In particular, it follows that if we join every point x of the $(n-1)$-dimensional sphere S, lying in E_n, with its antipode x^* by an acyclic continuum $\Phi(x)=\Phi(x^*)$ and if $\Phi(x)$ constitute a family over S, then the interior region of S is swept out by the sets $\Phi(x)$.

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Examples of sets definable by means of two and three quantifiers

by

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There are many categories of mathematical papers. On the one hand we have first-class papers which are read with interest by many mathematicians and which further the development of mathematical thought. On the other hand we have also papers which are studied exclusively by referees appointed for that task by editors of bibliographical journals and which even by these casual readers are put aside with a sigh "why do these people publish so much?"

The present paper belongs to the second rather than to the first category. I have collected in it a number of very special results which belong to the theory of recursive functions. More explicitly I consider fractions of the form $10^{-x}a(x,y)$ where a is a primitive recursive function and investigate the set of those integers y for which $\lim_x 10^{-x}a(x,y)$ exists and belongs to a preassigned class of real numbers. A typical result is given in the following theorem (cf. theorem 7 below): The set $Z_a^{(5)}$ of those y's for which $\lim_x 10^{-x}a(x,y)$ exists and is integral is the most general set of the class $\boldsymbol{Q}_3^{(1)}$, *i. e.*, the most general set definable in the form $\underset{y}{E}\prod_u\sum_v\prod_w R(y,u,v,w)$ with a recursive R. The expression "most general" means that if a runs over the set of primitive recursive functions, then the set $Z_a^{(5)}$ runs over the whole class $\boldsymbol{Q}_3^{(1)}$.

Investigating this example and other similar ones I encountered some phenomena which I found interesting. If, for example, we narrow down the variability of a's to the set of functions for which $\lim_x 10^{-x}a(x,y)$ always exists (*i. e.*, exists for $y=0,1,2,\ldots$), then the corresponding sets $Z_a^{(5)}$ cease to represent arbitrary sets of the class $\boldsymbol{Q}_3^{(1)}$. As a runs over the narrower class of functions, the set $Z_a^{(5)}$ runs over the whole class $\boldsymbol{Q}_2^{(1)}$ which is known to be different from $\boldsymbol{Q}_3^{(1)}$. No such reduction occurs if instead of $Z_a^{(5)}$ we consider sets $Z_a^{(6)}$ containing all such y's for which $\lim_x 10^{-x}a(x,y)$ exists and is irrational. In this case the set $Z_a^{(6)}$ runs over the whole class $Q_3^{(1)}$ both when α runs over the set of all primitive recursive functions and when it runs over the set of such functions α for which $\lim_x 10^{-x}\alpha(x,y)$ always exists.

One could ask what will happen of instead of integral or irrational values prescribed for the limits $\lim_x 10^{-x}\alpha(x,y)$ we consider limits belonging to an arbitrary class of real numbers. This general problem is not touched in the paper.

The results obtained allow us to construct effective examples of sets definable by means of two or three quantifiers but not definable by a smaller number of them. From theorem 7 quoted above it follows for instance that if $U(n,x,y)$ is a (general recursive) function universal for the class of primitive recursive functions with two arguments, then the set of pairs (n,y) such that $\lim_x 10^{-x}U(n,x,y)$ exists and has an integral value belongs to the class $Q_3^{(2)}$ but not to $P_3^{(2)}$. Thus the minimal number of quantifiers needed for defining this set is 3.

There are many interesting sets of integers for which the exact number of quantifiers needed in their definitions is not known. For some sets the determination of this number represents an important problem (*e. g.*, for sets encountered in the theory of constructive ordinals). It seems to me that the construction of effective examples in which the minimal number of quantifiers can be determined may contribute to the solution of the more serious problems mentioned above.

The lower case Greek letters always denote primitive recursive functions. The class of all these functions is denoted by Φ whereas Φ^* denotes the class of those functions $\varphi(x,y)$ for which $\lim_x 10^{-x}\varphi(x,y)$ exists for every y.

The logical symbols used in this paper are the same as in my paper [2]. We denote by $P_n^{(k)}$ the family of sets having the form

$$\underset{(x_1,\ldots,x_k)}{E}\{\underbrace{\sum_{y_1}\prod_{y_2}\cdots}_{n\text{ quantifiers}}[\varphi(x_1,\ldots,x_k,y_1,\ldots,y_n)=0]\}$$

and by $Q_n^{(k)}$ the family of sets having the form

$$\underset{(x_1,\ldots,x_k)}{E}\{\underbrace{\prod_{y_1}\sum_{y_2}\cdots}_{n\text{ quantifiers}}[\varphi(x_1,\ldots,x_k,y_1,\ldots,y_n)=0]\}\ ^{1}).$$

Definition 1. $Z_\gamma^{(1)} = \underset{y}{E}\sum_{x_0}\prod_{x>x_0}[\gamma(x,y)=0]$.

[1]) Each y_i in this and the immediately preceeding formula can be replaced by a complex $y_{j1}, y_{j2}, \ldots, y_{jk_j}$.

