqslant t$. Hence

$$(3)\qquad \varrho_s(X,Y)\leqslant\varrho_c(X,Y).$$

It follows that the identical mapping in 2^M induces a continuous mapping of 2^M_c onto 2^M_s. But the converse is not true. For instance, denoting by X_k the set constituted by the numbers $0,1/k,2/k,\ldots,(k-1)/k,1$, and by X_0 the closed interval $\langle 0,1\rangle$, we see at once that

$$\lim_{k\to\infty}\varrho_s(X_k,X_0)=0\qquad\text{and}\qquad \lim_{k\to\infty}\varrho_c(X_k,X_0)=\frac{1}{2}.$$

Using the terminology due to **Mazurkiewicz** [7]) we shall say that the metric ϱ_c is *more restrictive* than the metric ϱ_s.

Let us consider some examples to illustrate the contents of the metrics ϱ_s and ϱ_c. We denote by E_n the n-dimensional Euclidean space and by Q_n the n-dimensional cube, defined as the set of all points $(x_1,x_2,\ldots,x_n)\in E_n$ satisfying the inequalities

$$|x_i|\leqslant 1\qquad\text{for}\quad i=1,2,\ldots,n.$$

[6]) By ANR-*set* we understand here every compactum A such that if M is a metric space and B a subset of M homeomorphic with A, then there exists a neighbourhood U of B in M and a continuous function r (called *retraction* of U to B) mapping U onto B in such a manner that for every $x\in B$ it is $r(x)=x$.

[7]) See **Mazurkiewicz** [16].

Example 1. Setting

$$A_k = \underset{(x_1,x_2)}{E}\left[x_2 = \cos\frac{\pi}{x_1},\ \frac{1}{k} \leqslant x_1 \leqslant 1\right] \quad \text{for} \quad k = 1, 2, \ldots,$$

we obtain a sequence of simple arcs lying in Q_2. This sequence is convergent in the space $2_s^{Q_2}$, and also in the space $2_c^{Q_2}$, to the closure of the set $\underset{(x_1,x_2)}{E}[x_2 = \cos\pi x_1, 0 < x_1 \leqslant 1]$. It follows that in $2_c^{Q_2}$ the set composed of all locally connected continua, and also the set composed of all AR-sets (= absolute retracts)[8]), is not closed.

Example 2. Let B_0 denote the closure of the set

$$\underset{(x_1,x_2)}{E}\left[x_2 = \cos\frac{\pi}{x_1},\ 0 < |x_1| \leqslant 1\right] \subset Q_2.$$

Let us observe that for every arcwise connected continuum B we have $\varrho_c(B_0, B) \geqslant 1/2$. Otherwise there would exist a continuous mapping φ of B into B_0 such that $\varrho\big(p, \varphi(p)\big) < 1/2$ for every $p \in B$ and a continuous mapping ψ of B_0 into B such that $\varrho\big(q, \psi(q)\big) < 1/2$ for every $q \in B_0$. Then the abscissa of the point $\varphi\psi(1, -1)$ would be positive and the abscissa of the point $\varphi\psi(-1, -1)$ would be negative. But this is impossible, because the arcwise connected continuum $\varphi(B) \subset B_0$ joins the points $\varphi\psi(1, -1)$ and $\varphi\psi(-1, -1)$, and B_0 does not contain any simple arc joining two points of abscissae with different signs.

Example 3. Let us set

$$D_{kl} = \underset{t}{E}[2l \cdot 3^{-k} \leqslant t \leqslant (2l+1) \cdot 3^{-k}], \qquad k = 1, 2, \ldots, \quad l = 0, 1, \ldots, \frac{1}{2}(3^k - 1),$$

$$D_k = \sum_{l=0}^{(3^k-1)/2} D_{kl}, \qquad H_k = \langle 0; 1\rangle - D_k, \qquad F_k = D_k \cdot H_k,$$

$$A = (\langle 0; 1\rangle \times \langle 0; 1\rangle) - \sum_{k=1}^{\infty} (H \times H_k)^{9)}.$$

Evidently A is identical with the well known locally connected curve of Sierpiński[10]), universal for plane curves. Setting

$$A_k = A \cdot [(F_k \times \langle 0; 1\rangle) + (\langle 0; 1\rangle \times F_k)],$$

[8]) By AR-*set* we understand here every compactum A such that for every metric space if B is a subset of M homeomorphic with A, then there exists a continuous function r (*retraction* of M to B) mapping M onto B in such a manner that $r(x) = x$ for every $x \in B$.

[9]) The symbol $\times$ denotes the Cartesian multiplication.

[10]) See Sierpiński [17], p. 629.

we easily see that A_k is a 1-dimensional polyhedron, and that the sequence $\{A_k\}$ converges in $2_c^{Q_2}$ to A. It follows that in $2_c^{Q_2}$ the limit of a sequence of ANR-sets can be a continuum not locally contractible at each of its points.

Example 4. If $\{A_k\}$ is a sequence of compacta convergent in 2_s^M to a set A containing only one point, then $\{A_k\}$ converges to A also in the space 2_c^M.

Example 5. Let us set

$$A_0 = \underset{(x_1,x_2)}{E}[x_1 = \cos t; x_2 = \sin t; 0 \leqslant t \leqslant 2\pi],$$

$$A_k = \underset{(x_1,x_2)}{E}\left[x_1 = \cos t; x_2 = \sin t; \frac{1}{k} \leqslant t \leqslant 2\pi\right] \quad \text{for} \quad k = 1, 2, \ldots$$

Evidently $A_k \to A$ in the space $2_s^{Q_2}$, but $A_k \not\to A_0$ in the space $2_c^{Q_2}$ because every continuous mapping φ of the circle A_0 into $A_k \subsetneqq A_0$ is not homotopic to the identity, and consequently there exists a point $p \in A_0$ such that $\varrho\big(p, \varphi(p)\big) = 2$. Hence $\varrho_c(A_0, A_k) \geqslant 2$ for every $k = 1, 2, \ldots$

On the other hand it is evident that the sets A_k constitute in $2_c^{Q_2}$ a Cauchy sequence. It follows that the *space* $2_c^{Q_2}$ *is not complete*.

3. Properties of the metric ϱ_c. The examples just given show that the metric of continuity cannot be regarded as adequate for our aims. But the metric of continuity is simple and intuitive, and it will be useful to construct another metric (metric of homotopy). Therefore we shall state some of its properties.

It is known[11]) that for every n-dimensional compactum X there exists an $\varepsilon > 0$ such that for every continuous mapping φ of X satisfying the inequality

$$\varrho\big(x, \varphi(x)\big) < \varepsilon \quad \text{for every} \quad x \in X$$

we have $\dim \varphi(X) \geqslant n$. It follows that

(4) *If* $\dim X \geqslant n$, *then there exists an* $\varepsilon > 0$ *such that* $\varrho_c(X, Y) < \varepsilon$ *implies* $\dim Y \geqslant n$.

We infer that

(5) *The set composed by all* $X \in 2_c^M$ *with* $\dim X \geqslant n$ *is open.*

Now let us show that there exists a relation between homological properties of two compacta X and Y and their distance $\varrho_c(X, Y)$. Let n be a non-negative integer. We denote by $p^n(X)$ the upper bound of the

[11]) Cf. for instance Kuratowski [13], p. 64.

integers k for which there exists an $\varepsilon_k>0$ such that for every $\eta>0$ a system of k homologically ε_k-independent n-dimensional η-cycles lies in X[12]).

(6) *If $p^n(X)\geqslant m$, then there exists an $\varepsilon>0$ such that $\varrho_c(X,Y)<\varepsilon$ implies $p^n(Y)\geqslant m$.*

Proof. Let ε_m be a positive number such that for every $\eta>0$ there exists in X a system of m η-cycles homologically ε_m-independent in X. We shall show that for $\varepsilon=\varepsilon_m/6$ the proposition (6) is satisfied.

If $\varrho_c(X,Y)<\varepsilon$, then there exists a continuous mapping φ of X into Y and a continuous mapping ψ of Y into X such that

$$\varrho\big(x,\varphi(x)\big)<\varepsilon \quad \text{for every} \quad x\in X,$$
$$\varrho\big(y,\psi(y)\big)<\varepsilon \quad \text{for every} \quad y\in Y.$$

Since φ and ψ are uniformly continuous on the compacta X and Y, there exists a positive η such that

$$(7)\quad \varrho(x,x')<\eta \quad \text{implies} \quad \varrho\big(\varphi(x),\varphi(x')\big)<\tfrac{1}{3}\varepsilon_m \quad \text{and} \quad \varrho\big(\psi\varphi(x),\psi\varphi(x')\big)<\tfrac{1}{3}\varepsilon_m \quad \text{for every} \quad x,x'\in X,$$

$$(8)\quad \varrho(y,y')<\eta \quad \text{implies} \quad \varrho\big(\psi(y),\psi(y')\big)<\tfrac{1}{3}\varepsilon_m \quad \text{for every} \quad y,y'\in Y.$$

Let $\gamma_1,\gamma_2,\ldots,\gamma_m$ be a system of n-dimensional η-cycles homologically ε_m-independent in X. By (7) the function φ maps them onto some n-dimensional $\varepsilon_m/3$-cycles $\gamma_{1\varphi},\gamma_{2\varphi},\ldots,\gamma_{m\varphi}$ lying in Y, and the function ψ maps the last cycles onto some n-dimensional $\varepsilon_m/3$-cycles $\gamma_{1\psi\varphi},\gamma_{2\psi\varphi},\ldots,\gamma_{m\psi\varphi}$ lying in X. But

$$\varrho\big(\psi\varphi(x),x\big)\leqslant\varrho\big(\psi\varphi(x),\varphi(x)\big)+\varrho\big(\varphi(x),x\big)\leqslant 2\varepsilon=\tfrac{1}{3}\varepsilon_m \quad \text{for every} \quad x\in X.$$

It follows that the $\varepsilon_m/3$-cycle $\gamma_{i\psi\varphi}$ is ε_m-homologous in X with the cycle γ_i, for $i=1,2,\ldots,m$. Consequently, the $\varepsilon_m/3$-cycles $\gamma_{1\psi\varphi},\gamma_{2\psi\varphi},\ldots,\gamma_{m\psi\varphi}$ are homologically ε_m-independent in X. But then the cycles $\gamma_{1\varphi},\gamma_{2\varphi},\ldots,\gamma_{m\varphi}$ are homologically ε-independent in Y. For otherwise there would exist in Y an $(n+1)$-dimensional ε-chain $\varkappa$ bounded by a linear combination

[12]) A set composed of $\eta+1$ points of X, having the diameter $<\eta$ is called *n-dimensional η-simplex in X*. The notions of an *oriented η-simplex in X*, of an *η-chain in X* (with arbitrarily given coefficients) and of an *η-cycle in X* are introduced as usual. A system $\gamma_1,\gamma_2,\ldots,\gamma_k$ of n-dimensional η-cycles in X is said to be *homologically ε-independent in X*, provided that a linear combination $c_1\cdot\gamma_1+c_2\cdot\gamma_2+\ldots+c_k\cdot\gamma_k$ with integral coefficients constitute the boundary of an ε-chain in X only if all coefficients vanish.

$c_1\gamma_{1\varphi}+c_2\gamma_{2\varphi}+\ldots+c_m\gamma_{m\varphi}$, where not all coefficients c_i vanish. The function ψ maps the ε-chain $\varkappa$ onto a 3ε-chain $\varkappa_\psi$ in X with the boundary $c_1\gamma_{1\psi\varphi}+c_2\gamma_{2\psi\varphi}+\ldots+c_m\gamma_{m\psi\varphi}$. But this is impossible, because $3\varepsilon=\varepsilon_m/2$ and the cycles $\gamma_{1\psi\varphi},\gamma_{2\psi\varphi},\ldots,\gamma_{m\psi\varphi}$ are homologically ε_m-independent in X. Hence (6) is proved. It follows that

(9) *The set composed of all compacta $X\subset M$ with $p^n(X)\geqslant m$ is open in 2^M_c.*

We say that a compactum X has the *property* $\mathfrak{F}$ if for every continuous mapping f of X into itself there exists a point $x\in X$ such that $f(x)=x$. Let us show that

(10) *If $X\in 2^M$ does not have the property $\mathfrak{F}$, then there exists an $\varepsilon>0$ such that $\varrho_c(X,Y)<\varepsilon$ implies that also Y does not have property $\mathfrak{F}$.*

Proof. Let f be a continuous mapping of X into itself such that $f(x)\neq x$ for every $x\in X$. Since X is compact, there exists a positive η such that

$$\varrho\big(x,f(x)\big)\geqslant\eta \quad \text{for every} \quad x\in X.$$

Let us show that the proposition (10) is satisfied for the number $\varepsilon=\eta/2$.

