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The inhomogeneous minima of a sequence of symmetric Markov forms

by

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1. Introduction

Let

$$f(x, y) = ax^2 + bxy + cy^2 \tag{1.1}$$

be an indefinite binary quadratic form with real coefficients and discriminant $D = b^2 - 4ac > 0$, and write $\Delta = +\sqrt{D}$; and let $m(f)$ denote the homogeneous minimum of the form f,

$$m(f) = \inf[|f(x, y)|;\ x, y \text{ integral},\ (x, y) \neq (0, 0)].$$

If $P = (x_0, y_0)$ is any real point, we define

$$M(f; P) = M(f; x_0, y_0) = \inf[|f(x+x_0, y+y_0)|;\ x, y \text{ integral}];$$

the *inhomogeneous minimum*, $M(f)$, of f is now defined by

$$M(f) = \sup_P M(f; P), \tag{1.2}$$

where the supremum is taken over all real points P. (In fact, in (1.2) it is sufficient to consider any complete set of incongruent points (mod 1) because $P \equiv P'$ (mod 1) clearly implies $M(f; P) = M(f; P')$.)

If we define

$$M_2(f) = \sup_P [M(f; P);\ P \in C],$$

where

$$C = [P;\ M(f; P) \neq M(f)],$$

then

$$M_2(f) \leqslant M(f);$$

if strict inequality holds, we call $M_2(f)$ the *second minimum* of f.

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In this paper I examine the inhomogeneous minima of the forms g_n defined as follows:

$$g_n(x, y) = u_{2n+3}x^2 + v_{2n+3}xy - u_{2n+3}y^2 \quad (n \geqslant 1), \tag{1.3}$$

where u_r, $r = 0, 1, \ldots$, denote the Fibonacci numbers ($u_0 = 0$, $u_1 = 1$, $u_{r+1} = u_r + u_{r-1}$ for $r \geqslant 1$), and v_r, $r = 0, 1, \ldots$, denote the Lucas numbers ($v_0 = 2$, $v_1 = 1$, $v_{r+1} = v_r + v_{r-1}$ for $r \geqslant 1$). It is easily shown that the forms $\{g_n\}$ form a subsequence of the Markov forms $F_0, F_1, \ldots$ (see Dickson [9], Ch. VII). In fact, we have

$$g_1(x, y) = 5x^2 + 11xy - 5y^2 = F_2(x, y),$$

$$g_2(x, y) = 13x^2 + 29xy - 13y^2 = F_3(x, y),$$

$$g_3(x, y) = 34x^2 + 76xy - 34y^2 = F_4(x, y).$$

The first two Markov forms, F_0, F_1, do not belong to the sequence $\{g_n\}$ for $n \geqslant 1$, and will not be included in the general discussion of the forms g_n, because the continued fraction expansions, both simple and "semi-regular", of their "roots" are rather special; however the coefficients of F_0, F_1 are of the same shape as those of the g_n, and so we may write

$$g_{-1}(x, y) = x^2 + xy - y^2 = F_0(x, y),$$

$$g_0(x, y) = 2x^2 + 4xy - 2y^2 = F_1(x, y).$$

It is well known (see Davenport [4], [5], Varnavides [11]) that

$$M(g_{-1}) = \tfrac{1}{4} = \tfrac{1}{4}m(g_{-1}); \tag{1.4}$$

$$M(g_0) = 1, \quad M_2(g_0) = \tfrac{1}{2} = \tfrac{1}{4}m(g_0); \tag{1.5}$$

and Davenport [5] has shown that

$$M(g_1) = \tfrac{5}{4} = \tfrac{1}{4}m(g_1). \tag{1.6}$$

The main result of this paper is

THEOREM 1. *For $n \geqslant 11$ the following statements hold:*

(i) *if* $n \not\equiv 0 \pmod 3$, *then*

$$M(g_n) = \tfrac{1}{4}u_{2n+3} = \tfrac{1}{4}m(g_n);$$

(ii) *if* $n \equiv 0 \pmod 3$, *then*

$$M(g_n) = \tfrac{1}{4}(8u_{2n+3} - 3v_{2n+3}) > \tfrac{1}{4}m(g_n),$$

$$M_2(g_n) = \tfrac{1}{4}u_{2n+3} = \tfrac{1}{4}m(g_n).$$

For the proof of this theorem, I use the divided cell method for evaluating the inhomogeneous minimum which was devised by Barnes and Swinnerton-Dyer [1], and then extended and applied by Barnes [2]; for convenience of reference, I give an outline of this method in section 2. In section 3, I give a theorem on "I-reduced" forms which makes it possible to apply the method of section 2 to the forms g_n, and which is therefore the essential tool in the proof of Theorem 1. In section 4, I give the theoretical part of the proof of Theorem 1, but only a sample of the numerical part of the proof, as this is all of the same kind.