THEOREM 1. *The family* $\{Z_\gamma^{(1)}\}_{\gamma\in\Phi}$ *is identical with* $P_2^{(1)}$ [2]).

Proof. It immediately follows from the definition that $Z_\gamma^{(1)} \in P_2^{(1)}$. It remains therefore to show that for each ϱ in Φ with three arguments a function γ in Φ can be found such that

$$\sum_p\prod_s[\varrho(p,s,y)=0] \equiv \sum_{x_0}\prod_{x>x_0}[\gamma(x,y)=0]. \tag{1}$$

We define γ along with two auxiliary functions π and σ by a simultaneous induction:

$$\gamma(0,y)=\varrho(0,0,y),\qquad \pi(0,y)=0,\qquad \sigma(0,y)=0,$$
$$\gamma(x+1,y)=\varrho\big(\pi(x,y)+1,0,y\big)\cdot\big[1\dot{-}\big(1\dot{-}\gamma(x,y)\big)\big]+ \\ +\varrho\big(\pi(x,y),\sigma(x,y)+1,y\big)\cdot[1\dot{-}\gamma(x,y)],$$
$$\pi(x+1,y)=[\pi(x,y)+1]\cdot\big[1\dot{-}\big(1\dot{-}\gamma(x,y)\big)\big]+\pi(x,y)\cdot[1\dot{-}\gamma(x,y)],$$
$$\sigma(x+1,y)=[\sigma(x,y)+1]\cdot[1\dot{-}\gamma(x,y)].$$

To explain the meaning of these definitions let us arrange all the pairs (i,j) into an infinite system

$$\begin{array}{l}(0,0),(0,1),(0,2),\ldots\\(1,0),(1,1),(1,2),\ldots\\ \ldots\ldots\ldots\ldots\end{array}$$

and consider a variable point P_y which moves over the system in such a way that in the xth moment its coordinates are $\big(\pi(x,y),\sigma(x,y)\big)$. It is clear from the definitions of functions π and σ that in the $(x+1)$st moment the point P_y either moves one place to the right in the same row or jumps to the initial point of the next row. The first move occurs if $\gamma(x,y)=0$ and the second if $\gamma(x,y)\neq 0$. Furthermore, it is clear from the definitions that $\gamma(x,y)=\varrho\big(\pi(x,y),\sigma(x,y),y\big)$, *i. e.*, that $\gamma(x,y)$ gives the value of $\varrho(p,s,y)$ calculated for the coordinates (p,s) of the point P_y. Hence, if $\varrho(p,s,y)=0$, then P_y moves from the point (p,s) to the point $(p,s+1)$ and if $\varrho(p,s,y)\neq 0$, then P_y moves from the point (p,s) to the point $(p+1,0)$.

Now let us assume that $\sum_p\prod_s[\varrho(p,s,y)=0]$. Let p_0 be the smallest integer such that $\varrho(p_0,s,y)=0$ for $s=0,1,2,\ldots$ For each $i<p_0$ there is a (smallest) integer s_i such that $\varrho(i,s_i,y)\neq 0$. In this case the moves of the point P_y may be described as follows: starting at the point $(0,0)$ it moves s_0 places to the right, then jumps to the point $(1,0)$ and moves s_1

[2]) A theorem equivalent to theorem 1 has been proved independently by Markwald [1].

places to the right, jumps again to the point $(2,0)$ and so on; eventually it reaches the p_0th row and moves on it indefinitely. Hence $\gamma(x,y)=0$ for $x>s_0+s_1+\ldots+s_{p_0}$.

Let us now assume that $\prod_p\sum_s[\varrho(p,s,y)\neq 0]$ and let s_p denote the smallest integer such that $\varrho(p,s_p,y)\neq 0$. Repeating the previous argument we see that the point P_y passes through all points (p,s_p) and hence $\gamma(x,y)\neq 0$ for infinitely many x.

Formula (1) is thus proved.

Remark. A theorem similar to theorem 1 holds also for the class $\boldsymbol{P}_2^{(k)}$. In order to obtain this more general theorem we replace the "y" in theorem 1 by "$(y_1,\ldots,y_k)$".

Definition 2. $Z_\gamma^{(2)}=\underset{y}{E}\,[\overline{\lim}_{x\to\infty}10^{-x}\gamma(x,y)<1]$.

Theorem 2. *The family $\{Z_\gamma^{(2)}\}_{\gamma\in\Phi}$ is identical with $\boldsymbol{P}_2^{(1)}$.*

Proof. For each γ, $Z_\gamma^{(2)}\in\boldsymbol{P}_2^{(1)}$ since

$$\overline{\lim_x}\,10^{-x}\gamma(x,y)<1\equiv\sum_{q\neq 0}\sum_r\prod_{x>r}[q\gamma(x,y)<10^x(q-1)].$$

Now let us assume that $Z\in\boldsymbol{P}_2^{(1)}$. According to theorem 1 there is a primitive recursive function δ such that $Z=Z_\delta^{(1)}$. Put

$$\gamma(x,y)=10^x\Big[1\dot{-}\big(1\dot{-}\delta(x,y)\big)\Big].$$

If $y\in Z$, then $\delta(x,y)=0$ from a certain x on, and hence $\overline{\lim}_x 10^{-x}\gamma(x,y)=0$. If $y\notin Z$, then $\delta(x,y)\neq 0$ for infinitely many x and $10^{-x}\gamma(x,y)=1$ for infinitely many x. It follows that $\overline{\lim}_x 10^{-x}\gamma(x,y)=1$ and hence $Z=Z_\gamma^{(2)}$.