If $\varrho_c(X,Y)<\varepsilon$, then there exists a continuous mapping φ of X into Y and a continuous mapping ψ of Y into X such that

$$\varrho\big(\varphi(x),x\big)<\varepsilon \quad \text{for every} \quad x\in X,$$
$$\varrho\big(\psi(y),y\big)<\varepsilon \quad \text{for every} \quad y\in Y.$$

Then $\varphi f\psi$ is a continuous mapping of Y into itself, and for every $y\in Y$ we have

$$\varrho\big(\varphi f\psi(y),y\big)\geqslant\varrho\big(f\psi(y),\psi(y)\big)-\varrho\big(\varphi f\psi(y),f\psi(y)\big)-\varrho\big(\psi(y),y\big)>\eta-2\varepsilon=0.$$

Hence $\varphi f\psi(y)\neq y$ for every $y\in Y$, *i. e.* Y does not have the property $\mathfrak{F}$.

It follows by (10) that

(11) *The set composed of all compacta $X\subset M$ with the property $\mathfrak{F}$ is closed in 2^M_c.*

4. 2^M_c as a topological invariant of M. Problems. Let us show that the topological structure of the space 2^M_c depends only upon the topological structure of M, *i. e.*

(12) *If M and M' are homeomorphic, then also 2^M_c and $2^{M'}_c$ are homeomorphic.*

Proof. Let h be a homeomorphic mapping of M onto M'. If we assign to every compactum $X\in 2^M$ the compactum $h(X)\in 2^{M'}$, then we obtain a 1-1 mapping of 2_c^M onto $2_c^{M'}$. It remains to show that this mapping and its inverse are both continuous. Since our assumptions are symmetrical, it suffices to show that the mapping

$$X \to h(X)$$

is continuous at every point $X \in 2_c^M$.

Since h is continuous and X is compact, there exists for every $\varepsilon > 0$ an $\eta > 0$ such that

(13) $z \in X$, $z' \in M$, and $\varrho(z, z') < \eta$ imply together $\varrho(h(z), h(z')) < \varepsilon$.

If $Y \in 2_c^M$ and $\varrho_c(X, Y) < \eta$, then there exists a continuous mapping φ of X into Y such that

$$\varrho(x, \varphi(x)) < \eta \quad \text{for every} \quad x \in X.$$

Let h^{-1} denote the inverse of the homeomorphism h. Setting

$$\varphi'(x') = h\varphi h^{-1}(x') \quad \text{for every} \quad x' \in h(X),$$

we obtain a continuous mapping φ' of $h(X)$ into $h(Y)$. Since $h^{-1}(x') \in X$ and $\varrho(\varphi h^{-1}(x'), h^{-1}(x')) < \eta$, we infer by (13) that

(14) $\varrho(\varphi'(x'), x') = \varrho(h\varphi h^{-1}(x'), hh^{-1}(x')) < \varepsilon$ for every $x' \in h(X)$.

On the other hand, there exists a continuous mapping ψ of Y into X such that

$$\varrho(y, \psi(y)) < \eta \quad \text{for every} \quad y \in Y.$$

Setting

$$\psi'(y') = h\psi h^{-1}(y') \quad \text{for every} \quad y' \in h(Y),$$

we obtain a continuous mapping ψ' of $h(Y)$ into $h(X)$. Since $\psi h^{-1}(y') \in X$ and $\varrho(\psi h^{-1}(y'), h^{-1}(y')) < \eta$, we infer by (13) that

(15) $\varrho(\psi'(y'), y') = \varrho(h\psi h^{-1}(y'), hh^{-1}(y')) < \varepsilon$ for every $y' \in h(Y)$.

It follows by (14) and (15) that $\varrho_c(h(X), h(Y)) < \varepsilon$, *i. e.* the continuity of the mapping $X \to h(X)$ is proved.

PROBLEM 1. *Let M be a separable space. Is it true that the space 2_c^M is also separable?*

PROBLEM 2. *Does there exist an AR-set M such that the space 2_c^M is not connected?*

PROBLEM 3. *Does there exist an ANR-set M such that the space 2_c^M is not locally connected?*

5. Module of contractibility. Homotopical convergence [13]. For every metric space $A \neq 0$ and every $t \geqslant 0$ let us denote by $T_{A,t}$ the set composed of the number 1 and of all numbers $\tau \geqslant t$ such that every subset E of A with diameter $\leqslant t$ is contractible to a point in a subset of A with diameter $\leqslant \tau$. Let $\varphi_A(t)$ denote the lower bound of the set $T_{A,t}$. Evidently φ_A is a non-decreasing function of the variable $t \geqslant 0$, satisfying the conditions

(16) $\varphi_A(0) = 0, \quad \mathrm{Min}(1, t) \leqslant \varphi_A(t) \leqslant 1$ for every $t \geqslant 0$.

The function φ_A will be said to be the *module of the contractibility* of the space A. Evidently, if A is compact, then the continuity of φ_A at the point $t = 0$ is equivalent to the local contractibility [14]) of A.

A class $\mathfrak{A}$ of spaces $A \neq 0$ will be said to be *equally locally contractible* if there exists a continuous function $\varphi(t)$, defined for every $t \geqslant 0$ and such that

(17) $0 = \varphi(0) \leqslant \varphi_A(t) \leqslant \varphi(t)$ for every $t \geqslant 0$.

Let M be a metric space. We shall denote by 2_0^M the class of all non-empty ANR-sets lying in M. A sequence $\{A_n\} \subset 2_0^M$ will be said to be *homotopically convergent* to a set $A_0 \in 2_0^M$ if

1° $\lim_{n\to\infty} \varrho_s(A_n, A_0) = 0$,

2° *the class composed of the sets $A_0, A_1, \ldots$ is equally locally contractible.*

Evidently, by this last definition the class 2_0^M becomes a L^*-space (in the sense of Fréchet [6]). We shall show, and this is the main aim of this note, that if M is a finite dimensional compactum, then this L^*-space can be metrized in such a manner that the obtained metric space is complete. Before doing this (in section 16), we shall prove some lemmas.

[13]) Compare the notion of the *n-regular convergence* (in the sense of homology) introduced by G. T. Whyburn [19] and [20] and the notion of the *n-regular convergence space* (in the sense of homology) introduced by P. A. White [18]. See also E. G. Begle [2]. A notion of *homotopy-n-regular convergence* was recently introduced by M. L. Curtis [7]. Our homotopical convergence is equivalent (for compact spaces of finite dimension) with the homotopy-n-regular convergence for all $n = 0, 1, 2, \ldots$

[14]) A space M is said to be *locally contractible at the point* $a \in M$ if for every neighbourhood U of a (in M) there exists a neighbourhood U_0 of a (in M) contractible to a over U, *i. e.* there exists a continuous function $f(x, t)$, defined on the Cartesian product $U_0 \times \langle 0; 1 \rangle$, with values belonging to U and such that $f(x, 0) = x$ and $f(x, 1) = a$ for every $x \in U_0$. By a *locally contractible* space one understands a space locally contractible at each of its points.

6. Concave functions. A real valued function $y=f(x)$, defined in an interval I (closed or not, finite or infinite), is said to be *concave* if

$$f\big(tx_1+(1-t)x_2\big)\geqslant tf(x_1)+(1-t)f(x_2) \quad \text{for every} \quad x_1,x_2\in I \quad \text{and} \quad 0\leqslant t\leqslant 1. \tag{18}$$

Evidently, if f is concave in I, then it is concave also in every interval $I'\subset I$.

It is easy to observe that the condition (18) is equivalent to the following one:

If $x_1,x_1',x_2,x_2'\in I$; $x_1<x_2$, $x_1'<x_2'$, $x_1\leqslant x_1'$, $x_2\leqslant x_2'$ then

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\geqslant\frac{f(x_2')-f(x_1')}{x_2'-x_1'}. \tag{19}$$

It follows that if $I=\langle\alpha;\beta\rangle$ and f is a concave function in I, then the inequality $\alpha<\alpha_1\leqslant x_1<x_2\leqslant\beta_1<\beta$ implies that

$$\frac{f(\alpha_1)-f(\alpha)}{\alpha_1-\alpha}\geqslant\frac{f(x_2)-f(x_1)}{x_2-x_1}\geqslant\frac{f(\beta)-f(\beta_1)}{\beta_1-\beta}. \tag{20}$$

Consequently, f satisfies in the interval $\langle\alpha_1;\beta_1\rangle$ the condition of Lipschitz [15]. We conclude that f is continuous in the open interval $(\alpha;\beta)$.

We infer also by (19) that a function f, concave in an open interval $(\alpha;\beta)$, either is monotonic or has in the interval $(\alpha;\beta)$ exactly one maximum. Moreover, let us observe that

(21) *Every function f, concave in an infinite interval $\langle\alpha;\infty)$, either is not decreasing or $\lim\limits_{x\to\infty}f(x)=-\infty$.*

Finally, let us show that

(22) *If f and g are two non-decreasing concave functions, f in $\langle\alpha;\beta\rangle$ and g in $\langle f(\alpha);\infty)$, then gf is not decreasing and concave in $\langle\alpha;\beta\rangle$.*

It suffices to show that in $I=\langle\alpha;\beta\rangle$ the function gf satisfies the condition (19). Let $x_1,x_1',x_2,x_2'\in I$ be as in (19). If $f(x_1')=f(x_2')$, then $gf(x_2')-gf(x_1')=0$, and the inequality

$$\frac{gf(x_2)-gf(x_1)}{x_2-x_1}\geqslant\frac{gf(x_2')-gf(x_1')}{x_2'-x_1'} \tag{23}$$

is satisfied. If, however, $f(x_1')<f(x_2')$ then, by (19), $f(x_2)-f(x_1)>0$, and we infer that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\geqslant\frac{f(x_2')-f(x_1')}{x_2'-x_1'}>0. \tag{24}$$

[15]) A real valued function f, defined in an interval I, is said to *satisfy the condition of Lipschitz* if there exists a constant $\varkappa>0$ (*coefficient of Lipschitz*) such that for every $x,y\in I$ we have $|f(x)-f(y)|\leqslant\varkappa|x-y|$.

Since $f(x_1)<f(x_2)$, $f(x_1')<f(x_2')$, $f(x_1)\leqslant f(x_1')$ and $f(x_2)\leqslant f(x_2')$, we infer by the concavity of g that

$$\frac{gf(x_2)-gf(x_1)}{f(x_2)-f(x_1)}\geqslant\frac{gf(x_2')-gf(x_1')}{f(x_2')-f(x_1')}\geqslant 0. \tag{25}$$

To obtain the required inequality (23) it suffices to multiply the inequalities (24) and (25)

7. Function f_φ. Let φ be a real valued function, defined in an interval $\langle\alpha;\beta)$ and having a finite upper bound. For every $x\in\langle\alpha;\beta)$, let us denote by $\omega(x)$ the set composed of all ordered pairs (x_1,x_2) of numbers $x_1,x_2\in\langle\alpha;\beta)$ such that $x_1<x_2$ and $x\in\langle x_1;x_2\rangle$. If $(x_1,x_2)\in\omega(x)$, then there exists exactly one number $t\in\langle 0;1\rangle$ such that $x=tx_1+(1-t)x_2$.

Let us set

$$\varphi_{x_1x_2}(x)=t\varphi(x_1)+(1-t)\varphi(x_2)$$

and

$$f_\varphi(x)=\operatorname*{Sup}_{(x_1,x_2)\in\omega(x)}\varphi_{x_1x_2}(x). \tag{26}$$

Evidently,

$$f_\varphi(\alpha)=\varphi(\alpha) \quad \text{and} \quad \varphi(x)\leqslant f_\varphi(x)<\operatorname*{Sup}_{y\in\langle\alpha;\beta)}\varphi(y) \quad \text{for every} \quad x\in\langle\alpha;\beta), \tag{27}$$

and if φ is a concave function in $\langle\alpha;\beta)$, then $f_\varphi=\varphi$.