It is clear from (1.4), (1.5), and (1.6) that Theorem 1 holds for $n = -1, 0, 1$. These results can also be proved by the divided cell method; the proofs when $n = -1, 0$ take only a few lines, but the proof when $n = 1$ involves more numerical work. In section 5, I show that Theorem 1 also holds for $n = 2, 3$. This strongly suggests that the theorem may hold for all $n \geqslant -1$, but the numerical details of the proof for $4 \leqslant n \leqslant 10$ would be very tedious.

Davenport [6] has shown that there exists a constant k such that, if f is any indefinite binary quadratic form with discriminant $D > 0$ and $\Delta = +\sqrt{D}$, then $M(f) > k\Delta$. We may therefore define an absolute constant K by

$$K = \sup[k;\ M(f) > k\Delta],$$

where the supremum is taken over all forms. Cassels [3] has shown that $K \geqslant 1/45{,}2$, and, as shown in my thesis [10], this may be improved [1] to about $K \geqslant 1/39$. As a consequence of Theorem 1, we obtain an upper bound for K:

$$K \leqslant 1/12.$$

For, if n is arbitrarily large and $n \not\equiv 0 \pmod 3$, we have $M(g_n) = \frac{1}{4}m(g_n)$, and, since $m(g_n)$ tends down to $\Delta/3$ as $n \to \infty$, this means that there are forms g_n with $M(g_n)$ arbitrarily close to $\Delta/12$.

The details of proofs and the numerical work which have been omitted from this paper are given in full in may thesis [10].

I wish to thank Dr. E. S. Barnes, who was my supervisor in Sydney, very much for all his help; in particular, I am grateful to him for suggesting the application of the divided cell method to the forms g_n. I am also grateful to Professor L. J. Mordell for his advice on the preparation of this paper. The computations for sections 4 and 5 were done on a Brunsviga provided by the University of Sydney.

(1) [Added in November, 1958] V. Ennola has recently proved that $K \geqslant 1/30{,}69\ldots$; see Annales Universitatis Turkuensis, Ser. AI, 28 (1958), p. 9-58.

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2. The divided cell method

We suppose that the form f is given by (1.1) and that it does not represent zero; then we may write

$$f(x, y) = (\alpha x+\beta y)(\gamma x+\delta y),$$

where α/β, δ/γ are irrational, and $|\alpha\delta-\beta\gamma| = \Delta = \sqrt{D}$. We write

$$\xi_0 = \alpha x_0+\beta y_0, \qquad \eta_0 = \gamma x_0+\delta y_0,$$
$$\xi = \alpha x+\beta y+\xi_0, \qquad \eta = \gamma x+\delta y+\eta_0.$$

If the set of points (ξ, η), where x and y take all integral values, has no point on either of the axes $\xi = 0$, $\eta = 0$, then this set is called in this paper an *inhomogeneous lattice* corresponding to f and (x_0, y_0), and is denoted by $L = L(\xi_0, \eta_0)$; clearly

$$M(f; x_0, y_0) = \inf[|\xi\eta|; (\xi, \eta)\in L].$$

A parallelogram whose vertices are lattice points of L is called a *cell* of the lattice if it contains no lattice points other than its vertices; that is, if and only if its area is Δ. A cell is said to be *divided* if one of its vertices is in each of the four quadrants. Delauney [7] proved that every inhomogeneous lattice has at least one divided cell; also, since α/β, δ/γ are irrational, none of the lattice lines of L can be parallel to either of the axes $\xi = 0$, $\eta = 0$. The method of Barnes and Swinnerton-Dyer depends on Delauney's algorithm, which is based on these two facts, for constructing a doubly infinite chain of divided cells $\{S_n\}$ $(-\infty < n < \infty)$.

