additives uniformément bornées — une suite partielle convergente. Soit $\mu_{\lambda f}(\Delta)=\lim\limits_{n_k}\mu_{n_k}(\Delta)$. D'une manière analogue que dans le travail [2] on peut former l'expression suivante:

$$I(\tau)=2\lambda\int\limits_F f(\zeta)\,d\tau(\Delta_\zeta)+\iint\limits_F \log|z-\zeta|^{-1}\,d\tau(\Delta_z)\,d\tau(\Delta_\zeta),$$

(où τ représente une distribution quelconque de la masse unité sur F) et démontrer que

$$\log\big(1/d(F,\omega_\lambda)\big)=I(\mu_{\lambda f})\leqslant I(\tau).$$

D'après (7) on peut présenter la fonction $\log\Phi_{n_k}^{(0)}(z,\eta_\lambda^{(n_k)})$ sous la forme suivante:

$$\log\Phi_{n_k}^{(0)}(z,\eta_\lambda^{(n_k)})=\lambda F(z)+ \tag{11}$$
$$+\int\limits_F \log\big(|z-\zeta|\,\exp\{-\lambda[F(z)+f(\zeta)]\}\big)\,d\mu_{n_k}(\Delta_\zeta)-\log\big(\Delta_{n_k}^{(0)}(\eta_\lambda^{(n_k)})\big)^{1/n_k}.$$

Lorsque k tend vers ∞ on obtient de (11) l'égalité[2]):

$$\log\Phi(z,\lambda f)=\int\limits_F\log|z-\zeta|\,d\mu_{\lambda f}(\Delta_\zeta)-\lambda\int\limits_F f(\zeta)\,d\mu_{\lambda f}(\Delta_\zeta)+\log\big(1/(F,\omega_\lambda)\big). \tag{12}$$

Désignons par $U(z)$ la fonction

$$U(z)=\int\limits_F\log|z-\zeta|^{-1}\,d\mu_{\lambda f}(\Delta_\zeta)+\lambda F(z)$$

et posons

$$\gamma_{\lambda f}=\log\big(d(F,\omega_\lambda)\big)^{-1}-\lambda\int\limits_F f(\zeta)\,d\mu_{\lambda f}(\Delta_\zeta).$$

D'après (12), il suit de (5) que $U(z)=\gamma_{\lambda f}$ pour chaque point $z\in F_{\lambda f}$. Lorsque $z\in F$, on a $U(z)\geqslant\gamma_{\lambda f}$.

Dans le cas général $\lambda>0$ et $f(z)\neq\mathrm{const}$ l'ensemble $F-F_{\lambda f}$ n'est pas vide. Néanmoins, lorsque $\lambda\to 0$, on peut démontrer [4] que la distance d'un point quelconque z_0 de F à $F_{\lambda f}$ tend vers zéro.

Travaux cités

[1] O. Frostman, *Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Meddel. f. Lunds Univ. Mat. Sem. 3, p. 1-118.

[2] J. Górski, *Méthode des points extrémaux de résolution du problème de Dirichlet dans l'espace*, Ann. Pol. Math. 1 (1955), p. 418-429.

[3] F. Leja, *Sur une famille de fonctions analytiques extrémales*, Ann. Soc. Pol. Math. 25 (1952), p. 1-16.

[4] — *Polynômes extrémaux et la représentation conforme des domaines doublement connexes*, Ann. Pol. Math. 1 (1955), p. 13-28.

[5] C. de la Vallée Poussin, *Extension de la méthode du balayage de Poincaré et problème de Dirichlet*, Ann. de l'Inst. H. Poincaré 2 (1932), p. 169-232.

[2]) De la formule (11) découle, en particulier, d'après (4) l'existence de la limite (3).

On the epidermic effect for ordinary differential inequalities of the first order

by W. Mlak (Kraków)

The aim of this paper is to generalize the epidermic effect in the case of finite systems of ordinary differential inequalities. The proof given in this paper is different from the proof given in [3].

I am glad to express here my thanks to J. Szarski for many valuable remarks which helped me to obtain the above-mentioned generalization.