Let us show that f_φ is a concave function. Let $x_1<x_2$ be two numbers belonging to $\langle\alpha;\beta)$. It suffices to prove that for every two numbers $a_1<f_\varphi(x_1)$ and $a_2<f_\varphi(x_2)$ we have

$$f_\varphi\big(tx_1+(1-t)x_2\big)\geqslant ta_1+(1-t)a_2 \quad \text{for every} \quad 0\leqslant t\leqslant 1. \tag{28}$$

Since $a_1<f_\varphi(x_1)$, there exist in $\langle\alpha;\beta)$ two numbers $\xi_1<\xi_2$ and a $t_1\in\langle 0;1\rangle$ such that $x_1=t_1\xi_1+(1-t_1)\xi_2$ and

$$a_1<t_1\varphi(\xi_1)+(1-t_1)\varphi(\xi_2). \tag{29}$$

Similarly $a_2<f_\varphi(x_2)$ implies that there exist in $\langle\alpha;\beta\rangle$ two numbers $\xi_3<\xi_4$ and a $t_2\in\langle 0;1\rangle$ such that $x_2=t_2\xi_3+(1-t_2)\xi_4$ and

$$a_2<t_2\varphi(\xi_3)+(1-t_2)\varphi(\xi_4). \tag{30}$$

It follows by (29) and (30) that

$$ta_1+(1-t)a_2<t[t_1\varphi(\xi_1)+(1-t_1)\varphi(\xi_2)]+(1-t)[t_2\varphi(\xi_3)+(1-t_2)\varphi(\xi_4)],$$

i. e. the point $p=\big(tx_1+(1-t)x_2,ta_1+(1-t)a_2\big)$ of the Euclidean plane E_2 lies below the segment A joining the points $\big(x_1,t_1\varphi(\xi_1)+(1-t_1)\varphi(\xi_2)\big)$ and $\big(x_2,t_2\varphi(\xi_3)+(1-t_2)\varphi(\xi_4)\big)$. But the segment A lies in the convex quadruple with the vertices $p_\nu=\big(\xi_\nu,\varphi(\xi_\nu)\big)$, $\nu=1,2,3,4$. It follows that there exist two natural numbers $\mu,\nu\leqslant 4$ such that p lies below the segment joining p_μ and p_ν. It means that there exists a number $\tau \epsilon \langle 0;1\rangle$ such that $tx_1+(1-t)x_2=\tau\xi_\mu+(1-\tau)\xi_\nu$ and

$$ta_1+(1-t)a_2\leqslant\tau\varphi(\xi_\mu)+(1-\tau)\varphi(\xi_\nu).$$

But, by the definition of the function f_φ, we have

$$f_\varphi\big(tx_1+(1-t)x_2\big)=f_\varphi\big(\tau\xi_\mu+(1-\tau)\xi_\nu\big)\geqslant\tau\varphi(\xi_\mu)+(1-\tau)\varphi(\xi_\nu).$$

This implies (28), and thus the concavity of f_φ is proved.

8. Some properties of f_φ. Let us assume, as in section 7, that φ is a real function, defined in an interval $\langle\alpha;\beta)$ and having the finite upper bound. Then

(31) *If $\varphi(x)\leqslant\psi(x)$ for every $x\epsilon\langle\alpha;\beta)$ and if ψ is concave, then $f_\varphi(x)\leqslant\psi(x)$ for every $x\epsilon\langle\alpha;\beta)$.*

Otherwise there would exist an $x_0\epsilon\langle\alpha;\beta)$ and an $\varepsilon>0$ such that

(32) $$f_\varphi(x_0)-\varepsilon>\psi(x_0).$$

By the definition of f_φ there exist $x_1,x_2\epsilon\langle\alpha;\beta)$ and a $t\epsilon\langle 0;1\rangle$ such that $x_0=tx_1+(1-t)x_2$ and

(33) $$f_\varphi(x_0)-\varepsilon<t\varphi(x_1)+(1-t)\varphi(x_2).$$

But ψ is concave, whence

(34) $$\psi(x_0)\geqslant t\psi(x_1)+(1-t)\psi(x_2).$$

We infer by (30), (31) and (32) that

$$t\psi(x_1)+(1-t)\psi(x_2)<t\varphi(x_1)+(1-t)\varphi(x_2),$$

contrary to the assumption $\varphi(x_1)\leqslant\psi(x_1)$, $\varphi(x_2)\leqslant\psi(x_2)$ and $0\leqslant t\leqslant 1$.

Now let us prove that

(35) *If $\lim_{x\to\alpha}\varphi(x)=a$, then $\lim_{x\to\alpha}f_\varphi(x)=a$.*

Proof. Given $\varepsilon>0$, there exists an $\eta>0$ such that

$$a-\frac{\varepsilon}{2}\leqslant\varphi(x)\leqslant a+\frac{\varepsilon}{2}\quad\text{for}\quad \alpha\leqslant x\leqslant\alpha+\eta.$$

According to our hypothesis, φ has a finite upper bound c in the interval $\langle\alpha;\beta)$. Let n be a natural number such that $1/n(c-a)<\varepsilon/2$, and let $x\epsilon\langle\alpha;\alpha+\eta/n\rangle$. If $\alpha\leqslant x_1\leqslant x\leqslant x_2<\beta$, then there exists a $t\epsilon\langle 0;1\rangle$ such that

$$x=tx_1+(1-t)x_2.$$

If $x_2\leqslant\alpha+\eta$, then $\varphi(x_1)\leqslant a+\varepsilon/2$ and $\varphi(x_2)\leqslant a+\varepsilon/2$. Hence

$$t\varphi(x_1)+(1-t)\varphi(x_2)\leqslant t\Big(a+\frac{\varepsilon}{2}\Big)+(1-t)\Big(a+\frac{\varepsilon}{2}\Big)=a+\frac{\varepsilon}{2}<a+\varepsilon.$$

If, however, $x_2>\alpha+\eta$, then $\alpha+\eta/n\geqslant x=tx_1+(1-t)x_2\geqslant t\alpha+(1-t)(\alpha+\eta)=\alpha+(1-t)\eta$, whence $1-t\leqslant 1/n$. It follows that

$$t\varphi(x_1)+(1-t)\varphi(x_2)\leqslant t\Big(a+\frac{\varepsilon}{2}\Big)+(1-t)c$$

$$=ta+(1-t)a+t\cdot\frac{\varepsilon}{2}+(1-t)(c-a)\leqslant a+\frac{\varepsilon}{2}+\frac{1}{n}(c-a)<a+\varepsilon.$$

Hence, in both cases, for $x\epsilon\langle\alpha;\alpha+\eta/n\rangle$ and $\alpha\leqslant x_1\leqslant x\leqslant x_2<\beta$, we have

$$t\varphi(x_1)+(1-t)\varphi(x_2)<a+\varepsilon.$$

Consequently,

$$f_\varphi(x)\leqslant a+\varepsilon\quad\text{for every}\quad x\epsilon\Big\langle\alpha;\alpha+\frac{1}{n}\eta\Big\rangle.$$

On the other hand, $f_\varphi(x)\geqslant\varphi(x)\geqslant a-\varepsilon/2$ for $x\epsilon\langle\alpha;\alpha+\eta/n\rangle$. This implies (35).

Moreover, let us observe that if φ and ψ are two real functions defined in $\langle\alpha;\beta)$ and having finite upper bounds, then

(36) $$f_{\varphi+\psi}(x)\leqslant f_\varphi(x)+f_\psi(x)\quad\text{for every}\quad x\epsilon\langle\alpha;\beta).$$

In particular, if ψ satisfies, for an $\varepsilon>0$, the inequality $-\varepsilon\leqslant\psi(x)\leqslant\varepsilon$ for every $x\epsilon\langle\alpha;\beta)$, then we infer by (27) that $-\varepsilon\leqslant f_\psi(x)\leqslant\varepsilon$ for every $x\epsilon\langle\alpha;\beta)$, and we find by (36) that

(37) *If $|\psi(x)|\leqslant\varepsilon$ for every $x\epsilon\langle\alpha;\beta)$, then $|f_{\varphi+\psi}(x)-f_\varphi(x)|\leqslant\varepsilon$ for every $x\epsilon\langle\alpha;\beta)$.*

9. Indicatrices. By an *indicatrix* we understand a concave function $\lambda(t)$, defined in the interval $\langle 0;\infty)$ and satisfying the conditions

(38) $$\lambda(0)=0,\qquad 0\leqslant\lambda(t)\leqslant 1\quad\text{for every}\quad t>0.$$

It follows by (19) and (21) that

(39) *Every indicatrix is not decreasing in $\langle 0;\infty)$ and it is uniformly continuous in $(0;\infty)$.*

Applying (22), we infer that

(40) *If λ is an indicatrix and f a non-decreasing, concave function defined in $\langle 0;\infty)$ and such that $f(0)=0$, then the superposition λf is an indicatrix.*

Moreover, let us observe that

(41) *If λ_1, λ_2 are two continuous indicatrices such that $\lambda_1(t) \leqslant \lambda_2(t)$ for every $t \geqslant 0$, and ε, η two positive numbers such that $|t_1 - t_2| < \eta$ implies $|\lambda_2(t_1) - \lambda_2(t_2)| < \varepsilon$, then $|t_1 - t_2| < \eta$ implies also $|\lambda_1(t_1) - \lambda_1(t_2)| < \varepsilon$.*

For otherwise there would exist two numbers $t_1, t_2 \in \langle 0; \infty)$ such that $0 < t_2 - t_1 < \eta$ and that $\lambda_1(t_2) - \lambda_1(t_1) \geqslant \varepsilon$. It follows by (19) that

$$\frac{\lambda_1(t_2 - t_1) - \lambda_1(0)}{t_2 - t_1} \geqslant \frac{\varepsilon}{t_2 - t_1}$$

and consequently $\lambda_1(t_2 - t_1) \geqslant \varepsilon$. On the other hand $\lambda_2(t_2 - t_1) - \lambda_2(0) < \varepsilon$, *i. e.* $\lambda_2(t_2 - t_1) < \varepsilon$. It follows that $\lambda_1(t_2 - t_1) > \lambda_2(t_2 - t_1)$, contrary to our hypothesis.

We infer by (41) that

(42) *If λ_0 is a continuous indicatrix, then the family of all indicatrices λ satisfying the inequality $\lambda(t) \leqslant \lambda_0(t)$ for every $t > 0$ is equally continuous in $\langle 0; \infty)$* [16].

LEMMA. *Let $\{\lambda_n\}$ be a uniformly convergent sequence of continuous indicatrices. Then there exists a continuous indicatrix λ_0 such that*

$$\lambda_n(t) \leqslant \lambda_0(t) \quad \textit{for every} \quad t \geqslant 0. \tag{43}$$

Proof. Setting

$$\varphi_n(t) = \operatorname*{Sup}_{k \leqslant n} \lambda_n(t),$$

we see at once that φ_n constitute a sequence uniformly convergent in the interval $\langle 0; \infty)$, and that the function

$$\varphi(t) = \lim_{n \to \infty} \varphi_n(t)$$

is continuous and satisfies the inequality

$$\lambda_n(t) \leqslant \varphi(t) \leqslant 1 \quad \text{for every} \quad t \geqslant 0 \quad \text{and} \quad n = 1, 2, \ldots$$

Setting

$$\lambda_0(t) = f_\varphi(t),$$

we obtain a concave function satisfying (43) and having values $\leqslant 1$. Applying (35), we infer that $\lim_{t \to 0} \lambda_0(t) = \lambda_0(0) = 0$, and consequently λ_0 is a continuous indicatrix.

Remark. Let us observe that in the case where, for a t_0, $\lambda_n(t) = c$ for every $t \geqslant t_0$ we have also $\varphi(t) = c$ for every $t \geqslant t_0$, and consequently $\lambda_0(t) = c$ for every $t \geqslant t_0$.