Suppose A_0, B_0, C_0, D_0 are the vertices of the divided cell S_0, and are *either* in the first, fourth, third, and second quadrants respectively, *or* in the third, second, first, and fourth quadrants respectively, so that A_0D_0, B_0C_0 intersect the η-axis. Then S_1, with vertices A_1, B_1, C_1, D_1, is the cell defined by taking A_1B_1 as the unique lattice step in the line A_0D_0 which cuts the ξ-axis, and C_1D_1 as the unique lattice step in the line B_0C_0 which cuts the ξ-axis. Similarly, S_{-1} is the cell defined by taking the unique lattice steps in the lines A_0B_0, C_0D_0 which cut the η-axis. It is clear that S_1, S_{-1} are again divided cells, and therefore the same constructions may be applied to obtain divided cells S_2, S_{-2}, and so on indefinitely. In this way, starting from S_0, we get a doubly infinite chain of divided cells $\{S_n\}$ $(-\infty < n < \infty)$ of the lattice L, and if we then apply this process to any particular cell S_n of the chain, we shall obtain exactly the same chain. It is shown in my thesis that, if $\{S_n\}$ is such a chain of divided cells of an inhomogeneous lattice L, then $\{S_n\}$ includes all the divided cells of L; that is, the chain of divided cells of an inhomogeneous lattice is unique.

Barnes and Swinnerton-Dyer [1] showed that, if L is an inhomogeneous lattice corresponding to f and to the point (x', y'), and if $\{S_n\}$ $(-\infty < n < \infty)$ is the chain of divided cells of L, where, for each n, S_n has vertices A_n, B_n, C_n, D_n, and

$$\frac{\pi_n}{4} = \min[|\xi\eta|; (\xi, \eta) = A_n, B_n, C_n, D_n], \tag{2.1}$$

then

$$M(f; x', y') = \inf[|\xi\eta|; (\xi, \eta)\in L] = \inf_n \frac{\pi_n}{4}. \tag{2.2}$$

Thus, in order to evaluate $M(f; x', y')$, it is sufficient to consider the values of $|\xi\eta|$ corresponding to points (ξ, η) which are vertices of divided cells of L.

Barnes [2] showed further that there is a one-to-one correspondence between chains of divided cells $\{S_n\}$ and pairs of chains of integers, $\{a_n\}$, $\{\varepsilon_n\}$, with certain properties. In order to state this correspondence explicitly, we must introduce some further definitions.

Let $\{a_n\}$ $(n \geqslant 1)$ be a sequence of integers such that $|a_n| \geqslant 2$ for all n and a_n is not constantly equal to 2 or to -2 for large n, and let

$$p_0 = 1, \quad q_0 = 0; \quad p_1 = a_1, \quad q_1 = 1;$$
$$p_{n+1} = a_{n+1}p_n-p_{n-1} \quad (n \geqslant 1),$$
$$q_{n+1} = a_{n+1}q_n-q_{n-1} \quad (n \geqslant 1).$$

It is easily shown that the sequence $\{p_n/q_n\}$ converges to a limit α such that $|\alpha| > 1$, and we define $\alpha = [a_1, a_2, a_3, \ldots]$ by

$$\alpha = [a_1, a_2, a_3, \ldots] = \lim_{n\to\infty}\frac{p_n}{q_n}.$$

Clearly

$$\alpha = a_1-\frac{1}{a_2-}\,\frac{1}{a_3-}\cdots,$$

so that $\alpha = [a_1, a_2, a_3, \ldots]$ can be transformed into a classical semi-regular continued fraction

$$\alpha = a_1+\frac{\mu_2}{|a_2|+}\,\frac{\mu_3}{|a_3|+}\cdots \quad (\mu_i = \pm 1),$$

whose convergents have the same values (though the signs of p_n and q_n may be different). We shall not use classical semi-regular continued fractions of this type here, and so without confusion we may call $\alpha = [a_1, a_2, a_3, \ldots]$ a *semi-regular* continued fraction expansion of α, to distinguish it from the simple continued fraction expansion of α. Any irrational α with $|\alpha| > 1$ has infinitely many semi-regular continued fraction expansions. We note that, if

$$2 \leqslant k < \alpha < k+1,$$

then there exist expansions $\alpha = [a_1, a_2, a_3, \ldots]$ both with $a_1 = k$ and with $a_1 = k+1$, but if

$$1 < \alpha < 2,$$

then *every* expansion of α must have $a_1 = 2$.