Definition 3. $Z_\gamma^{(3)}=\underset{y}{E}\,[\lim_{x\to\infty}10^{-x}\gamma(x,y)$ exists and is equal to $0]$.

Theorem 3. *The family $\{Z_\gamma^{(3)}\}_{\gamma\in\Phi}$ is identical with $\boldsymbol{Q}_3^{(1)}$.*

Proof. For each γ, $Z_\gamma^{(3)}\in\boldsymbol{Q}_3^{(1)}$ since

$$\lim_x 10^{-x}\gamma(x,y)=0\equiv\prod_{p\neq 0}\sum_q\prod_{x>q}[p\gamma(x,y)<10^x].$$

Now let us assume that $Z\in\boldsymbol{Q}_3^{(1)}$, *i. e.* that

$$y\in Z\equiv\prod_p\sum_q\prod_r[\varrho(p,q,r,y)=0],\qquad \varrho\in\Phi.$$

Since the set $\underset{(y,p)}{E}\sum_q\prod_r[\varrho(p,q,r,y)=0]$ is in $\boldsymbol{P}_2^{(2)}$, there exists by theorem 1 a primitive recursive function $\gamma(p,x,y)$ such that

(2) $$y\in Z\equiv\prod_p\sum_{x_0}\prod_{x>x_0}[\gamma(p,x,y)=0].$$

We denote, as usual, by $[m/n]$ the integral part of m/n and put $[m/0]=1$. Furthermore, we put

$$\gamma'(p,x,y)=[10^x/p]\Big(1\dot{-}\big(1\dot{-}\gamma(p,x,y)\big)\Big)$$

and define by induction an auxiliary function $\pi(x,y)$:

$$\pi(0,y)=0,$$

$$\pi(x+1,y)=\{1\dot{-}[1\dot{-}\sum_{v\leqslant\pi(x,y)}\gamma'(v,x+1,y)]\}\cdot(\mu v)_{\pi(x,y)}[\gamma'(v,x+1,y)\neq 0]+$$
$$+[\pi(x,y)+1]\cdot[1\dot{-}\sum_{v\leqslant\pi(x,y)}\gamma'(v,x+1,y)].$$

Here $(\mu v)_t[\ldots]$ denotes the least integer $\leqslant t$ such that ... or 0 if no such v exists.

The above definition of π is equivalent to the following one: If there is a $v\leqslant\pi(x,y)$ such that $\gamma'(v,x+1,y)\neq 0$, then $\pi(x+1,y)$ is equal to the smallest such v:

(3) $$\pi(x+1,y)=(\mu v)_{\pi(x,y)}[\gamma'(v,x+1,y)\neq 0].$$

If no such v exists, then

(4) $$\pi(x+1,y)=\pi(x,y)+1.$$

We shall prove the equivalence

(5) $$y\in Z\equiv\underline{\lim_x}\,\pi(x,y)=\infty.$$

First let us assume that $y\in Z$. By (2) there exists for each p a smallest integer x_p such that $\gamma(p,x,y)=0$ for $x>x_p$. Let p be arbitrary and $x>\max_{j\leqslant p}x_j$. If $\pi(x,y)\leqslant p$, then from $x>\max_{j\leqslant p}x_j$ it follows that

$$\gamma'(v,x+1,y)=0\qquad\text{for}\quad v\leqslant\pi(x,y),$$

and hence, by (4), $\pi(x+1,y)=\pi(x,y)+1$. If $\pi(x,y)+1\leqslant p$, then by the same argument $\pi(x+2,y)=\pi(x,y)+2$; if $\pi(x,y)+2$ is still less than or equal to p, then repeating the same argument we obtain $\pi(x+3,y)=\pi(x,y)+3$. It is clear that after at most $p+1$ steps we shall obtain an x_0 such that

$$\max_{j\leqslant p}x_j<x_0\leqslant p+2+\max_{j\leqslant p}x_j\qquad\text{and}\qquad\pi(x_0,y)>p.$$

We shall show that $\pi(x,y)>p$ for all $x\geqslant x_0$. This is evident for $x=x_0$. Let us assume that $\pi(x,y)>p$ for an $x\geqslant x_0$. If $\pi(x+1,y)$ is defined by means of (4), then evidently $\pi(x+1,y)>p$; if $\pi(x+1,y)$ is defined by means of (3), then again $\pi(x+1,y)>p$ since $\gamma'\big(\pi(x+1,y),x+1,y\big)\neq 0$ by formula (3), whereas $\gamma'(v,x+1,y)=0$ for $v\leqslant p$ since $x+1>x_v$ and hence $\gamma(v,x+1,y)=0=\gamma'(v,x+1,y)$.