10. A lemma on indicatrices. Let us prove the following

LEMMA. *Let λ be a continuous indicatrix. For every $\varepsilon > 0$ there exists an $\eta > 0$ such that if φ is a function satisfying the inequality*

$$0 \leqslant \varphi(x) \leqslant \lambda(x) \quad \textit{for every} \quad x \geqslant 0 \tag{44}$$

and u a continuous function, defined in $\langle 0; \infty)$ and satisfying the conditions

$$|u(x)| < \eta \quad \textit{and} \quad x + u(x) \geqslant 0 \quad \textit{for every} \quad x \geqslant 0, \tag{45}$$

then, setting $\psi(x) = \varphi(x + u(x))$, we have

$$|f_\varphi(x) - f_\psi(x)| < \varepsilon \quad \textit{for every} \quad x \geqslant 0.$$

Proof. Since $\lim_{x \to 0} \lambda(x) = 0$, it follows by (44) that $\lim_{x \to 0} \varphi(x) = 0$, and we infer by (35) that $\lim_{x \to 0} f_\varphi(x) = 0$. Applying (35) and (39), we conclude that f_φ is uniformly continuous in the interval $\langle 0; \infty)$. Consequently, there exists a positive η_1 such that

$$|x - x'| < \eta_1 \quad \text{implies} \quad |f_\varphi(x) - f_\varphi(x')| < \frac{1}{6}\varepsilon. \tag{46}$$

First we prove that the inequality

$$|u(x)| < \eta_1 \quad \text{for every} \quad x > 0 \tag{47}$$

implies the inequality

$$f_\psi(x) < f_\varphi(x) + \frac{1}{3}\varepsilon \quad \text{for every} \quad x > 0. \tag{48}$$

By the definition of f_ψ, given in section 7, there exist x_1, x_2 and t such that $0 < x_1 \leqslant x \leqslant x_2$, $0 \leqslant t \leqslant 1$, $x = tx_1 + (1-t)x_2$ and that

$$f_\psi(x) < t\psi(x_1) + (1-t)\psi(x_2) + \frac{\varepsilon}{6}.$$

Since

$$\begin{aligned} t\psi(x_1) + (1-t)\psi(x_2) &= t\varphi[x_1 + u(x_1)] + (1-t)\varphi[x_2 + u(x_2)] \\ &\leqslant f_\varphi\Big[t\big(x_1 + u(x_1)\big) + (1-t)\big(x_2 + u(x_2)\big)\Big] \\ &= f_\varphi[tx_1 + (1-t)x_2 + tu(x_1) + (1-t)u(x_2)] = f_\varphi[x + tu(x_1) + (1-t)u(x_2)], \end{aligned}$$

[16]) A family F of functions mapping a metric space M into another metric space is said to be *equally continuous at a point $p \in M$*, if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for every function $f \in F$ and for every $q \in M$ with $\varrho(p, q) < \eta$ we have $\varrho(f(p), f(q)) < \varepsilon$. If F is equally continuous at every point $p \in M$, then we say that F is *equally continuous in M*.

and since (by (47))

$$|tu(x_1)+(1-t)u(x_2)|\leqslant t|u(x_1)|+(1-t)|u(x_2)|<\eta_1,$$

we have

$$f_\psi(x)<f_\varphi[x+tu(x_1)+(1-t)u(x_2)]+\frac{\varepsilon}{6}<f_\varphi(x)+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}=f_\varphi(x)+\frac{\varepsilon}{3},$$

and thus the proof of the inequality (48) is complete.

Now we shall show that there exists an $\eta>0$ such that

$$(49)\qquad |x-x'|<\eta \quad\text{implies}\quad |f_\psi(x)-f_\psi(x')|<\frac{1}{2}\varepsilon.$$

First let us suppose that $x>\eta/2$. Then $f_\psi(x)\geqslant f_\psi(\eta_1/2)$ and (because f_ψ is concave) none of the points $(\xi,0)$ with $\xi>0$ lies on the straight line joining the points $\big(\eta_1/2, f_\psi(\eta_1/2)\big)$ and $\big(x, f_\psi(x)\big)$. It follows that

$$(50)\qquad f_\psi(x)<\frac{2x}{\eta_1}\cdot f_\psi\left(\frac{1}{2}\eta_1\right).$$

We infer by (48) and (50) that for $x,x'>\eta_1/2$

$$f_\psi(x)<\frac{x}{\eta_1}\cdot\left[2f_\varphi\left(\frac{1}{2}\eta_1\right)+\varepsilon\right]\quad\text{and}\quad f_\psi(x')<\frac{x'}{\eta_1}\cdot\left[2f_\varphi\left(\frac{1}{2}\eta_1\right)+\varepsilon\right].$$

Since $f_\psi(0)=0$ we infer by (19) that

$$(51)\qquad x,x'>\frac{1}{2}\eta_1 \quad\text{implies}\quad |f_\psi(x)-f_\psi(x')|<\left[2f_\varphi\left(\frac{1}{2}\eta_1\right)+\varepsilon\right]\cdot\frac{|x-x'|}{\eta_1}.$$

Since $f(\eta_1/2)<\lambda(\eta_1/2)$, we conclude by (51) that in the interval $(\eta_1/2\,;\infty)$ the function f_ψ satisfies the condition of Lipschitz with the coefficient

$$(52)\qquad \varkappa=\frac{1}{\eta_1}\cdot\left[2\lambda\left(\frac{1}{2}\eta_1\right)+\varepsilon\right].$$

Let us observe that $\varkappa$ does not depend on the particular choice of the function φ (satisfying the inequality (44)) and of the function $u(s)$ (satisfying the inequality (45)).

Setting

$$(53)\qquad \eta=\operatorname{Min}\left(\frac{1}{2}\eta_1\,;\frac{\varepsilon}{2\varkappa}\right),$$

we see at once that for every $x,x'\geqslant 0$ such that $|x-x'|<\eta$ either $x,x'\leqslant\eta_1$ or $x,x'>\eta_1/2$. In the first case we have, by (46) and (48),

$$0<f_\psi(x)<\frac{\varepsilon}{6}+\frac{\varepsilon}{3}=\frac{\varepsilon}{2}\quad\text{and}\quad 0<f_\psi(x')<\frac{\varepsilon}{6}+\frac{\varepsilon}{3}=\frac{\varepsilon}{2},$$

whence $|f_\psi(x)-f_\psi(x')|<\varepsilon/2$. In the second case the last inequality follows by (51). Thus the proof of (49) is complete.

After these preliminaries we can complete the proof of our lemma in a few steps. By (48) it suffices to show that (45) and (53) imply

$$(54)\qquad f_\varphi(x)<f_\psi(x)+\varepsilon \quad\text{for every}\quad x>0.$$

If $x\leqslant\eta_1$, then (46) implies that $f_\varphi(x)\leqslant\varepsilon/6$, whence (54) is satisfied. If, however, $x>\eta_1$, then there exist three numbers x_1,x_2,t such that $x_1\neq x_2$, $0<x_1\leqslant x\leqslant x_2$, $0\leqslant t\leqslant 1$, $x=tx_1+(1-t)x_2$ and that

$$(55)\qquad f_\varphi(x)<t\varphi(x_1)+(1-t)\varphi(x_2)+\frac{1}{3}\varepsilon.$$

Now we distinguish the following two cases:

1. *There exists a positive x' such that $x_1=x_1'+u(x_1')$.*

Since the continuous function $x+u(x)$ takes arbitrarily large values, then there exists an $x_2'>0$ such that

$$(56)\qquad x_2=x_2'+u(x_2').$$

Then

$$|tx_1+(1-t)x_2-[tx_1'+(1-t)x_2']|=|tu(x_1')+(1-t)u(x_2')|<\eta,$$

and we infer, by (55), (48) and (49) that $f_\varphi(x)<t\varphi(x_1)+(1-t)\varphi(x_2)+\varepsilon/3$ $=t\psi(x_1')+(1-t)\psi(x_2')+\varepsilon/3\leqslant f_\psi[tx_1'+(1-t)x_2']+\varepsilon/3<f_\psi[tx_1+(1-t)x_2]$ $+\varepsilon/2+\varepsilon/3<f_\psi(x)+\varepsilon$. Hence (54) holds.

2. *$x_1'+u(x_1')\neq x_1$ for every $x_1'>0$.*

Since every number $y\geqslant\eta_1/2$ belongs (by (45) and (53)) to the set of values of the continuous function $x+u(x)$, we infer that $x_1<\eta_1/2$. It follows by (46) that $\varphi(x_1)\leqslant f_\varphi(x_1)<\varepsilon/6$. Since $x_2\geqslant x>\eta_1$, there exists a number $x_2'>0$ satisfying (56). Then $\varphi(x_2)=\psi(x_2')$ and $\varphi(x_1)<\varepsilon/6$, whence

$$(57)\qquad t\varphi(x_1)+(1-t)\varphi(x_2)<t\psi(x_1)+(1-t)\psi(x_2')+\frac{\varepsilon}{6}\leqslant f_\psi[tx_1+(1-t)x_2']+\frac{\varepsilon}{6}.$$

But

$$|tx_1+(1-t)x_2-[tx_1+(1-t)x_2']|=(1-t)|u(x_2')|<\eta,$$

and consequently we infer by (49) that

$$f_\psi[tx_1+(1-t)x_2']<f_\psi[tx_1+(1-t)x_2]+\frac{1}{2}\varepsilon=f_\psi(x)+\frac{\varepsilon}{2}.$$

It follows by (55) and (57) that

$$f_\varphi(x)<f_\psi(x)+\frac{\varepsilon}{3}+\frac{\varepsilon}{6}+\frac{\varepsilon}{2}=f_\psi(x)+\varepsilon.$$

Thus the inequality (54) is proved, *i.e.* the proof of our lemma is complete.

11. Indicatrix of contractibility. Let φ_A denote the module of the contractibility (defined in section 5) of a metric space $A \neq 0$. We shall understand by the *indicatrix of the contractibility* of A the concave function $\lambda_A(t)$, defined for $t \geqslant 0$ by the formula

(58) $$\lambda_A(t) = f_{\varphi_A}(t).$$

We infer by (16) and (27) that

(59) $$\lambda_A(0) = 0 \quad \text{and} \quad \operatorname{Min}(1, t) \leqslant \lambda_A(t) \leqslant 1 \quad \text{for every} \quad t \geqslant 0.$$

It follows that λ_A is an indicatrix. By (39) and (35) it is continuous (and also uniformly continuous) if and only if $\lim\limits_{t\to 0} \varphi_A(t) = 0$. We have already observed (in section 5) that the last condition is equivalent to the local contractibility of A. It follows that

(60) *The indicatrix of a locally contractible compactum is uniformly continuous.*

Finally, let us observe that a class $\mathfrak{A}$ of spaces $A \neq 0$ is equally locally contractible (in the sense of section 5) if and only if there exists a continuous indicatrix λ such that

(61) $$\lambda_A(t) \leqslant \lambda(t) \quad \text{for every} \quad A \in \mathfrak{A} \quad \text{and} \quad t \geqslant 0.$$

In fact, the condition (61) is evidently sufficient for the equal local contractibility of $\mathfrak{A}$. On the other hand, if $\mathfrak{A}$ is equally locally contractible and if $\varphi(t)$ is a continuous function satisfying (17), then, setting $\lambda(t) = f_\varphi(t)$, we obtain a continuous indicatrix λ satisfying condition (61).

Example 6. Let A be a convex set lying in a vectorial space [17]). Then $\varphi_A(t) = t$ for $0 \leqslant t \leqslant 1$ and $\varphi_A(t) = 1$ for $t \geqslant 1$. In this case φ_A is a concave function, whence $\lambda_A(t) = f_{\varphi_A}(t) = \varphi_A(t)$ for every $t \geqslant 0$. The function λ_A has the graph shown in fig. 1.

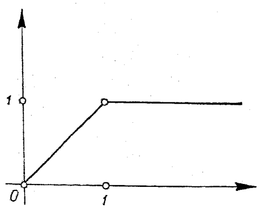

Fig. 1

Example 7. Let S_r be the circle defined in the Euclidean plane E_2 by the equation

$$x_1^2 + x_2^2 = r^2.$$

One sees easily that for $0 < r \leqslant 1/2$ the function φ_{S_r} has the graph shown in fig. 2 and the indicatrix λ_{S_r} has either the graph shown in fig. 3 (if $1/2r \geqslant 2/\sqrt{3}$) or that shown in fig. 4 (if $1/2r < 2/\sqrt{3}$). Let us observe that λ_{S_0} has the graph shown in fig. 1 and is not identical with $\lim\limits_{r\to 0} \lambda_{S_r}$.

[17]) General reference: Banach [1].

Example 8. Let A_ε denote, for $\varepsilon \leqslant \pi/2$ the set of all points $(\frac{1}{2}\cos\vartheta, \frac{1}{2}\sin\vartheta) \in E_2$ with $\varepsilon \leqslant \vartheta \leqslant 2\pi$ (see fig. 5). Fig. 6 represents the graph

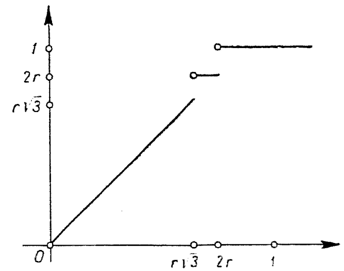

Fig. 2

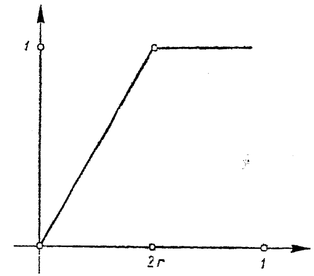

Fig. 3

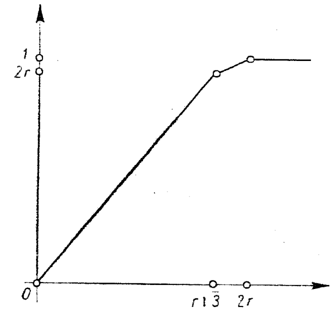

Fig. 4

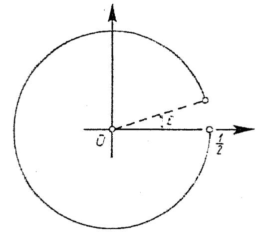

Fig. 5

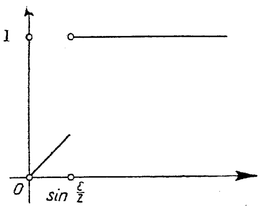

Fig. 6

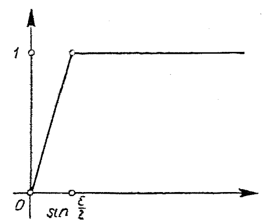

Fig. 7

of φ_{A_ε} and fig. 7 — the graph of λ_{A_ε}. Let us observe that $\lambda_{S_{1/2}}$ given by the graph in fig. 4 for $r = 1/2$, is not identical with $\lim\limits_{\varepsilon\to 0} \lambda_{A_\varepsilon}$.