An indefinite binary quadratic form F with discriminant $D > 0$ is called *inhomogeneously reduced*, or I-*reduced*, if it can be written in the shape

$$F(x, y) = \pm \frac{\Delta}{|\theta\varphi - 1|}(\theta x + y)(x + \varphi y), \tag{2.3}$$

where $\Delta = +\sqrt{D}$ and

$$|\theta| > 1, \qquad |\varphi| > 1.$$

It is well known (see Dickson [8], Ch. V) that, corresponding to any indefinite binary quadratic form $f(x, y)$ which does not represent zero, there is an equivalent *Gauss-reduced* form, that is, a form (2.3) which satisfies the more stringent conditions

$$\theta < -1, \qquad \varphi > 1.$$

Hence there certainly exists an I-reduced form equivalent to any indefinite binary quadratic form f which does not represent zero.

Let

$$f_0(x, y) = \pm \frac{\Delta}{|\theta_0\varphi_0 - 1|}(\theta_0 x + y)(x + \varphi_0 y)$$

be any I-reduced form equivalent (under an integral unimodular linear transformation) to the given form f. Let $\{a_n\}$ $(-\infty < n < \infty)$ be a chain of integers such that

$$\theta_0 = [a_0, a_{-1}, a_{-2}, \ldots], \qquad \varphi_0 = [a_1, a_2, a_3, \ldots];$$

then $\{a_n\}$ is called an *a-chain* of the form f. The chain of equivalent I-reduced forms $\{f_n\}$ $(-\infty < n < \infty)$ corresponding to $\{a_n\}$ is defined by

$$f_n(x, y) = \pm \frac{\Delta}{|\theta_n\varphi_n - 1|}(\theta_n x + y)(x + \varphi_n y),$$

where

$$\theta_n = [a_n, a_{n-1}, a_{n-2}, \ldots], \qquad \varphi_n = [a_{n+1}, a_{n+2}, a_{n+3}, \ldots];$$

$\{f_n\}$ is called the *form-chain* of f corresponding to $\{a_n\}$. By our definition of semi-regular continued fractions, any a-chain $\{a_n\}$ of f must satisfy the condition:

(A) $\left\{\begin{array}{l} |a_n| \geqslant 2 \textit{ for all } n, \textit{ and } a_n \textit{ is not constantly equal to } 2 \textit{ or to } -2 \textit{ for} \\ \textit{large } n \textit{ of either sign.} \end{array}\right.$

Let $\{\varepsilon_n\}$ $(-\infty < n < \infty)$ be a chain of integers which satisfies the following conditions:

(E)
- (i) $|\varepsilon_n| \leqslant |a_{n+1}| - 2$ *and* ε_n *has the same parity as* a_{n+1};
- (ii) *neither* $a_{n+1} + \varepsilon_n$ *nor* $a_{n+1} - \varepsilon_n$ *is constantly equal to* -2 *for large* n *of either sign;*
- (iii) *for any* n, *the relation*

$$a_{n+2r+1} + \varepsilon_{n+2r} = a_{n+2r+2} - \varepsilon_{n+2r+1} = 2$$

does not hold either for all $r \geqslant 0$ *or for all* $r \leqslant 0$.

Then $\{\varepsilon_n\}$ is called an *ε-chain* corresponding to $\{a_n\}$ (or, equivalently, to $\{f_n\}$), and $\{a_n\}, \{\varepsilon_n\}$ are called a *chain-pair* of the form f.

For a given chain-pair, $\{a_n\}, \{\varepsilon_n\}$, we define

$$\sigma_n = \varepsilon_{n-1} + \sum_{r=1}^{\infty} (-1)^r \frac{\varepsilon_{n-r-1}}{\theta_{n-1}\theta_{n-2}\ldots\theta_{n-r}}, \tag{2.4}$$

$$\tau_n = \varepsilon_n + \sum_{r=1}^{\infty} (-1)^r \frac{\varepsilon_{n+r}}{\varphi_{n+1}\varphi_{n+2}\ldots\varphi_{n+r}}. \tag{2.5}$$

It is easily shown that the series (2.4), (2.5) are absolutely convergent and that $|\sigma_n| < |\theta_n| - 1$, $|\tau_n| < |\varphi_n| - 1$.