We have thus proved that $\prod_{x\geqslant x_0}[\pi(x,y)>p]$ and, since p was arbitrary, this proves that $\lim_{x\to\infty}\pi(x,y)=\infty$.

Next we assume that $y\notin Z$. By (2) there exists a smallest p_0 such that there are infinitely many values of x such that $\gamma(p_0,x,y)\neq 0$. We denote these values by $x_1,x_2,\ldots$ and have therefore

$$\gamma'(p_0,x_j,y)\neq 0 \quad \text{for} \quad j=1,2,3,\ldots \tag{6}$$

Exactly as before we can show that $\pi(x,y)\geqslant p_0$ from a certain x_0 on. We can assume that x_0 is chosen so large that $\gamma(v,x,y)=0$ for $v<p_0$ and $x>x_0$. If $x_j>x_0+1$, then the relation between $\pi(x_j,y)$ and $\pi(x_j-1,y)$ is expressed by formula (3) and the smallest $v\leqslant\pi(x_j-1,y)$ satisfying $\gamma'(v,x_j,y)\neq 0$ is equal to p_0. Hence $\pi(x_j,y)=p_0$ for infinitely many j, which proves that

$$\lim_j \pi(x_j,y)=p_0. \tag{7}$$

Formula (5) is thus proved.

From the definition of γ' we easily obtain the inequalities

$$\gamma'(p,x,y)\leqslant 10^x/p \quad \text{for arbitrary} \quad p,x,y, \tag{8a}$$

$$-1+10^x/p<\gamma'(p,x,y) \quad \text{if} \quad \gamma'(p,x,y)\neq 0 \tag{8b}$$

(if $p=0$, then fractions with the denominator p must be taken equal to 1).

Now let $\delta(x,y)=\gamma'\big(\pi(x,y),x,y\big)$. If $y\in Z$, then from (5) and (8a) we obtain $\lim_x 10^{-x}\delta(x,y)=0$. If $y\notin Z$, then we use (6), (7) and (8b) and infer that $10^{-x_j}\delta(x_j,y)\geqslant -10^{-x_j}+1/p_0$ for infinitely many x_j; hence $\lim_x 10^{-x}\delta(x,y)$ either does not exist or is different from 0. This proves that $Z=Z_\delta^{(3)}$, q. e. d.

THEOREM 4. *The family $\{Z_\gamma^{(3)}\}_{\gamma\in\Phi^*}$ is identical with $\boldsymbol{Q}_2^{(1)}$.*

Proof. If $\gamma\in\Phi^*$, then $Z_\gamma^{(3)}\in\boldsymbol{Q}_2^{(1)}$ since

$$[\lim_x 10^{-x}\gamma(x,y)=0]\equiv\prod_{p\neq 0}\sum_x[10^{-x}\gamma(x,y)<1/p].$$

Let us now assume that $Z\in\boldsymbol{Q}_2^{(1)}$ and let Z' be the complement of Z. Since $Z'\in\boldsymbol{P}_2^{(1)}$, we can apply theorem 1 and obtain a function δ such that

$$y\in Z'\equiv\sum_{x_0}\prod_{x>x_0}[\delta(x,y)=0].$$

Let

$$\beta(x,y)=1+\sum_{t\leqslant x}\delta(t,y), \qquad \gamma(x,y)=[10^x/\beta(x,y)].$$

It is evident that

$$\beta(x,y)\leqslant\beta(x+1,y),$$

$$y\in Z\equiv[\lim_x\beta(x,y)=\infty],$$

$$-10^{-x}+1/\beta(x,y)<10^{-x}\gamma(x,y)\leqslant 1/\beta(x,y).$$

From these formulas we easily obtain $\gamma\in\Phi^*$ and $Z=Z_\gamma^{(3)}$.

Remarks. 1. It follows from the above proof that if $y\in Z$, then $\lim_x 10^{-x}\gamma(x,y)$ is rational but not integral since $\beta(x,y)>1$ and $\beta(x,y)$ is constant from a certain x on.

2. Let us call a real number λ recursive if there are functions $\alpha,\beta\in\Phi$ such that $\lambda=\lim_x\alpha(x)/\beta(x)$. Theorems 3 and 4 remain true if we replace 0 in the definition 3 by a recursive real number λ.

Definition 4. Let Φ^{**} be the subclass of Φ^* containing all the functions φ such that among the numbers $\lim_{x\to\infty}10^{-x}\varphi(x,y)$, $y=0,1,2,\ldots$ there are only finitely many different numbers.