We introduce the following

Assumption H. *The functions $f_i(t,y_1,\ldots,y_n)$ $(i=1,\ldots,n)$, continuous in the open set Ω of the space of points $(t,y_1,\ldots,y_n)$, satisfy the following condition:*

(M) *If $A_i=(t,a_1,\ldots,a_{i-1},c,a_{i+1},\ldots,a_n)$, $B_i=(t,b_1,\ldots,b_{i-1},c,b_{i+1},\ldots,b_n)$ and $a_\nu\leqslant b_\nu$ for $\nu=1,2,\ldots,i-1,i+1,\ldots,n$, then $f_i(A_i)\leqslant f_i(B_i)$.*

Let us consider the system of differential equations

$$y_i'=f_i(t,y_1,\ldots,y_n)\quad(i=1,2,\ldots,n). \tag{1}$$

It is known (see [2], p. 122, theorem I) that, if the functions f_i fulfil the assumption H in the open set Ω, then for every point $(t_0,y_1^0,\ldots,y_n^0)$ there exists a right maximal integral $\tau_1(t),\ldots,\tau_n(t)$ of the system (1) such that $y_i^0=\tau_i(t_0)$.

The integral $\tau_1(t),\ldots,\tau_n(t)$ may be prolonged to the boundary of Ω.

One can easily prove the following

Lemma. *Let $\tau_1(t),\ldots,\tau_n(t)$ be the right maximal integral of the system* (1) *valid in the interval $[t_0,t_1]$. Suppose that the system* (1) *satisfies* H *and the system of functions $\tau_1^\nu(t),\ldots,\tau_n^\nu(t)$ forms a right maximal integral of the system*

$$y_i'=f_i(t,y_1,\ldots,y_n)+1/\nu\quad(i=1,2,\ldots,n;\ \nu=1,2,3,\ldots)$$

such that $\tau_i^\nu(t_0)=\tau_i(t_0)+1/\nu$ $(i=1,2,\ldots,n)$.

Under these assumptions, for ν sufficiently large, the functions $\tau_i^\nu(t)$ are determined in $[t_0,t_1]$ and $\tau_i^\nu(t)\to\tau_i(t)$ uniformly in $[t_0,t_1]$ $(i=1,2,\ldots,n)$. At the same time, for $t\in[t_0,t_1]$ and $i=1,2,\ldots,n$, we have $\tau_i^\nu(t)>\tau_i(t)$ for ν sufficiently large.

Now we formulate the epidermic theorem for systems of differential inequalities.

THEOREM. *Suppose that the following conditions are fulfilled:*

1. *The sequence $\tau_1(t),\ldots,\tau_n(t)$ forms a right maximal integral, valid in $[t_0,t_0+a)$ $(a>0)$, of the system* (1). *The right members of* (1) *fulfil* H.

2. *The functions $\varphi_1(t),\ldots,\varphi_n(t)$ are continuous in $\Delta=[t_0,t_0+a)$ and*

$$\big(t,\varphi_1(t),\ldots,\varphi_n(t)\big)\in\Omega \quad \text{for} \quad t\in\Delta.$$

3. *The functions $\varepsilon_i(t)$ $(i=1,2,\ldots,n)$ are continuous in Δ and $\varepsilon_i(t)>0$ for $t\in\Delta$.*

4. *For $i=1,2,\ldots,n$ we have $\varphi_i(t_0)\leqslant\tau_i(t_0)$.*

5. *Let k be an arbitrary number of the sequence $1,2,\ldots,n$. For $t\in\Delta$ the inequality $\tau_k(t)<\varphi_k(t)<\tau_k(t)+\varepsilon_k(t)$ implies the inequality*

$$\overline{D}_-\varphi_k(t)\leqslant f_k\big(t,\varphi_1(t),\ldots,\varphi_n(t)\big).$$

Under the assumptions given above the inequalities

$$\varphi_i(t)\leqslant\tau_i(t) \qquad (i=1,2,\ldots,n)$$

hold for $t\in[t_0,t_0+a)$.