Barnes [2] showed that, if $\{S_n\}$ is the chain of divided cells of any inhomogeneous lattice corresponding to f and a given point P, and if π_n is defined by (2.1), then there is a unique chain-pair, $\{a_n\}$, $\{\varepsilon_n\}$, of f such that, for each n,

$$\pi_n = \tag{2.6}$$

$$= \frac{\Delta}{|\theta_n\varphi_n - 1|} \min \begin{bmatrix} |(1+\theta_n+\sigma_n)(1+\varphi_n+\tau_n)|, & |(-1+\theta_n+\sigma_n)(1-\varphi_n+\tau_n)|, \\ |(-1-\theta_n+\sigma_n)(-1-\varphi_n+\tau_n)|, & |(1-\theta_n+\sigma_n)(-1+\varphi_n+\tau_n)| \end{bmatrix};$$

and, conversely, that, corresponding to any chain-pair $\{a_n\}$, $\{\varepsilon_n\}$ of f, there is a point P (unique mod 1) such that, if $\{S_n\}$ is the chain of divided cells of any inhomogeneous lattice corresponding to f and P, and if π_n is defined by (2.6), then (2.1) holds for each n.

We define (x_n, y_n) $(-\infty < n < \infty)$ by

$$\begin{aligned} \theta_n x_n + y_n &= \tfrac{1}{2}(-1-\theta_n+\sigma_n), \\ x_n + \varphi_n y_n &= \tfrac{1}{2}(-1-\varphi_n+\tau_n). \end{aligned} \tag{2.7}$$

Using (2.7), we can combine our results in the following theorem.

THEOREM 2. *If $\{a_n\}$, $\{\varepsilon_n\}$ is a chain-pair of f, and π_n is defined by* (2.6), *and if we put*

$$M(\{a_n\}, \{\varepsilon_n\}) = \inf_n \frac{\pi_n}{4}, \tag{2.8}$$

then there exists a point P such that (for each r)

$$M(P) = M(f; P) = M(f_r; x_r, y_r) = M(\{a_n\}, \{\varepsilon_n\});$$

and if $M(f)$ is the inhomogeneous minimum of f, then

$$M(f) = \sup M(\{a_n\}, \{\varepsilon_n\}),$$

where the supremum is taken over all possible chain-pairs of f.

Thus we can evaluate $M(f)$ by examining the chain-pairs of f, without explicitly considering the divided cells at all. The success of this approach to the problem of the inhomogeneous minimum depends on the rapid convergence of the series (2.4), (2.5). Estimates of the error made in replacing these series by partial sums will be needed for computations and are given below.

We here introduce the permanent notation $\|x\|$ for a quantity whose modulus does not exceed $|x|$.

LEMMA 2.1. *If $\{a_n\}$ is an a-chain of f, and $\{\varepsilon_n\}$ $(-\infty < n < \infty)$ is a chain of integers which satisfies* (E) (i), *then*

$$\sigma_n = \varepsilon_{n-1} - \frac{\varepsilon_{n-2}}{\theta_{n-1}} + \ldots + (-1)^r \frac{\varepsilon_{n-r-1}}{\theta_{n-1}\ldots\theta_{n-r}} + \left\| \frac{1}{\theta_{n-1}\ldots\theta_{n-r}} \left(1 - \frac{1}{|\theta_{n-r-1}|}\right) \right\|,$$

$$\tau_n = \varepsilon_n - \frac{\varepsilon_{n+1}}{\varphi_{n+1}} + \ldots + (-1)^r \frac{\varepsilon_{n+r}}{\varphi_{n+1}\ldots\varphi_{n+r}} + \left\| \frac{1}{\varphi_{n+1}\ldots\varphi_{n+r}} \left(1 - \frac{1}{|\varphi_{n+r+1}|}\right) \right\|.$$

The proof of this lemma is given in [2], § 3.