THEOREM 5. *The family $\{Z_\gamma^{(3)}\}_{\gamma\in\Phi^{**}}$ is identical with $\boldsymbol{P}_2^{(1)}\cap\boldsymbol{Q}_2^{(1)}$.*

Proof. First let us assume that $\gamma\in\Phi^{**}$. We denote by $y_1,y_2,\ldots,y_k$ integers satisfying the conditions

$$\lim_x 10^{-x}\gamma(x,y_j)\neq\lim_x 10^{-x}\gamma(x,y_h) \quad \text{for} \quad j\neq h,$$

$$\lim_x 10^{-x}\gamma(x,y_j)\neq 0 \quad \text{for} \quad j=1,2,\ldots,k,$$

$$\text{if } \lim_x 10^{-x}\gamma(x,y)\neq 0, \text{ then } \sum_{j\leqslant k}[\lim_x 10^{-x}\gamma(x,y)=\lim_x 10^{-x}\gamma(x,y_j)].$$

The existence of integers $y_1,\ldots,y_k$ follows from the assumption $\gamma\in\Phi^{**}$. We do not exclude the possibility $k=0$; in this case $\lim 10^{-x}\gamma(x,y)=0$ for all y.

From definitions 3 and 4 we obtain

$$y\in Z_\gamma^{(3)}\equiv\prod_{p\neq 0}\sum_x[10^{-x}\gamma(x,y)<1/p],$$

$$y\notin Z_\gamma^{(3)}\equiv\sum_{j\leqslant k}\prod_{p\neq 0}\sum_x\Big[10^{-x}\big(\gamma(x,y)-\gamma(x,y_j)\big)<1/p\Big].$$

It follows from these formulas that $Z_\gamma^{(3)}\in\boldsymbol{P}_2^{(1)}\cap\boldsymbol{Q}_2^{(1)}$. This result holds true also in the case of $k=0$ since then $Z_\gamma^{(3)}$ contains all the integers.

Now we assume that $Z\in\boldsymbol{P}_2^{(1)}\cap\boldsymbol{Q}_2^{(1)}$. By theorem 4 there exist two functions $\alpha_1,\alpha_2\in\Phi^*$ such that

$$y\in Z\equiv\lim_x 10^{-x}\alpha_1(x,y)=0, \tag{9}$$

$$y\in Z\equiv\lim_x 10^{-x}\alpha_2(x,y)=0. \tag{10}$$

Let

$$\gamma(x,y)=\left[\frac{10^x a_1(x,y)}{a_1(x,y)+a_2(x,y)}\right].$$

Since

$$\frac{a_1(x,y)}{a_1(x,y)+a_2(x,y)}-10^{-x}<10^{-x}\gamma(x,y)\leqslant\frac{a_1(x,y)}{a_1(x,y)+a_2(x,y)},$$

we easily infer from (9) and (10) that $\gamma\in\Phi^{**}$ and $Z=Z_\gamma^{(3)}$.

Definition 5. $Z_\gamma^{(4)}=\underset{y}{E}\,[\lim\limits_{x} 10^{-x}\gamma(x,y)$ exists].

THEOREM 6. *The family* $\{Z_\gamma^{(4)}\}_{\gamma\in\Phi}$ *is identical with* $\boldsymbol{Q}_3^{(1)}$.

Proof. $Z_\gamma^{(4)}\in\boldsymbol{Q}_3^{(1)}$ since

$$y\in Z_\gamma^{(4)}\equiv\prod_{p\neq0}\sum_{q}\prod_{x_1,x_2}\{(x_1>q)(x_2>q)\rightarrow|10^{-x_1}\gamma(x_1,y)-10^{-x_2}\gamma(x_2,y)|<1/p\}.$$

If $Z\in\boldsymbol{Q}_3^{(1)}$, then by theorem 3 there is a $\gamma\in\Phi$ such that $Z=Z_\gamma^{(3)}$. Putting $\beta(2x,y)=10^{-x}\gamma(x,y),\beta(2x+1,y)=0$ we obtain $Z=Z_\beta^{(4)}$, q. e. d.

Definition 6. $Z_\gamma^{(5)}=\underset{y}{E}\,[\lim\limits_{x} 10^{-x}\gamma(x,y)$ exists and is integral].

THEOREM 7. *The family* $\{Z_\gamma^{(5)}\}_{\gamma\in\Phi}$ *is identical with* $\boldsymbol{Q}_3^{(1)}$.

Proof. We denote by $\{a\}$ the distance from a to the nearest integer and put

$$[p/10^x]=(\mu n)_p(10^x n>p)-1,$$

$$(p,x)=\min\left(p-10^x[p/10^x],-p+10^x+10^x[p/10^x]\right).$$

It is then obvious that

$$\{p/10^x\}=10^{-x}(p,x).$$

Since $\{a\}$ is a continuous function of a, we have the equivalence

$$y\in Z_\gamma^{(5)}\equiv\left(\lim_{x\to\infty}\{10^{-x}\gamma(x,y)\}=0\right)\equiv\left(\overline{\lim_{x\to\infty}}\,10^{-x}\big(\gamma(x,y),x\big)=0\right)$$

$$\equiv\prod_{p}\sum_{q}\prod_{x>q}\left(10^{-x}\big(\gamma(x,y),x\big)<1/p\right),$$

which proves that $Z_\gamma^{(5)}\in\boldsymbol{Q}_3^{(1)}$.