Proof. Condition 5 we call N condition. The assumptions of the lemma are satisfied. Using the notation of the lemma we need only prove that, for every interval $\Delta(t_1)=[t_0,t_1]$ $(t_0<t_1<t_0+a)$ and for ν sufficiently large, the inequalities

$$(2) \qquad \varphi_i(t)<\tau_i^\nu(t) \qquad (i=1,2,\ldots,n)$$

are satisfied for $t\in\Delta(t_1)$. Because of the continuity of $\varepsilon_i(t)$ and the uniform convergence of $\tau_i^\nu(t)$ there exists ν_0 such that for $\nu\geqslant\nu_0$

$$(3) \qquad 0<\tau_i^\nu(t)-\tau_i(t)<\varepsilon_i(t) \quad \text{for} \quad t\in\Delta(t_1),\quad i=1,2,\ldots,n.$$

Suppose that there exists $\nu'\geqslant\nu_0$ such that for some k $(1\leqslant k\leqslant n)$ and $\bar{t}\in\Delta(t_1)$ the following inequality holds:

$$\tau_k^{\nu'}(\bar{t})\leqslant\varphi_k(\bar{t}).$$

Therefore the set

$$E=\sum_{i=1}^{n}\mathop{E}_{t}\{\tau_i^{\nu'}(t)\leqslant\varphi_i(t),\ t\in\Delta(t_1)\}$$

is not empty. We put $\xi=\inf E$. Then we have $t_0<\xi$

$$(4) \qquad \varphi_i(t)<\tau_i^{\nu'}(t) \quad \text{for} \quad t_0\leqslant t<\xi,\ i=1,2,\ldots,n,$$

$$(5) \qquad \varphi_i(\xi)\leqslant\tau_i^{\nu'}(\xi) \quad \text{for} \quad i=1,2,\ldots,n.$$

There is such an s $(1\leqslant s\leqslant n)$ that

$$(6) \qquad \varphi_s(\xi)=\tau_s^{\nu'}(\xi).$$

By (3) and (6) we have

$$(7) \qquad \tau_s(\xi)<\varphi_s(\xi)<\tau_s(\xi)+\varepsilon_s(\xi).$$

According to the condition N and to (7) we have

$$\overline{D}_-\varphi_s(\xi)\leqslant f_s\big(\xi,\varphi_1(\xi),\ldots,\varphi_n(\xi)\big).$$

Because of (5), (6) and (M), according to the definition of $\tau_i^{\nu'}(t)$, we get

$$\overline{D}_-\varphi_s(\xi)<f_s\big(\xi,\tau_1^{\nu'}(\xi)\ldots\tau_n^{\nu'}(\xi)\big)+1/\nu'=\overline{D}_-\tau_s^{\nu'}(\xi).$$

Consequently, according to (6), the inequality $\tau_s^{\nu'}(t)<\varphi_s(t)$ holds for $t<\xi$ and t sufficiently near ξ. This contradicts (4). Therefore E is empty and the inequalities (2) hold for $\nu\geqslant\nu_0$.

Remark 1. Let us consider the function

$$\psi_k(t)=\varphi_k(t)-\int_{t_0}^{t}f_k\big(z,\varphi_1(z),\ldots,\varphi_n(z)\big)\,dz.$$

The condition N means that the inequality $\overline{D}_-\psi_k(t)\leqslant 0$ is true in the open set

$$Z_k=\mathop{E}_{t}\{\tau_k(t)<\varphi_k(t)<\tau_k(t)+\varepsilon_k(t),\ t\in\Delta\}.$$

On the other hand the following theorem is true ([1], p. 203): if for a function continuous in a given interval, one of Dini's derivatives is non-positive at every point of that interval with the exception of a denumerable set of points, then the function is non-increasing in the given interval.

Applying this theorem to the function $\psi_k(t)$ and the components of the set Z_k, one can see that the condition N is equivalent to the following one.

Condition N_1. In every component of the set Z_k, with the exception of a denumerable set of points, one of the inequalities

$$\overline{D}_+\varphi_k(t)\leqslant f_k\big(t,\varphi_1(t),\ldots,\varphi_n(t)\big),$$

$$\overline{D}_-\varphi_k(t)\leqslant f_k\big(t,\varphi_1(t),\ldots,\varphi_n(t)\big),$$

$$\underline{D}_+\varphi_k(t)\leqslant f_k\big(t,\varphi_1(t),\ldots,\varphi_n(t)\big),$$

$$\underline{D}_-\varphi_k(t)\leqslant f_k\big(t,\varphi_1(t),\ldots,\varphi_n(t)\big)$$

is satisfied.

In particular, in different components of Z_k inequalities for different Dini's derivatives may be satisfied. The epidermic theorem remains true if we replace the condition N by N_1.