LEMMA 2.2. *Let $\{a_n\}$, $\{\varepsilon_n\}$ be given as in Lemma 2.1. Then*

(i) *if further $\theta_{n-r-1}, \theta_{n-r-2}$ differ in sign (i. e. if a_{n-r-1}, a_{n-r-2} differ in sign), then*

$$\sigma_n = \varepsilon_{n-1} - \frac{\varepsilon_{n-2}}{\theta_{n-1}} + \ldots + (-1)^r \frac{\varepsilon_{n-r-1}}{\theta_{n-1}\ldots\theta_{n-r}} + \left\| \frac{1}{\theta_{n-1}\ldots\theta_{n-r}} \left(1 - \frac{1}{|\theta_{n-r-1}|} - \frac{2}{|\theta_{n-r-1}\theta_{n-r-2}|}\right) \right\|;$$

(ii) *if further $\varphi_{n+r+1}, \varphi_{n+r+2}$ differ in sign (i. e. if a_{n+r+2}, a_{n+r+3} differ in sign), then*

$$\tau_n = \varepsilon_n - \frac{\varepsilon_{n+1}}{\varphi_{n+1}} + \ldots + (-1)^r \frac{\varepsilon_{n+r}}{\varphi_{n+1}\ldots\varphi_{n+r}} + \left\| \frac{1}{\varphi_{n+1}\ldots\varphi_{n+r}} \left(1 - \frac{1}{|\varphi_{n+r+1}|} - \frac{2}{|\varphi_{n+r+1}\varphi_{n+r+2}|}\right) \right\|.$$

This lemma is given in my thesis; it is easily proved there in the same way as Lemma 2.1, by using the fact that, if φ_r, φ_{r+1} differ in sign, then

$$|\varphi_r| = |a_{r+1}| + \frac{1}{|\varphi_{r+1}|}.$$

When we wish to evaluate $M(f)$, we try first to find and reject those a-chains (with the corresponding form-chains) for which $M(\{a_n\}, \{\varepsilon_n\})$ is small for any corresponding ε-chain, and then to examine the other chains more closely. At this preliminary stage, strict inequalities are not needed, and it is unnecessary to decide whether $\{\varepsilon_n\}$ satisfies the conditions (E) (ii) and (E) (iii); therefore it is convenient in Lemmas 2.1 and 2.2 to assume only that $\{\varepsilon_n\}$ satisfies (E) (i). It is possible to eliminate a large number of form-chains $\{f_n\}$ with all corresponding ε-chains by using Lemmas 2.3 and 2.4 below.

First we introduce the following definition: for any indefinite binary quadratic form f given by

$$f(x, y) = ax^2 + bxy + cy^2,$$

we define

$$\lambda = \lambda(f) = \min|a \pm b + c| = \min|f(1, \pm 1)|.$$

LEMMA 2.3. *If $\{f_n\}$ is a form-chain of f, and $\{a_n\}$ is the corresponding a-chain, then, for every corresponding ε-chain and for every r, we have*

$$M(f; P) = M(\{a_n\}, \{\varepsilon_n\}) = M(f_r; x_r, y_r) \leqslant \frac{\pi_r}{4} \leqslant \frac{\lambda(f_r)}{4}.$$

This lemma is proved in [2] (Lemmas 3.2, 3.3).

LEMMA 2.4. *In Lemma* 2.3, *we can have*

$$M(f;P) = \frac{\lambda(f_r)}{4} \tag{2.9}$$

if and only if conditions (i) *and* (ii) *are both satisfied*:

(i) $\lambda(f_r) = \inf_n \lambda(f_n)$;

(ii) $(x_r, y_r) \equiv (\frac{1}{2}, \frac{1}{2}) \pmod 1$.

The condition (ii) *implies that*:

(iii) *the chain* $\{a_n\}$ *is even,* (*i. e.* a_n *is even for all* n);

(iv) $\varepsilon_n = 0$ *for all* n;

(v) $P \equiv (\frac{1}{2}, 0)$ *or* $(0, \frac{1}{2})$ *or* $(\frac{1}{2}, \frac{1}{2}) \pmod 1$.