We assume now that $Z\in\boldsymbol{Q}_3^{(1)}$. By theorem 3 there is a function $\gamma\in\Phi$ such that $Z=Z_\gamma^{(3)}$. Putting

$$\beta(2x,y)=\gamma(x,y),\qquad\beta(2x+1,y)=0,$$

we obtain a function β such that $Z=Z_\gamma^{(3)}=Z_\beta^{(5)}$.

THEOREM 8. *The family* $\{Z_\gamma^{(5)}\}_{\gamma\in\Phi^*}$ *is identical with* $\boldsymbol{Q}_2^{(1)}$.

Proof. $Z_\gamma^{(5)}\in\boldsymbol{Q}_2^{(1)}$ for $\gamma\in\Phi^*$ since

$$y\in Z_\gamma^{(5)}\equiv\lim_{x}\{10^{-x}\gamma(x,y)\}=0\equiv\prod_{p}\sum_{x}\left(10^{-x}\big(\gamma(x,y),x\big)<1/p\right).$$

If $Z\in\boldsymbol{Q}_2^{(1)}$, then it follows from remark 1 on p. 265 that there is a function $\gamma\in\Phi^*$ such that $Z=Z_\gamma^{(5)}$.

Definition 7. $Z_\gamma^{(6)}=\underset{y}{E}\,[\lim\limits_{x} 10^{-x}\gamma(x,y)$ exists and is irrational].

THEOREM 9. *The family* $\{Z_\gamma^{(6)}\}_{\gamma\in\Phi}$ *is identical with* $\boldsymbol{Q}_3^{(1)}$.

Proof. $Z_\gamma^{(6)}\in\boldsymbol{Q}_3^{(1)}$ since

$$y\in Z_\gamma^{(6)}\equiv\left(\lim_{x}10^{-x}\gamma(x,y)\text{ exists}\right)\cdot\prod_{p}\prod_{q\neq0}\sum_{n}\prod_{x}[|10^{-x}\gamma(x,y)-p/q|>1/n]$$

$$\equiv\prod_{s\neq0}\sum_{t}\prod_{x_1>t}\prod_{x_2>t}[|10^{-x_1}\gamma(x_1,y)-10^{-x_2}\gamma(x_2,y)|<1/s].$$

$$\prod_{p}\prod_{q\neq0}\sum_{n}\prod_{x}[|10^{-x}\gamma(x,y)-p/q|>1/n].$$

Let us now assume that $Z\in\boldsymbol{Q}_3^{(1)}$, *i. e.*, $Z=Z_\gamma^{(3)}$ for a $\gamma\in\Phi$. Let $\sqrt{2}=1+\sum\limits_{n}10^{-n}c_n$ and $\varphi(x)=\sum\limits_{n=1}^{x}10^{x-n}c_n$. The function φ is primitive recursive. Further let $\beta(2x,y)=\gamma(x,y)+\varphi(x)$, $\beta(2x+1,y)=\varphi(x)$. It is evident that $\lim\limits_{x}10^{-x}\beta(x,y)$ exists if and only if $\lim\limits_{x\to\infty}10^{-x}\gamma(x,y)$ exists and is equal 0, *i. e.*, for $y\in Z$. In that case $\lim\limits_{x}10^{-x}\beta(x,y)=\sqrt{2}-1$ and hence $Z=Z_\beta^{(6)}$.

THEOREM 10. *The family* $\{Z_\gamma^{(6)}\}_{\gamma\in\Phi^*}$ *is identical with* $\boldsymbol{Q}_3^{(1)}$.

Proof. In view of theorem 9 it is sufficient to show that for each $Z\in\boldsymbol{Q}_3^{(1)}$ there is a $\gamma\in\Phi^*$ such that $Z=Z_\gamma^{(6)}$. Let us assume therefore that

$$y\in Z\equiv\prod_{j}\sum_{p}\prod_{s}[\varrho(p,s,j,y)=0],\qquad\varrho\in\Phi. \tag{11}$$

We now repeat the construction carried out in the proof of theorem 1 replacing the function $\varrho(p,s,y)$ by the function $\varrho(p,s,j,y)$ occurring in (11) and treating j as a new parameter. We obtain a function $\pi(x,j,y)$ with the following properties:

$$\pi(x+1,j,y)\geqslant\pi(x,j,y), \tag{12}$$

(13) if $\prod\limits_{p}\sum\limits_{s}[\varrho(p,s,j,y)\neq0]$, then $\lim\limits_{x\to\infty}\pi(x,j,y)=\infty$,

(14) if p_0 is a smallest integer such that $\prod\limits_{s}[\varrho(p_0,s,j,y)=0]$, then

$$\lim_{x\to\infty}\pi(x,j,y)=p_0.$$

We put $\pi'(x,j,y)=\pi(x,j,y)+1$ and denote by $a(x,j,y)$ and $\beta(x,j,y)$ the numerator and the denominator of the jth convergent of the continuous fraction

$$\frac{1|}{|\pi'(x,0,y)}+\frac{1|}{|\pi'(x,1,y)}+\ldots$$

To this effect we take

$$\alpha(x,0,y)=1,\qquad \beta(x,0,y)=\pi'(x,0,y),$$
$$\alpha(x,1,y)=\pi'(x,1,y),\qquad \beta(x,1,y)=\pi'(x,0,y)\cdot\pi'(x,1,y)+1,$$
$$\alpha(x,j+2,y)=\pi'(x,j+2,y)\cdot\alpha(x,j+1,y)+\alpha(x,j,y),$$
$$\beta(x,j+2,y)=\pi'(x,j+2,y)\cdot\beta(x,j+1,y)+\beta(x,j,y).$$