12. Indicatrix of contractibility and homeomorphisms. Evidently the indicatrix of contractibility is not invariant under homeomorphisms. However, the following lemma holds:

LEMMA. *Let g be a homeomorphism mapping a compactum M onto another compactum M'. For every continuous indicatrix $\lambda(t)$ there exists a continuous indicatrix $\lambda'(t)$ such that if $A \in 2^M$ and $\lambda_A(t) \leqslant \lambda(t)$ for every $t \geqslant 0$, then $\lambda_{g(A)}(t) \leqslant \lambda'(t)$ for every $t \geqslant 0$.*

Proof. Let us denote, for every $t \geqslant 0$, by $a(t)$ the upper bound of the diameters of the sets $g(A)$, where $A \in 2^M$ and $\delta(A) \leqslant t$, and by $a'(t)$ the upper bound of the diameters of the sets $g^{-1}(A')$, where $A' \in 2^{M'}$ and $\delta(A') \leqslant t$. Moreover, let us set

$$(62) \qquad \alpha(t) = \mathrm{Min}\left[1, \mathrm{Max}\left(t, a(t)\right)\right], \qquad \alpha'(t) = \mathrm{Max}\left(t, a'(t)\right).$$

Evidently, both functions, $\alpha(t)$ and $\alpha'(t)$, vanish for $t=0$, are not decreasing and are positive for $t>0$. Let us show that, setting

$$(63) \qquad \lambda'(t) = f_\alpha \lambda f_{\alpha'}(t) \quad \text{for every} \quad t \geqslant 0,$$

we obtain a continuous indicatrix satisfying the required condition.

It follows by (62) and (40) that λ' is an indicatrix. We have to prove that if $A' = g(A)$ and

$$(64) \qquad \lambda_A(t) \leqslant \lambda(t) \quad \text{for every} \quad t \geqslant 0,$$

then

$$(65) \qquad \lambda_{A'}(t') \leqslant \lambda'(t') \quad \text{for every} \quad t' \geqslant 0.$$

If $\lambda'(t')=1$, then (65) is evident. Hence we can assume that $\lambda'(t')<1$ By (62) and (63) $\lambda'(t') \geqslant \alpha\lambda f_{\alpha'}(t) \geqslant \mathrm{Min}[1, \lambda f_{\alpha'}(t')]$. Hence $\lambda f_{\alpha'}(t')<1$ and consequently also (by (64))

$$(66) \qquad \lambda_A f_{\alpha'}(t') < 1.$$

It follows that every subset of A with the diameter $\leqslant f_{\alpha'}(t')$ is contractible in a subset of A with diameter $\leqslant \lambda_A f_{\alpha'}(t')$.

Consider now a set $E' \in 2^{A'}$ with $\delta(E') \leqslant t'$. Then the diameter of the set $E = g^{-1}(E')$ is $\leqslant a'(t') \leqslant \alpha'(t') \leqslant f_{\alpha'}(t')$. Hence there exists a set $F \subset A$ with $\delta(F) \leqslant \lambda_A f_{\alpha'}(t')$ such that E is contractible in F. Then E' is contractible in the set $F' = g(F)$. But $\delta(F) \leqslant \lambda_A f_{\alpha'}(t')$ implies that $\delta(F') \leqslant a\lambda_A f_{\alpha'}(t')$. Moreover, $\lambda'(t') = f_\alpha \lambda f_{\alpha'}(t') < 1$ implies that $\alpha\lambda f_{\alpha'}(t') < 1$ and, by (62), we infer that

$$\alpha\lambda f_{\alpha'}(t') = \mathrm{Max}[\lambda f_{\alpha'}(t'), a\lambda f_{\alpha'}(t')] \geqslant a\lambda f_{\alpha'}(t') \geqslant a\lambda_A f_{\alpha'}(t').$$

Hence $\delta(F') \leqslant a\lambda_A f_{\alpha'}(t') \leqslant \lambda'(t')$ and the proof of the lemma is complete.

13. Contractibility and retraction. Let A be a compact non-empty subset of the n-dimensional Euclidean space E_n. We denote by $U_\eta(A)$, for every $\eta > 0$, the η-neighbourhood of A in E_n, *i. e.* the set of all points $p \in E_n$ at a distance $<\eta$ from A. We shall prove the following

LEMMA. *Let n be a positive integer and $\lambda(t)$ a continuous indicatrix. There exists an increasing and continuous function $\alpha(\varepsilon)$, defined for $0 < \varepsilon \leqslant 1$, satisfying the inequality*

$$(67) \qquad 0 < \alpha(\varepsilon) \leqslant \varepsilon$$

and such that for every $A \in 2^{E_n}$ the inequality

$$(68) \qquad \lambda_A(t) \leqslant \lambda(t) \quad \textit{for every} \quad t \geqslant 0$$

implies that there exists a retraction r_A of $U_{\alpha(1)}(A)$ to A satisfying the inequality

$$(69) \qquad \varrho\big(x, r_A(x)\big) < \varepsilon \quad \textit{for every} \quad x \in U_{\alpha(\varepsilon)}(A).$$

Proof. Without loss of generality we may assume that the indicatrix λ satisfies the conditions

$$(70) \qquad \begin{aligned} t \leqslant \lambda(t) &\quad \text{for every} \quad 0 \leqslant t \leqslant 1 \\ \lambda(t) = 1 &\quad \text{for every} \quad t \geqslant 1. \end{aligned}$$

Let $\mu(t)$ denote the n-th iteration of the function $\lambda(2t)$. Evidently, $\mu(t)$ is a continuous, non-decreasing function of the variable $t \geqslant 0$ such that (by (70))

$$(71) \qquad \mu(0) = 0 \quad \text{and} \quad \mu(t) \geqslant \lambda(t) \quad \text{for every} \quad t \geqslant 0.$$

It is clear that there exists an increasing, continuous function $\alpha(\varepsilon) > 0$ satisfying for $0 < \varepsilon \leqslant 1$ the following two conditions:

$$(72) \qquad \text{If} \quad 0 < t \leqslant 4\alpha(\varepsilon) \quad \text{then} \quad \mu(t) < \frac{1}{2}\varepsilon,$$

$$(73) \qquad \alpha(\varepsilon) < \frac{1}{4}\varepsilon.$$

Now we consider a simplicial decomposition Σ of the set $E_n - A$, such that for every $0 < \varepsilon \leqslant 1$, the diameter of every simplex $\Delta \in \Sigma$ containing a point at a distance $<\alpha(\varepsilon)$ from A is $<\alpha(\varepsilon)/2$. Let us assign to every vertex p of Σ a point $r(p) \in A$ such that

$$\varrho\big(p, r(p)\big) = \varrho(p, A).$$

By our hypothesis, if $\Delta \in \Sigma$ contains a point at a distance $<\alpha(\varepsilon)$ from A, then for every vertex p of Δ we have $\varrho(p, A) < 3\alpha(\varepsilon)/2$. It follows that the diameter of the set $F_{0,\Delta}$ of all points $r(p)$, where p runs through all vertices of Δ, is $<\alpha(\varepsilon) + 2\cdot 3\alpha(\varepsilon)/2 = 4\alpha(\varepsilon)$. It follows by (71) and (72)

that
$$\lambda[4\alpha(\varepsilon)]\leqslant\mu[4\alpha(\varepsilon)]<\varepsilon/2<1$$
and consequently $F_{0,\Delta}$ is contractible to a point in a subset of A with diameter $<\lambda[4\alpha(\varepsilon)]$. We infer that the mapping r can be extended continuously over the 1-faces of Δ in such a manner that the values lie in A and the diameter of the image of every 1-face of Δ is $<\lambda[4\alpha(\varepsilon)]$.

Thus we have shown (for $n\geqslant 1$) that the extended function r maps the sum of all 1-faces of Δ onto a set $F_{1,\Delta}$ with diameter $<2\lambda[4\alpha(\varepsilon)]$. It follows by (72) and by the definition of the function μ that the set $F_{1,\Delta}$ is contractible to a point in a subset of A with diameter $<\lambda\Big[2\lambda\big(4\alpha(\varepsilon)\big)\Big]$. We infer that r can be extended continuously over the 2-faces of Δ (if $n>1$) in such a manner that the values lie in A and the diameter of the image of every 2-face of Δ is $<\lambda\Big[2\lambda\big(4\alpha(\varepsilon)\big)\Big]$.

Since $\mu(t)$ denotes the n-th iteration of $\lambda(2t)$, we infer easily, by an immediate induction, that r can be continuously extended step by step to 3-faces, 4-faces, ..., $(n-1)$-faces, n-face $=\Delta$, for every simplex $\Delta\in\Sigma$ containing at least one point at a distance $<\alpha(\varepsilon)$ from A. Moreover, this extension can be done in such a manner that the diameter of the image of such a simplex is $\leqslant\mu[4\alpha(\varepsilon)]<\varepsilon/2$. In particular we infer that the extended function r is defined (and continuous) in the set $U_{\alpha(1)}(A)-A$, being a subset of the sum of all simplexes Δ containing at least one point at a distance $<\alpha(1)$ from A. Moreover, if $x\in U_{\alpha(1)}(A)-A$ and the distance $\varrho(x,A)$ is $<\alpha(\varepsilon)$, then
$$\varrho\big(x,r(x)\big)<\delta(\Delta)+\alpha(\varepsilon)+\mu\big(4\alpha(\varepsilon)\big)<\varepsilon/4+\varepsilon/4+\varepsilon/2=\varepsilon.$$

It follows that, setting $r(x)=x$ for every $x\in A$, we do not damage the continuity of r. Evidently, the mapping r, defined in this way in the set $U_{\alpha(1)}$, satisfies the inequality (69) and is a retraction of $U_{\alpha(1)}$ to A. Thus the proof is finished.

14. Generalization. Now we generalize the lemma of section 13 in the following manner:

LEMMA. *Let Q be a space homeomorphic to the Euclidean n-dimensional cube Q_n, and let λ be a continuous indicatrix. Then there exists an increasing and continuous function $\alpha(\varepsilon)>0$, defined for $0<\varepsilon\leqslant 1$, such that for every $A\in 2^Q$ the inequality*

(74) $$\lambda_A(t)\leqslant\lambda(t)\quad \textit{for every}\quad t\geqslant 0$$

implies that there exists a retraction r_A of $U_{\alpha(1)}(A)$ to A satisfying the inequality
$$\varrho\big(x,r_A(x)\big)<\varepsilon\quad \textit{for every}\quad x\in U_{\alpha(\varepsilon)}(A).$$

Proof. Let g be a homeomorphic mapping of Q onto Q_n. By the lemma of section 12 there exists a continuous indicatrix $\lambda'(t)$ such that for every $A\in 2^Q$ the inequality (74) implies
$$\lambda_{g(A)}(t)\leqslant\lambda'(t)\quad\text{for every}\quad t\geqslant 0.$$
Now, applying the lemma of section 13, we infer that there exists an increasing and continuous function $\alpha'(\varepsilon)>0$, where $0<\varepsilon\leqslant 1$, such that for every set $A'\in 2^{E_n}$ the inequality
$$\lambda_{A'}(t)\leqslant\lambda'(t)\quad\text{for every}\quad t\geqslant 0$$
implies that there exists a retraction $r_{A'}$ of $U_{\alpha'(1)}(A')$ to A' satisfying the inequality

(75) $$\varrho\big(x',r_{A'}(x')\big)<\varepsilon\quad\text{for every}\quad x'\in U_{\alpha'(\varepsilon)}(A').$$

Since Q is compact, there exist two non-decreasing positive functions $\omega(\eta)$ and $\omega'(\eta)$, defined for $\eta>0$ and such that for $x,y\in Q$ the inequality $\varrho(x,y)<\omega(\eta)$ implies the inequality $\varrho\big(g(x),g(y)\big)<\eta$ and for $x',y'\in Q_n$ the inequality $\varrho(x',y')<\omega'(\eta)$ implies the inequality $\varrho\big(g^{-1}(x'),g^{-1}(y')\big)<\eta$.