Proof. For each r,

$$\lambda(f_r) = \frac{\Delta}{|\theta_r\varphi_r - 1|}\min[|(\theta_r-1)(\varphi_r-1)|, |(\theta_r+1)(\varphi_r+1)|].$$

Hence it is clear from (2.8) that (2.9) holds if and only if (i) and (ii) hold) if (ii) holds, then, as $|\sigma_r| < |\theta_r|-1$, $|\tau_r| < |\varphi_r|-1$, it follows from (2.7) that $\sigma_r = \tau_r = 0$, which implies (iii) and (iv); (v) also follows from (iir since P must be the image of (x_r, y_r) under an integral unimodular linea; transformation.

Finally, to avoid unnecessary enumeration of cases, we need another lemma.

LEMMA 2.5. *If* $\{a_n\}, \{\varepsilon_n\}$ *is any chain-pair, the value of*

$$M(P) = M(\{a_n\}, \{\varepsilon_n\})$$

is unaltered by any of the following operations:

(i) *reversing the chains* $\{a_{n+1}\}, \{\varepsilon_n\}$ *about the same point*;

(ii) *changing the signs of all* ε_n;

(iii) *changing the signs of all* a_n *and of alternate* ε_n.

This lemma is proved in [2] (Lemma 3.1).

3. Equivalent I-reduced forms

In order to use the method of section 2 for evaluating the inhomogeneous minimum of an indefinite binary quadratic form g which does not represent zero, we must be able to determine all the a-chains of g. I now turn to the problem of determining all possible chains of I-reduced forms $\{f_n\}$ of g (and hence all a-chains of g).

Barnes [2] showed that there is only a finite number of I-reduced forms equivalent to a form with integral coefficients; hence, if g is proportional to a form with integral coefficients, we can obtain all the I-reduced forms equivalent to g by a finite number of trials. However this method becomes laborious if the discriminant of g is large, and breaks down altogether if the number of I-reduced forms equivalent to g is unbounded or infinite. Thus another method is needed.

The natural thing to do is to start from a particular I-reduced form equivalent to g, say f, where

$$f(x, y) = \pm\frac{\Delta}{|\theta\varphi - 1|}(\theta x + y)(x + \varphi y) \qquad (|\theta| > 1,\ |\varphi| > 1),$$

and, by expanding θ, φ in all possible ways as semi-regular continued fractions, to obtain all the chains $\{f_n\}$ to which f belongs. The questions then arise, whether all the form-chains of g are included among these chains, and whether every I-reduced form equivalent to g belongs to at least one of them.

Before discussing the answers to these questions, we need some definitions and notations.

If the form F is equivalent to the form f under an integral unimodular linear transformation

$$T = \begin{bmatrix} t & u \\ v & w \end{bmatrix},$$

where t, u, v, w are integral, and $tw - uv = \pm 1$, that is, if

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t & u \\ v & w \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$$

gives

$$f(x, y) = F(X, Y),$$

we shall write

$$F = fT = f\begin{bmatrix} t & u \\ v & w \end{bmatrix}.$$

With this notation, $(fT_1)T_2 = f(T_1T_2)$. If $tw - uv = +1$, the forms will be called *properly equivalent*; and if neither of the statements $t = w = 0$, $u = v = 0$ holds, the transformation T will be called *non-trivial*.

We denote by $f = (a, b, c)$ an I-reduced form which does not represent zero:

$$f(x, y) = ax^2 + bxy + cy^2,$$

where $D = b^2 - 4ac > 0$ is the discriminant of the form and $+\sqrt{D} = \Delta$. Clearly, $f(x, y)$ is I-reduced if and only if $f(x, -y)$, $f(y, x)$, $f(y, -x)$ are

I-reduced; also, by Lemma 2.5, any chain containing one of these forms can be converted into a chain containing $f(x, y)$ by reversing the chain $\{a_n\}$ ($\{a_{-n}\}$ is the *reverse* of the chain $\{a_n\}$), or by replacing $\{a_n\}$ by $\{-a_n\}$ (its *negative*), or by both. It would therefore be sufficient to consider only those I-reduced forms (a, b, c) with $b > 0$, $|a| \leqslant |c|$. Here we adopt the convention of considering only I-reduced forms with $b > 0$; with this convention, the I-reduced form $f = (a, b, c)$ can be factorized as

$$f(x, y) = ax^2 + bxy + cy^2 = \pm \frac{\Delta}{|r_1 r_2 - 1|}(r_1 x + y)(x + r_2 y), \tag{3.1}$$

where

$$r_1 = \frac{b + \Delta}{2c}, \quad r_2 = \frac{b + \Delta}{2a},$$

$$b > 0, \quad |r_1| > 1, \quad |r_2| > 1, \tag{3.2}$$

and r_1, r_2 are irrational. We shall call r_1 and r_2 the *first* and *second roots* of f respectively.