Remark 2. The assertion of our theorem remains true if $\varphi_i(t)$ are ACG in Δ. In that case the condition N may be replaced by the following one. For almost all points of the set

$$Z_k = \mathop{E}_t\{\tau_k(t) < \varphi_k(t) < \tau_k(t)+\varepsilon_k(t),\ t\in\Delta\}$$

the following inequality holds:

$$\dot{\varphi_k}(t) \leqslant f_k(t,\varphi_1(t),\ldots,\varphi_n(t)). \tag{8}$$

$\dot{\varphi_k}(t)$ denotes here the approximative derivative of the function $\varphi_k(s)$ at the point t. It is easy to see that the condition given above implies the condition N. In fact, if $\varphi_k(t)$ is ACG in Δ, so is

$$\psi_k(t) = \varphi_k(t) - \int_{t_0}^{t} f_k(z,\varphi_1(z),\ldots,\varphi_n(z))\,dz.$$

The inequality (8) states that almost everywhere in Z_k the approximative derivative of $\psi_k(t)$ is non-positive. Therefore one can conclude ([1], p. 225) that the function $\psi_k(t)$ is non-increasing in every component of Z_k. But it is sufficient for the inequality

$$\overline{D}_-\varphi_k(t) \leqslant f_k(t,\varphi_1(t),\ldots,\varphi_n(t))$$

to be satisfied in Z_k. Thus the condition N holds.

Remark 3. What has been proved for the right-hand maximal integral can also be proved in a similar way for the remaining extreme integrals.

In the case of the left-hand integrals one must introduce a suitable condition instead of (M) for the right-hand members of differential equations ([2], p. 137).

References

[1] S. Saks, *Theory of integral*, Warszawa-Lwów 1937.

[2] T. Ważewski, *Systèmes des équations et des inégalités différentielles ordinaires aux deuxièmes membres monotones et leurs applications*, Ann. Soc. Pol. Math. 23 (1950), p. 112-166.

[3] — *Certaines propositions de caractère „épidermique" relatives aux inégalités différentielles*, Ann. Soc. Pol. Math. 24 (1951), p. 1-12.

Equations satisfied by the extremal schlicht functions with a pole

by J. Zamorski (Wrocław)

In the present paper I give the differential functional equations which must be satisfied by extremal schlicht functions with a pole. These equations are analogical to the Schaeffer-Spencer equations for the regular schlicht functions.

Consider the class Σ of functions regular and schlicht for $0<|z|<1$, which have the expansion of the form $F(z)=z^{-1}+b_1z+b_2z^2+\ldots$

The first p coefficients of each function of this form determine a point of the real $2p$-dimensional Euclidean space. To the function $F(z)=z^{-1}+$ $+b_1z+b_2z^2+\ldots$ corresponds the point with the coordinates $(x_1,y_1,x_2,y_2,$ $\ldots,x_p,y_p)$ where $b_j=x_j+iy_j$. To the schlicht functions corresponds a certain set D in this space. This set contains the origin of the coordinate system, because the function $F(z)=z^{-1}$ belongs to the class Σ, it is connected, since together with the function $F(z)=z^{-1}+b_1z+b_2z^2+\ldots$ the function $\varrho F(z\varrho)=z^{-1}+b_1\varrho^2z+\ldots$ also belongs to the class Σ, which means that every point of the set D may be joined to the origin of the system by a curve lying in D. From the surface-theorem ([1], p. 72-76) it follows that the set D is bounded and from the normality [2] of the family Σ it follows that it is closed.

Now consider the region G in this space, which contains the set D and define in it an arbitrary real-valued function $E(x_1,y_1,x_2,y_2,\ldots,$ $x_p,y_p)$ of $2p$ arguments, continuous together with its first partial derivatives and satisfying the condition

$$\sum_{k=1}^{p}\left((\partial E/\partial x_k)^2+(\partial E/\partial y_k)^2\right) > 0$$

for every point of the set D.

The function E may be treated as a functional defined for the functions belonging to the class Σ. Let us introduce additional symbols

$$E_k = \tfrac{1}{2}\{\partial E/\partial x_k - i\partial E/\partial y_k\},\qquad \overline{E}_k = \tfrac{1}{2}\{\partial E/\partial x_k + i\partial E/\partial y_k\}.$$

Since the derivatives do not vanish in the interior of the set D, therefore the function E has the extreme value only on its boundary.