It is easy to observe that there exists a continuous function $\alpha(\varepsilon)$ such that

(76) $$0<\alpha(\varepsilon)<\operatorname{Min}\Big[\omega\Big(\alpha'\big(\omega'(\varepsilon)\big)\Big),\omega\big(\alpha'(\varepsilon)\big)\Big]\quad\text{for every}\quad \varepsilon>0.$$

Now, if $A\in 2^Q$ satisfies (72), then setting $A'=g(A)$ we have by (76)

(77) $$g[U_{\alpha(\eta)}(A)]\subset U_{\alpha'(\eta)}(A')\quad\text{for every}\quad 0<\eta\leqslant 1.$$

In particular $g[U_{\alpha(1)}(A)]\subset U_{\alpha'(1)}(A')$. But there exists a retraction $r_{A'}$ of $U_{\alpha'(1)}(A')$ to A' satisfying the inequality (75). Setting
$$r_A(x)=g^{-1}r_{A'}g(x)\quad\text{for every}\quad x\in U_{\alpha(1)}(A),$$
we obtain a retraction of $U_{\alpha(1)}(A)$ to A. Moreover, if $x\in U_{\alpha(\varepsilon)}(A)$ then $g(x)\in U_{\alpha'(\omega'(\varepsilon))}(A')$, whence $\varrho\big(g(x),r_{A'}g(x)\big)<\omega'(\varepsilon)$. It follows by (76) that $\varrho\big(x,g^{-1}r_{A'}g(x)\big)<\varepsilon$, i. e. $\varrho\big(x,r_A(x)\big)<\varepsilon$ for every $x\in U_{\alpha(\varepsilon)}(A)$.

15. Metric of homotopy ϱ_h. Let M be a metric space. We shall denote by 2^M_h the class of all non-empty ANR-sets lying in M, this class being considered as a metric space with the distance-function ϱ defined by the formula

(78) $$\varrho_h(X,Y)=\varrho_c(X,Y)+\operatorname*{Sup}_t|\lambda_X(t)-\lambda_Y(t)|\quad\text{for}\quad X,Y\in 2^M_h.$$

Evidently ϱ_h satisfies the usual axioms for the distance-function and we have

(79) $$\varrho_h(X,Y) \geqslant \varrho_c(X,Y) \geqslant \varrho_s(X,Y).$$

It is also evident that if M' is a closed subset of M, then the space $2_h^{M'}$ is a closed subset of the space 2_h^M.

It follows by (79) that the propositions (4), (5), (6), (9), (10) and (11) hold also if we replace the space 2_c^M by the space 2_h^M.

In our further study of the space 2_h^M we limit ourselves to the case in which M is a compactum of a finite dimension. Then the ANR-sets lying in M coincide with the locally contractible compacta[18]), *i. e.* with the compacta with continuous indicatrices of contractibility. We begin with the following

LEMMA. *Let Q be a metric space homeomorphic with the n-dimensional Euclidean cube Q_n, and let λ be a continuous indicatrix. The subset of 2_h^Q, composed of all sets A satisfying the condition*

$$\lambda_A(t) \leqslant \lambda(t) \quad \textit{for every} \quad t \geqslant 0,$$

is compact.

Proof. Without loss of generality we may assume that Q is, as set, identical with the cube Q_n itself, metrized by a metric ϱ, which in general is not Euclidean. By the *η-neighbourhood of a set* $A \subset Q$ we shall understand the η-neighbourhood of A in the sense of the metric ϱ, *i. e.* the set

$$U_\eta(A) = \underset{x}{E}\,[x \in Q\,;\, \varrho(x,A) < \eta].$$

By a *segment* in Q with endpoints $x_0, y_0 \in Q$ we mean the set $\overline{x_0 y_0} \subset Q = Q_n$ which is the segment with endpoints x_0, y_0 in the Euclidean metric. Hence

$$\overline{x_0 y_0} = \underset{x}{E}\,[x = (1-t)x_0 + t y_0, \text{ with } 0 \leqslant t \leqslant 1],$$

addition and multiplication by a number being understood for points of $Q = Q_n$ in the Euclidean sense.

Let $\{A_k\}$ be a sequence of compact non-empty subsets of Q satisfying the inequality

(80) $$\lambda_{A_k}(t) \leqslant \lambda(t) \quad \text{for every} \quad t \geqslant 0 \quad \text{and} \quad k = 1,2,\ldots$$

We infer by the lemma of section 14 that there exists a positive function $\alpha(\varepsilon)$, defined for $\varepsilon > 0$, such that for $k = 1,2,\ldots$ there exists a retraction r_{A_k} of $U_{\alpha(1)}(A_k)$ to A_k satisfying the condition

(81) $$\varrho\big(x, r_{A_k}(x)\big) < \varepsilon \quad \text{for every} \quad x \in U_{\alpha(\varepsilon)}(A_k).$$

[18]) See, for instance, Kuratowski [13], p. 289.

It is evident that the function α can be replaced by any other function with positive but smaller values. Hence we may assume that α satisfies also the conditions

(82) $$\alpha(\varepsilon) \leqslant \varepsilon \quad \text{for every} \quad \varepsilon > 0\,;$$

(83) if $\varrho(x,y) < \alpha(\varepsilon)$, then the diameter of the segment $\overline{xy}$ is $< \varepsilon$.

Since the space 2_s^M is compact[19]), we can assume (replacing $\{A_k\}$ by a suitably chosen subsequence) that

(84) $$\varrho_s(A_k, A_l) < \frac{1}{2}\alpha(1) \quad \text{for every} \quad k,l = 1,2,\ldots$$

It follows that

(85) $$U_{\frac{1}{2}\alpha(1)}(A_k) \subset U_{\alpha(1)}(A_l) \quad \text{for every} \quad k,l = 1,2,\ldots$$

and we infer that the set

(86) $$U = \prod_{k=1}^{\infty} U_{\alpha(1)}(A_k)$$

constitutes a neighbourhood for the set $\sum_{k=1}^{\infty} A_k$.

Now we consider a non-increasing sequence $\{\beta_k\}$ of positive numbers, defined by induction in the following manner:

(87) $$\beta_1 = \alpha\left(\frac{1}{2}\right),$$

(88) $$\beta_{k+1} = \alpha\left[\frac{1}{2}\cdot\alpha\left(\frac{1}{2}\beta_k\right)\right] \quad \text{for} \quad k = 1,2,\ldots$$

It follows by (82), (87) and (88) that $\beta_1 \leqslant 1/2$ and $\beta_{k+1} < \beta_k/2$, whence

(89) $$\beta_k \leqslant 2^{-k} \quad \text{for} \quad k = 1,2,\ldots$$

Let us set

(90) $$\eta_k = 4^{-k}\cdot\beta_{k+1}\,; \qquad \eta_k' = \frac{1}{3}\eta_k \quad \text{for} \quad k = 1,2,\ldots$$

Evidently $\eta_k \leqslant \alpha(1/2)$ for $k = 1,2,\ldots$

Replacing $\{A_k\}$ by a suitably chosen subsequence, we may assume that

(91) $$\varrho_s(A_k, A_{k+1}) < \frac{1}{4}\eta_k' \quad \text{for} \quad k = 1,2,\ldots$$

[19]) See, for instance, Kuratowski [13], p. 21.

Now we set

(92) $$U_k = U_{\eta_k}(A_k), \qquad V_k = U_{\eta'_k}(A_k) \quad \text{for} \quad k = 1, 2, \ldots$$

Then, for every $x \in U_{k+1}$, we have

$$\varrho(x, A_k) \leqslant \varrho(x, A_{k+1}) + \varrho_s(A_{k+1}, A_k) < \eta_{k+1} + \frac{1}{4}\eta'_k$$

$$= 4^{-(k+1)} \cdot \beta_{k+2} + 4^{-1} \cdot 3^{-1} \cdot 4^{-k} \cdot \beta_{k+1} \leqslant 4^{-(k-1)} \cdot \beta_{k+1} \cdot \left(1 + \frac{1}{3}\right) = \frac{1}{3} \cdot 4^{-k} \cdot \beta_{k+1} = \eta'_k.$$

It follows by (85), (90) and (92) that

(93) $$U_{k+1} \subset V_k \subset \overline{V}_k \subset U_k \subset U \quad \text{for} \quad k = 1, 2, \ldots$$

Let us define a sequence $\{r_k\}$ such that r_k is a retraction of U to A_k. We set

(94) $$r_1 = r_{A_1},$$

and, supposing that the retraction r_k is already defined and that it satisfies the condition

(95) $$r_k(x) = r_{A_k}(x) \quad \text{for every} \quad x \in V_k,$$

we define the retraction r_{k+1} in the following manner:

Let us observe that for $x \in U_{k+1} \subset V_k \subset U_k$ we have $r_k(x) \in A_k$. Applying (92), (90), (88) and (82), we conclude that

$$\varrho(x, A_k) < \eta_k < \beta_{k+1} \leqslant \alpha\left(\frac{1}{2}\beta_k\right) \quad \text{for every} \quad x \in U_{k+1}.$$

It follows by (81) that

(96) $$\varrho\big(x, r_{A_k}(x)\big) < \frac{1}{2}\beta_k \quad \text{for every} \quad x \in U_{k+1}.$$

Moreover, by (92) and (90) we have

$$\varrho(x, A_{k+1}) < \eta_{k+1} = 4^{-k-1} \cdot \beta_{k+2} < \alpha\left(\frac{1}{2}\beta_k\right) \quad \text{for every} \quad x \in U_{k+1}.$$

We infer by (96), (88) and (83) that for every $x \in U_{k+1}$ the diameter of the segment $\overline{x r_{A_k}(x)}$ is $< \alpha(^1/_2\beta_{k-1})/2$. Consequently, for every $y \in \overline{x r_{A_k}(x)}$ we have

$$\varrho(y, A_{k+1}) < \frac{1}{2}\alpha\left(\frac{1}{2}\beta_{k-1}\right) + \eta_{k+1} < \frac{1}{2}\alpha\left(\frac{1}{2}\beta_{k-1}\right) + \frac{1}{2}\alpha\left(\frac{1}{2}\beta_k\right) = \alpha\left(\frac{1}{2}\beta_{k-1}\right).$$

Applying (81) and (89), we conclude that

(97) $$\varrho\big(y, r_{A_{k+1}}(y)\big) < \frac{1}{2}\beta_{k-1} < 2^{-k} \quad \text{for every} \quad y \in \overline{x r_{A_k}(x)} \quad \text{with} \quad x \in U_{k+1}.$$

Now we consider in U a continuous function $f(x)$, with values belonging to the interval $\langle 0; 1\rangle$, which satisfies the conditions

$$f(x) = 0 \quad \text{for every} \quad x \in U - U_{k+1},$$
$$f(x) = 1 \quad \text{for every} \quad x \in V_{k+1}.$$

Setting

$$r_{k+1}(x) = r_{A_{k+1}}\Big[\big(1 - f(x)\big) r_k(x) + f(x) x\Big] \quad \text{for every} \quad x \in U,$$

we obtain a continuous mapping of U into A_{k+1}. The function r_{k+1} is a retraction of U to A_{k+1}, because

(98) $$r_{k+1}(x) = r_{A_{k+1}}(x) \quad \text{for every} \quad x \in V_{k+1}.$$

Moreover,

(99) $$r_{k+1}(x) = r_{A_{k+1}}[r_k(x)] \quad \text{for every} \quad x \in U - U_{k+1}.$$

Let us observe that for every element $x \in \overline{U}_{k+1} - V_{k+1}$ the point

$$y = \big(1 - f(x)\big) \times r_k(x) + f(x) x$$

lies on the segment $\overline{x r_k(x)}$. By (93) $\overline{U}_{k+1} \subset \overline{V}_k$, whence (95) implies that $r_k(x) = r_{A_k}(x)$. We infer by (96), (88) and (83) that the diameter of $\overline{x r_{A_k}(x)}$ is $< \alpha(\beta_{k-1})/2$. Moreover, for every $y \in \overline{x r_k(x)}$, we infer by (97) that $\varrho\big(y, r_{A_{k+1}}(y)\big) < 2^{-k}$. Consequently,

(100) $$\varrho\big(x, r_{k+1}(x)\big) \leqslant \varrho(x, y) + \varrho\big(y, r_{A_{k+1}}(y)\big) \leqslant \frac{1}{2}\alpha(\beta_{k-1}) + 2^{-k} < 2^{-k+1}$$

for every $x \in \overline{U}_{k+1} - V_{k+1}$.