It is evident that $\lim\limits_j \alpha(x,j,y)=\lim\limits_j \beta(x,j,y)=\infty$. We put

$$\varphi(x,y)=\alpha(x,x,y),\qquad \psi(x,y)=\beta(x,x,y)$$

and have therefore

$$\varphi(x,y)/\psi(x,y)=\frac{1|}{|\pi'(x,0,y)}+\frac{1|}{|\pi'(x,1,y)}+\ldots+\frac{1|}{|\pi'(x,x,y)}. \tag{15}$$

We shall now calculate $\lim\limits_x\big(\varphi(x,y)/\psi(x,y)\big)$. Let us first assume that $y\in Z$, *i. e.*, that for each j there is a smallest $p_0=p_0(j)$ such that $\prod\limits_s[\varrho(p_0,s,j,y)=0]$. According to (14) we have

$$\lim_x \pi'(x,j,y)=p_0(j)+1=p_0'(j)\qquad\text{for}\quad j=0,1,2,\ldots \tag{16}$$

Let λ be the irrational number

$$\lambda=\frac{1|}{|p_0'(0)}+\frac{1|}{|p_0'(1)}+\ldots$$

and let $R_n=P_n/Q_n$ be the nth convergent of this continuous fraction. As is well known, $|\lambda-R_n|<1/Q_nQ_{n-1}$ whence it follows that if $q(0),q(1),\ldots$ is a (finite or infinite) sequence which in its first n terms coincides with the sequence $p_0'(0),p_0'(1),\ldots$, and if $\mu=\frac{1|}{|q(0)}+\frac{1|}{|q(1)}+\ldots$, then $|\mu-R_n|<1/Q_nQ_{n-1}$ and hence

$$\left|\lambda-\left(\frac{1|}{|q(0)}+\frac{1|}{|q(1)}+\ldots\right)\right|<2/Q_nQ_{n-1}.$$

Let $\varepsilon>0$ be arbitrary; choose n so large that $2/Q_nQ_{n-1}<\varepsilon$. By (16) there is an x_0 such that for $x\geqslant x_0$

$$\pi'(x,j,y)=p_0'(j)\qquad\text{for}\quad j=0,1,\ldots,n-1.$$

Thus the first n terms of the sequence $\pi'(x,0,y),\pi'(x,1,y),\ldots$ coincide with the first n terms of the sequence $p_0'(0),p_0'(1),\ldots$ and hence for $x>\max(x_0,n)$

$$\left|\lambda-\left(\frac{1|}{|\pi'(x,0,y)}+\ldots+\frac{1|}{|\pi'(x,x,y)}\right)\right|<\varepsilon.$$

This inequality together with (15) yields

$$|\varphi(x,y)/\psi(x,y)-\lambda|<\varepsilon$$

and, since ε was arbitrary,

$$\lim_{x\to\infty}\big(\varphi(x,y)/\psi(x,y)\big)=\lambda.$$

Next we assume that $y\notin Z$, *i. e.*, that there is a (smallest) j_0 such that $\prod\limits_p\sum\limits_s[\varrho(p,s,j_0,y)\neq 0]$. Thus for $j<j_0$ we have formula (16) whereas from (13) it follows that

$$\lim_x \pi'(x,j_0,y)=\infty.$$

We obviously have

$$0<\frac{1|}{|\pi'(x,j_0,y)}+\ldots+\frac{1|}{|\pi'(x,x,y)}<\frac{1}{\pi'(x,j_0,y)}\qquad(x>j_0),$$

whence

$$\lim_{x\to\infty}\left(\frac{1|}{|\pi'(x,j_0,y)}+\ldots+\frac{1|}{|\pi'(x,x,y)}\right)=0.$$

Using (15) and (16) we obtain therefore

$$\lim_{x\to\infty}\big(\varphi(x,y)/\psi(x,y)\big)=\lim_{x\to\infty}\frac{1|}{|\pi'(x,0,y)}+\ldots+\frac{1|}{|\pi'(x,j_0-1,y)}$$
$$=\frac{1|}{|p_0'(0)}+\ldots+\frac{1|}{|p_0'(j_0-1)}$$

and hence $\lim\limits_x\big(\varphi(x,y)/\psi(x,y)\big)$ is rational.