It follows by (91), (90), (88) and (87) that

$$\varrho\big(r_k(x), A_{k+1}\big) < \frac{1}{4}\eta'_k < \beta_{k+1} \leqslant \alpha\left(\frac{1}{2}\beta_k\right) \quad \text{for every} \quad x \in U.$$

We infer by (81) that

$$\varrho\big(r_{A_{k+1}} r_k(x), r_k(x)\big) < \frac{1}{2}\beta_k \leqslant 2^{-k-1} \quad \text{for every} \quad x \in U.$$

Applying (99) we get

(101) $$\varrho\big(r_{k+1}(x), r_k(x)\big) < 2^{-k-1} \quad \text{for every} \quad x \in U - U_{k+1}.$$

Moreover, using (92), (90), (88), (87) and (81), we obtain

$$\varrho\big(x, r_{A_{k+1}}(x)\big) < \frac{1}{2}\beta_{k+1} < 2^{-k-2} \quad \text{for every} \quad x \in V_{k+1} \subset U_{k+1}.$$

By (98) and (100) we conclude that

(102) $$\varrho\big(x, r_{k+1}(x)\big) < 2^{-k+1} \quad \text{for every} \quad x \in \overline{U}_{k+1} \subset \overline{U}_k.$$

Consequently,

(103) $$\varrho\big(r_{k+1}(x), r_k(x)\big) \leqslant \varrho\big(r_{k+1}(x), x\big) + \varrho\big(x, r_k(x)\big) \leqslant 2^{-k+1} + 2^{-k+2} < 2^{-k+3}$$

for every $x \in \overline{U}_{k+1}$.

It follows by (101) and (102) that

$$\varrho\big(r_{k+1}(x), r_k(x)\big) < 2^{-k+3} \quad \text{for every} \quad x \in U,$$

and we infer that the sequence $\{r_k(x)\}$ is uniformly convergent in U.

Setting

(104) $$r(x) = \lim_{k\to\infty} r_k(x) \quad \text{for every} \quad x \in U,$$

we get a continuous mapping retracting U to a compact set $A \subset Q$. It remains to show that $\lim\limits_{k\to\infty} \varrho_h(A_k, A) = 0$.

Let η be an arbitrarily given positive number. Since $r(x) = x$ for every $x \in A$, we infer that there exists a neighbourhood U' of A in Q such that

(105) $$\varrho\big(r(x), x\big) < \eta \quad \text{for every} \quad x \in U'.$$

Let η' be another positive number $\leqslant \eta$ and so small that

(106) $$\text{if} \quad \varrho(x, y) < \eta' \quad \text{then} \quad \delta(\overline{xy}) < \eta,$$

(107) $$\text{if} \quad x \in A \quad \text{then} \quad \varrho(x, Q - U') > 2\eta'.$$

Since the sequence $\{r_k(x)\}$ converges uniformly to $r(x)$, we infer by (107) that there exists a number N such that for $k > N$

(108) $$\overline{U_{\eta'}(A + A_k)} \subset U',$$

(109) $$\varrho\big(r(x), x\big) < \eta' \quad \text{for every} \quad x \in A_k,$$

(110) $$\varrho\big(r_k(x), x\big) < \eta' \quad \text{for every} \quad x \in A.$$

It follows by (105) and (110) that

$$\lim_{k\to\infty} \varrho_c(A_k, A) = 0.$$

In view of (78) it remains to show that the indicatrices of the contractibility λ_{A_k} of the sets A_k converge uniformly to the indicatrix of the contractibility λ_A of the set A, and that $\lambda_A(t) \leqslant \lambda(t)$ for every $t \geqslant 0$.

Let φ_A and φ_{A_k} be defined as in section 11. If $0 \neq E \subset A$ and $\delta(E) \leqslant t$, then, setting

$$h_1(x, u) = (1-u)x + u r_k(x) \quad \text{for every} \quad x \in A \quad \text{and} \quad 0 \leqslant u \leqslant 1,$$

we get, by (106) and (110), a homotopical deformation of the set E into the set $r_k(E)$ with the values $h_1(x, u)$ lying in a set $F_1 \subset U'$. By (106) the diameter of F_1 is $\leqslant t + 2\eta$. If $\varphi_{A_k}(t + 2\eta) < 1$, then there exists a set $F_2 \subset A_k$ with the diameter $\leqslant \varphi_{A_k}(t + 2\eta)$, such that $r_k(E)$ is contractible to a point in F_2. It follows that the set E is contractible to a point in the set $F_1 + F_2$. Consequently, E is also contractible in the set $r(F_1 + F_2)$. But $F_1 \subset u_{\eta'}[r(E)] \subset u_{\eta'}(F_2) \subset u_\eta(F_2)$ whence, by (105), (108) and (109) the diameter of $r(F_1 + F_2)$ is $\leqslant 2\eta + \delta(F_2) + 2\eta \leqslant \varphi_{A_k}(t + 2\eta) + 4\eta$. It follows that

(111) $$\varphi_A(t) \leqslant \varphi_{A_k}(t + 2\eta) + 4\eta.$$

Evidently, the last inequality, proved under the hypothesis that $\varphi_{A_k}(t + 2\eta) < 1$, holds also if $\varphi_{A_k}(t + 2\eta) \geqslant 1$. Hence it holds for every $t \geqslant 0$.

In an analogous manner we show that

$$\varphi_{A_k}(t') \leqslant \varphi_A(t' + 2\eta) + 4\eta \quad \text{for every} \quad t' \geqslant 0.$$

If $t \geqslant 2\eta$, then setting $t' = t - 2\eta$, we conclude that

(112) $$\varphi_{A_k}(t - 2\eta) \leqslant \varphi_A(t) + 4\eta \quad \text{for every} \quad t \geqslant 2\eta.$$

Now let us set

$$u_1(t) = \begin{cases} -2\eta & \text{if} \quad t \geqslant 2\eta, \\ -t & \text{if} \quad t \leqslant 2\eta, \end{cases}$$

$$u_2(t) = 2\eta \quad \text{for every} \quad t \geqslant 0.$$

We infer by (111) and (112) that

(113) $$\varphi_{A_k}\big(t + u_1(t)\big) - 4\eta \leqslant \varphi_A(t) \leqslant \varphi_{A_k}\big(t + u_2(t)\big) + 4\eta \quad \text{for every} \quad t \geqslant 0$$

and that

(114) $$|u_\nu(t)| \leqslant 2\eta \quad \text{for} \quad \nu = 1, 2.$$

Applying the lemma of section 10, we infer that for arbitrarily given $\varepsilon > 0$ there exists a positive number $\eta < \varepsilon$ such that each of the two functions

$$\psi_\nu(t) = \varphi_{A_k}\big(t + u_\nu(t)\big), \qquad \nu = 1, 2,$$

satisfies the inequality

$$|f_{\varphi_{A_k}}(t) - f_{\psi_\nu}(t)| < \varepsilon \quad \text{for every} \quad t \geqslant 0, \quad \nu = 1, 2.$$

But $f_{\varphi_{A_k}}(t)=\lambda_{A_k}(t)$, whence

(115) $$|\lambda_{A_k}(t)-f_{\varphi_\nu}(t)|<\varepsilon \quad \text{for every} \quad t\geqslant 0 \quad \text{and} \quad \nu=1,2.$$

It follows that

(116) $$|f_{\varphi_1}(t)-f_{\varphi_2}(t)|<2\varepsilon \quad \text{for every} \quad t\geqslant 0.$$

Moreover, the inequality (113) and the inequality $\eta<\varepsilon$ imply that

(117) $$f_{\varphi_1}(t)-4\varepsilon\leqslant\lambda_A(t)\leqslant f_{\varphi_2}(t)+4\varepsilon \quad \text{for every} \quad t\geqslant 0.$$

Combining (115), (116) and (117), we obtain for $k>N$ the inequality

$$|\lambda_A(t)-\lambda_{A_k}(t)|<5\varepsilon.$$

Since ε is arbitrarily small, we infer that the sequence $\lambda_{A_k}(t)$ converges uniformly to $\lambda_A(t)$. Moreover, the inequality (80) implies that $\lambda_A(t)\leqslant\lambda(t)$ for every $t\geqslant 0$. This concludes the proof of our lemma.

16. Main theorem. Now we can state the main result of this note.

THEOREM. *Let M be a finite dimensional compactum. A sequence $\{A_k\}\subset 2_h^M$ is convergent if and only if it is convergent in the space 2_s^M and there exists a continuous indicatrix λ such that*

(118) $$\lambda_{A_k}(t)\leqslant\lambda(t) \quad \textit{for every} \quad t\geqslant 0 \quad \textit{and} \quad k=1,2,\ldots$$

Proof. Necessity. Let us suppose that $\{A_k\}$ is convergent in 2_h^M. It follows, by (79), that $\{A_k\}$ is convergent also in 2_s^M. Moreover by (78), the indicatrices λ_{A_k} constitute a uniformly convergent sequence. Applying the lemma of section 9 we infer that there exists a continuous indicatrix λ satisfying the inequality (118).

Sufficiency. By the well known theorem by Menger and Nöbeling [20]) M is homeomorphic with a subset of the n-dimensional Euclidean cube Q_n with $n>2\dim M$. It follows, by a theorem of Hausdorff [21]), that there exists a metric space Q homeomorphic with Q_n and containing (metrically) M.

Let us suppose that $\{A_k\}$ is convergent in 2_s^M to a set A and that there exists a continuous indicatrix λ satisfying the inequality (118). If $\{A_k\}$ would be not convergent in 2_h^M then we infer, by the lemma of section 15, that there exists two subsequences $\{A_{k'_\nu}\}$ and $\{A_{k''_\nu}\}$ convergent in 2_h^Q to two different sets A' and A''. It follows by (79) that $\{A_{k'_\nu}\}$ converges to A' and $\{A_{k''_\nu}\}$ converges to A'' also in 2_s^Q. Hence $A'=A''=A\subset M$, and this contradicts the assumption $A'\neq A''$.

Thus the theorem is proved.

[20]) See, for instance, Kuratowski [13], p. 69.

[21]) Hausdorff [10], p. 353.

As we have observed in section 11 the existence of a continuous indicatrix $\lambda(t)$ satisfying (118) is equivalent to the equally contractibility of $\{A_k\}$. It follows that the last theorem can be formulated also in the following manner:

THEOREM. *For finite dimensional compacta M a sequence $\{A_k\}\subset 2_h^M$ is convergent in 2_h^M if and only if it is convergent in 2_s^M and the sets $\{A_k\}$ are equally locally contractible.*

Recalling the definition of the homotopical convergence given in section 5, we obtain the following

COROLLARY 1. *If M is a finite dimensional compactum then the topology induced in 2_h^M by the metric ϱ_h is equivalent with the topology induced by the homotopical convergence.*

Moreover we have the following

COROLLARY 2. *If M is a finite dimensional compactum then putting*

$$\varrho_h'(X,Y)=\varrho_s(X,Y)+\operatorname*{Sup}_t|\lambda_X(t)-\lambda_Y(t)| \quad \textit{for} \quad X,Y\in 2_h^M$$

we obtain a metric ϱ_h' equivalent with the metric of homotopy ϱ_h.

Proof. Evidently ϱ_h' satisfies the usual axioms for distance function and the inequality

$$\varrho_h(X,Y)\geqslant\varrho_h'(X,Y)\geqslant\varrho_s(X,Y).$$

It follows that the convergence $A_k\to A$ in the sense of the metric ϱ_h implies the analogous convergence in the sense of the metric ϱ_h'. On the other hand, the convergence $A_k\to A$ in the sense of the metric ϱ_h' implies that $\varrho_s(A_k,A)\to 0$ and that the indicatrices λ_{A_k} converge uniformly to λ_A. By lemma of section 9 there exists a continuous indicatrix λ such that (118) holds. Consequently $A_k\to A$ also in the sense of the metric ϱ_h.

COROLLARY 3. *If M is a finite dimensional compactum then the space 2_h^M is separable.*

Proof. By the corollary 2, the space 2_h^M is homeomorphic with the same space provided with metric ϱ_h'. But this last space is isometric with a subset of the Cartesian product of the separable space 2_s^M and of the separable space of all continuous real functions $\lambda(t)$ defined on the interval $0\leqslant t\leqslant\delta(M)$ metrized by the formula

$$\varrho(\lambda_1,\lambda_2)=\operatorname*{Sup}_t|\lambda_1(t)-\lambda_2(t)|.$$

COROLLARY 4. *If M is a compactum of a finite dimension then the space 2_h^M is complete.*

Proof. If $\{A_k\}$ is a Cauchy sequence in 2_h^M then, by (79), $\{A_k\}$ is also a Cauchy sequence in 2_s^M and the indicatrices λ_{A_k} constitute a Cauchy sequence in the space $\langle 0;\infty)^{\langle 0;\infty)}$ of all continuous bounded functions mapping $\langle 0;\infty)$ into itself. But 2_s^M and $\langle 0;\infty)^{\langle 0;\infty)}$ are complete [22]). It follows that $\{A_k\}$ is convergent in 2_s^M and $\{\lambda_k(t)\}$ is uniformly convergent in $\langle 0;\infty)$. We infer by the lemma of section 9, that there exists a continuous indicatrix λ satisfying the inequality (118). It follows by the last theorem that $\{A_k\}$ is convergent in 2_h^M.

COROLLARY 5. *If M and M' are two homeomorphic compacta of a finite dimension, then the spaces 2_h^M and $2_h^{M'}$ are homeomorphic.*

Proof. Let g be a homeomorphism mapping M onto M'. Then g induces a homeomorphism mapping 2_s^M onto $2_s^{M'}$. It follows that, for every sequence $\{A_k\}\subset 2_h^M$, if $\{A_k\}$ is convergent in 2_s^M, then the sequence $\{A_k'\}$, where $A_k'=g(A_k)$, is convergent in $2_s^{M'}$. Moreover, if there exists a continuous indicatrix λ satisfying the inequality (118), then, by the lemma of section 12, there exists a continuous indicatrix satisfying the inequality

$$\lambda_{A_k'}(t)\leqslant\lambda'(t)\quad \text{for every}\quad t\geqslant 0,\ k=1,2,\ldots$$

It follows, by the last theorem, that the convergence of $\{A_k\}$ in 2_h^M implies the convergence of $\{A_k'\}$ in $2_h^{M'}$. In an analogous manner we infer that the convergence of $\{A_k'\}$ in $2_h^{M'}$ implies the convergence of $\{A_k\}$ in 2_h^M. Hence the 1-1 correspondence between 2_h^M and $2_h^{M'}$ induced by g is a homeomorphism.

COROLLARY 6. *Suppose M is a compactum of a finite dimension. A closed subset $\mathfrak{A}$ of 2_h^M is compact if and only if there exists a continuous indicatrix λ such that*

$$(119)\qquad \lambda_A(t)\leqslant\lambda(t)\quad \textit{for every}\quad A\in\mathfrak{A}\quad\textit{and}\quad t\geqslant 0.$$

Proof. Sufficiency. If the condition is satisfied and if $\{A_k\}\subset\mathfrak{A}$, then there exists a subsequence $\{A_{k_\nu}\}$ convergent in 2_s^M and $\lambda_{A_{k_\nu}}(t)\leqslant\lambda(t)$ for every $t\geqslant 0$, and $\nu=1,2,\ldots$ It follows from the last theorem that $\{A_{k_\nu}\}$ is convergent in 2_h^M. But $\mathfrak{A}$ is closed in 2_h^M, whence $\{A_{k_\nu}\}$ is also convergent in $\mathfrak{A}$.

Necessity. Let us suppose that for every $\varepsilon>0$ there exists an $\eta=\eta(\varepsilon)>0$ such that

$$(120)\qquad \lambda_A(t)\leqslant\varepsilon\quad \text{for every}\quad 0<t\leqslant\eta(\varepsilon)\quad\text{and}\quad A\in\mathfrak{A}.$$

[22]) See, for instance, Kuratowski [11], p. 315.

Let us set

$$\varphi(t)=\operatorname*{Sup}_{A\in\mathfrak{A}}\lambda_A(t)\quad \text{for every}\quad t\geqslant 0.$$

Then

$$(121)\qquad \varphi(0)=0,$$

$$(122)\qquad 0<\varphi(t)\leqslant 1\quad \text{for every}\quad t>0.$$

Moreover, we infer by (120) that for every $\varepsilon>0$ the inequality $0\leqslant t\leqslant\eta(\varepsilon)$ implies $\varphi(t)\leqslant\varepsilon$. Hence

$$(123)\qquad \lim_{t\to 0}\varphi(t)=0.$$

Using the notations of section 7, let us set

$$\lambda(t)=f_\varphi(t)\quad \text{for every}\quad t\geqslant 0.$$

Then $\lambda(t)$ is a concave function in $\langle 0;\infty)$ such that

$$\lambda(0)=0\quad\text{and}\quad 0\leqslant\lambda_A(t)\leqslant\lambda(t)\leqslant 1\quad \text{for every}\quad A\in\mathfrak{A}\quad\text{and}\quad t\geqslant 0.$$

Moreover, by (123) and (35), the function λ is continuous. Hence λ is a continuous indicatrix satisfying the condition (119).

Thus to prove the necessity, *i. e.* to finish the proof, it remains to show that the existence of an $\varepsilon>0$ such that for every $k=1,2,\ldots$ there exists a set $A_k\in\mathfrak{A}$ with the indicatrix λ_{A_k} satisfying the inequality

$$\lambda_{A_k}\left(\frac{1}{k}\right)\geqslant\varepsilon$$

implies that $\mathfrak{A}$ is not compact. In this case, for every natural k_0, we can find a natural l_0 such that

$$\lambda_{A_{k_0}}\left(\frac{1}{l_0}\right)\leqslant\frac{\varepsilon}{2}.$$

Then for every $l\geqslant l_0$ we have $\lambda_{A_l}(1/l_0)\geqslant\lambda_{A_l}(1/l)\geqslant\varepsilon$, whence

$$\operatorname*{Sup}_{t\geqslant 0}|\lambda_{A_{k_0}}(t)-\lambda_{A_l}(t)|\geqslant\frac{\varepsilon}{2}\quad \text{for every}\quad l\geqslant l_0.$$

It follows by (78) that for every natural k_0 the inequality $\varrho_h(A_{k_0},A_l)\geqslant\varepsilon/2$ is satisfied for almost all natural l, whence $\{A_k\}$ contains no convergent subsequence. Hence $\mathfrak{A}$ is not compact.

17. Homotopy types in the space 2_h^M. Problems. Following Hurewicz [23]) I shall refer to two compacta A and B as being of the same *homotopy type* if there exists a continuous mapping φ of A into B and a continuous mapping ψ of B into A such that the mapping $\psi\varphi$ of A

[23]) Cf. Hurewicz [11], p. 124.

into itself is homotopic to the identity and also the mapping $\varphi\psi$ of B into itself is homotopic to the identity.

Let M be a metric space and $A \in 2_h^M$. We denote by $[A]_M$ the subset of 2_h^M composed by all $B \in 2_h^M$ having the same homotopy type as A. We prove the following

THEOREM [24]). *Suppose M is a compactum of finite dimension. For each $A_0 \in 2_h^M$ the set $[A_0]_M$ is open in 2_h^M.*

Proof. In view of the imbedding theorem of Menger-Nöbeling and the corollary 5 of section 16 we can assume that M lies in the Euclidean n-dimensional cube Q_n. It suffices to prove that if $\{A_k\}$ is a sequence of sets convergent in 2_h^M to a set A_0 then, for almost all indices k, the homotopy type of A_k is the same as the homotopy type of A_0.

If $\{A_k\}$ converges in 2_h^M to A_0, then, by (79) and (78),

$$\lim_{k\to\infty} \varrho_s(A_k, A_0) = 0 \tag{124}$$

and

$$\lambda_{A_k}(t) \text{ converge uniformly to } \lambda_{A_0}(t). \tag{125}$$

It follows by (125) and by the lemma of section 9 that there exists a continuous indicatrix λ such that

$$\lambda_{A_k}(t) \leqslant \lambda(t) \quad \text{for every} \quad t \geqslant 0 \quad \text{and} \quad k = 0, 1, \ldots \tag{126}$$

Applying the lemma of section 13 we infer that there exists an increasing and continuous function $\alpha(\varepsilon)$, defined for $\varepsilon > 0$, satisfying the inequality

$$0 < \alpha(\varepsilon) \leqslant \varepsilon \tag{127}$$

and such that for every $k = 0, 1, \ldots$ there exists a retraction r_k of the $\alpha(1)$-neighbourhood $U_{\alpha(1)}(A_k)$ of A_k to A_k satisfying the condition

$$\varrho\big(x, r_k(x)\big) < \varepsilon \quad \text{for every} \quad x \in U_{\alpha(\varepsilon)}(A_k) \quad \text{and} \quad 0 < \varepsilon \leqslant 1. \tag{128}$$

By (124) there exists a k_0 such that

$$\varrho_s(A_k, A_0) < \alpha[\alpha(1)] \quad \text{for every} \quad k \geqslant k_0.$$

Hence for $x \in A_k$ with $k \geqslant k_0$ we have $\varrho(x, A_0) < \alpha[\alpha(1)] \leqslant \alpha(1)$; consequently $x \in U_{\alpha(1)}(A_0)$, and $r_0(x)$ is defined. Moreover, $\varrho[x, r_0(x)] < \alpha(1)$, whence the segment $\overline{xr_0(x)}$ lies in $U_{\alpha(1)}(A_k)$. It follows that, setting

$$r(x,t) = r_k\big(tx + (1-t) r_0(x)\big) \quad \text{for every} \quad x \in A_k \quad \text{and} \quad 0 \leqslant t \leqslant 1,$$

we get a continuous function with values belonging to A Since

$$r(x,0) = r_k r_0(x), \qquad r(x,1) = r_k(x) \quad \text{for every} \quad x \in A_k,$$

we infer that the mapping $r_k r_0$ of A_k into itself is homotopic to the identity.

By an analogous argument we show that, setting

$$\bar{r}(x,t) = r_0\big(tx + (1-t) r_k(x)\big) \quad \text{for every} \quad x \in A_0 \quad \text{and} \quad 0 \leqslant t \leqslant 1,$$

we obtain (for $k \geqslant k_0$) a homotopy in A_0 joining the mapping $r_0 r_k$ with the identity. It follows that for $k \geqslant k_0$ the homotopy type of A_k is the same as the homotopy type of A_0. Thus the theorem is proved.

PROBLEM 4. *Does the theorem of section 16 (and the corollaries 1-6) remain true for compacta of infinite dimension and, more generally, for all complete spaces?*

PROBLEM 5. *Let M be a polytope. Is it true that the set $\mathfrak{P}$ composed of all subpolytopes of M is dense in 2_h^M? What is the category (in the sense of Baire) of $\mathfrak{P}$?*

PROBLEM 6. *What is the category (in the sense of Baire) in the space $2_h^{Q_n}$ of the set composed of all ANR-sets lying in Q_n and having the singularity of Brouwer, of Mazurkiewicz or of Peano?* [25]).

[24]) Comp. M. L. Curtis [7], p. 242, th. 4.2.

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[25]) A compactum A is said to have the *singularity of Brouwer* if for a natural n the n-dimensional number of Betti of A is $\geqslant 2$, but the n-dimensional number of Betti of each subcompactum of A vanishes. A compactum A is said to have the *singularity of Mazurkiewicz* if for a natural n the n-dimensional number of Betti of A vanishes but a finite decomposition of A into proper subcompacta with vanishing n-dimensional Betti numbers does not exist. Finally a compactum A is said to have the *singularity of Peano* if there exists in A a subcompactum E contractible to a point over A and such that for every set $F \subset A$ such that E is contractible to a point over F we have $\dim F \geqslant 1 + \dim E$. Comp. Borsuk [3], p. 31. It is known that for polytopes none of these three singularities is possible. On the other hand, there exist ANR-sets, having such singularities. See Borsuk [4] and [5], Borsuk and Mazurkiewicz [6] and Lubański [14]. The solutions of problems 6 and 7 would give some information about the role of polytopes in the class of all ANR-sets.

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Elementary properties of Abelian groups *)

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Table of contents

Introduction

The present study is of meta-mathematical origin. In its initial stage the main aim was to solve the decision problem for the *elementary theory of Abelian groups* [1]). By the elementary theory of Abelian groups we understand that part of the general theory of Abelian groups in which we concern ourselves exclusively with group elements and fundamental group operations without involving any set-theoretical notions (like those of subgroup, isomorphism, etc.). Speaking more technically it is that part of general group theory which can be formalized within elementary logic (*i. e.*, the lower predicate calculus) [2]). In an analogous sense we speak of the *elementary theory* of any other kind of algebraic systems — besides the term "elementary theory" we also use in the same sense the term *arithmetic*. The decision problem for the elementary theory of Abelian groups is the problem of existence of a procedure which permits us to decide in each particular case whether a given sentence formulated in terms of the theory holds in all Abelian groups, *i. e.*, whether this sentence is a logical consequence of postulates characterizing the notion

*) This paper includes all results of the Doctoral Dissertation done by author at the University of California in 1950.

[1]) Previously the decision problem has been solved affirmatively for some special Abelian groups; see [4].

[2]) [2] and [10] may be consulted for various logical and meta-mathematical notions and results involved in our discussion.