Now let $\gamma(x,y)=[10^x\varphi(x,y)/\psi(x,y)]$. It is evident that this function is primitive recursive and

$$10^{-x}\gamma(x,y)\leqslant\varphi(x,y)/\psi(x,y)<10^{-x}\gamma(x,y)+10^{-x}.$$

It follows that $\lim\limits_x 10^{-x}\gamma(x,y)=\lim\limits_x\big(\varphi(x,y)/\psi(x,y)\big)$. Hence $\lim\limits_x 10^{-x}\gamma(x,y)$ exists for an arbitrary y and is rational for $y\notin Z$ and irrational for $y\in Z$. Hence $\gamma\in\Phi^*$ and $Z=Z_\gamma^{(6)}$.

We conclude by stating an open problem. Let X be a recursively enumerable set of non-negative rational numbers (*i. e.*, there are functions $\varphi,\psi\in\Phi$ such that $r\in X$ if and only if it can be represented in the form $r=\varphi(x)/\psi(x)$). Let $Z_\gamma(X)=\underset{y}{E}[\lim\limits_x 10^{-x}\gamma(x,y)\in X]$. The problem is to determine the family $\{Z_\gamma(X)\}_{\gamma\in\Phi^*}=F(X)$.

From theorems 4, 8, and 10 we obtain the following partial answers to that problem: if X_0 contains only the number 0, X_1 is the set of all non-negative integers, and X_2 is the set of all non-negative rationals, then $F(X_0)=\boldsymbol{Q}_2^{(1)}$, $F(X_1)=\boldsymbol{Q}_2^{(1)}$, $F(X_2)=\boldsymbol{P}_3^{(1)}$.

References

[1] W. Markwald, *Zur Eigenschaft primitiv-rekursiver Funktionen, unendlich viele Werte anzunehmen*, this volume, p. 166-167.

[2] A. Mostowski, *On definable sets of positive integers*, Fund. Math. 34 (1947), p. 81-112.

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Contributions to the theory of definable sets and functions

by

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In this paper we collect some scattered results concerning sets and functions definable in elementary arithmetic. We shall use consistently the terminology and notations of the paper [2], with which, we assume, the reader is acquainted. In particular we denote by R_k the set of k-ples $(x_1,x_2,\ldots,x_k)=\mathfrak{m}$, where the x_j's are non-negative integers, and by $\boldsymbol{P}_n^{(kl)}$ (or $\boldsymbol{Q}_n^{(kl)}$) the set of functions from R_k to R_l whose graphs are in $\boldsymbol{P}_n^{(k+l)}$ (or in $\boldsymbol{Q}_n^{(k+l)}$).

1. We begin by establishing some simple properties of the classes $\boldsymbol{P}_n^{(k1)}$ and $\boldsymbol{Q}_n^{(k1)}$.

THEOREM 1. $\boldsymbol{P}_n^{(k1)}\subset\boldsymbol{Q}_n^{(k1)}$.

Proof. The theorem is evident in case $n=0$. Let us, therefore, assume that $n>0$ and $f\in\boldsymbol{P}_n^{(k1)}$. It follows from the definitions that there exists a set $B\in\boldsymbol{Q}_{n-1}^{(k+2)}$ such that

$$\{f(\mathfrak{m})=m\}\equiv\sum_x\{(\mathfrak{m},m,x)\in B\}.$$

Hence

$$\{f(\mathfrak{m})\neq m\}\equiv\sum_{p,x}\{[(\mathfrak{m},p,x)\in B]\cdot(p\neq m)\}$$

which proves that the graph of f is in $\boldsymbol{Q}_n^{(k+1)}$, q. e. d.

THEOREM 2. *If* $n\geqslant 1$, *then* $\boldsymbol{P}_{n+1}^{(k1)}-\boldsymbol{Q}_n^{(k1)}\neq 0\neq\boldsymbol{Q}_n^{(k1)}-\boldsymbol{P}_n^{(k1)}$.

Proof. It is well known that there are sets M which belong to $\boldsymbol{P}_{n+1}^{(k)}\cdot\boldsymbol{Q}_{n+1}^{(k)}$ without belonging to $\boldsymbol{Q}_n^{(k)}$. Let f be the characteristic function of such a set M. The graph of f is in $\boldsymbol{P}_{n+1}^{(k+1)}$, since

$$\{y=f(\mathfrak{m})\}\equiv\{(y=0)\cdot(\mathfrak{m}\notin M)+(y=1)\cdot(\mathfrak{m}\in M)\}.$$

As $\{\mathfrak{m}\in M\}\equiv\{f(\mathfrak{m})=1\}$, the graph of f is not in $\boldsymbol{Q}_n^{(k+1)}$. Hence $f\in\boldsymbol{P}_{n+1}^{(k1)}-\boldsymbol{Q}_n^{(k1)}$.

Slightly more intricate is the proof that $\boldsymbol{Q}_n^{(k1)}-\boldsymbol{P}_n^{(k1)}\neq 0$. Let $C\in\boldsymbol{Q}_n^{(k)}-\boldsymbol{P}_n^{(k)}$ and let B be a set in $\boldsymbol{Q}_{n-1}^{(k+1)}$ such that

$$\mathfrak{m}\notin C\equiv\sum_x[(\mathfrak{m},x)\in B].$$