An a-chain $\{a_n\}$ $(-\infty < n < \infty)$ of f such that

$$r_1 = [a_0, a_{-1}, a_{-2}, \ldots], \quad r_2 = [a_1, a_2, a_3, \ldots]$$

will be called an *a-chain from* f, and the corresponding form-chain $\{f_n\}$ will be described as *from* f.

If $f = (a, b, c)$, we shall call

$$(c, b, a) = f \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

the *reverse* of f, and $(-a, b, -c)$ the *negative* of f.

We now return to the problem of determining all the I-reduced forms equivalent to a given I-reduced form f.

We note that it is not always possible to obtain all the I-reduced forms equivalent to f by taking all the forms in all the chains from f. For example, the Gauss-reduced form

$$g = (1, \sqrt{5}, -1) \tag{3.3}$$

has roots

$$r_1 = -r_2,$$

$$r_2 = \frac{3 + \sqrt{5}}{2} = 3 - \frac{1}{r_2} = [3, r_2] = [2, -2, -r_2],$$

while the equivalent form

$$G = g \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = (1, 2 + \sqrt{5}, \sqrt{5})$$

has roots

$$R_1 = \frac{1 + \sqrt{5}}{2},$$

$$R_2 = \frac{5 + \sqrt{5}}{2} > \frac{9 - \sqrt{5}}{2} = [3, r_1].$$

Clearly G cannot belong to any form-chain from g.

Also, even for integral Gauss-reduced forms, it is not always possible to obtain all the chains by taking all the chains from just one form. For example (as we shall show), the first and second roots of the form g_n defined by (1.3) satisfy

$$R_1 = -R_2,$$

$$R_2 = [3_n, 2, -2, R_2] = \left[3_n, \frac{5R_2 + 2}{2R_2 + 1}\right] = [3_{n+1}, 2, R_2 + 1]$$

(where 3_n means n successive 3's), while the equivalent form

$$g_n \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

has first and second roots

$$\frac{R_1}{R_1 + 1} = \frac{R_2}{R_2 - 1},$$

$$R_2 + 1;$$

the a-chain determined by the expansion

$$R_2 + 1 = [4, 3_n, 2, R_2 + 1]$$

cannot be an a-chain from g_n, as g_n has roots of opposite signs.

We shall prove the following theorem.

THEOREM 3. *Let $f = (a, b, c)$ $(b > 0)$ be a Gauss-reduced form given by (3.1) (so that $r_1 < -1$, $r_2 > 1$), and let $F = (A, B, C)$ $(B > 0)$ be an I-reduced form which is equivalent to f under the non-trivial linear transformation*

$$T = \begin{bmatrix} t & u \\ v & w \end{bmatrix}, \text{ where } t, u, v, w \text{ are integral, } w \geqslant 0, \text{ and } tw - uv = 1. \tag{3.4}$$

Then any form-chain from F must contain at least one of the three forms

$$f = (a, b, c),$$

$$(3.5)\qquad f\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = (a,\ 2a+b,\ a+b+c),$$

$$(3.6)\qquad f\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} = (a-b+c,\ b-2c,\ c).$$

It is easily shown that the forms (3.5), (3.6) are always I-reduced when f is Gauss-reduced. Since $f(-T) = fT$, there is no loss of generality in assuming in (3.4) that $w \geqslant 0$.

If F is equivalent to f under a non-trivial transformation

$$\begin{bmatrix}t & u\\ v & w\end{bmatrix}$$

for which $tw-uv = -1$, then F is properly equivalent to the reverse of f under the non-trivial transformation

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}t & u\\ v & w\end{bmatrix}.$$

Hence we can include the case of improper equivalence by replacing "f" by "the reverse of f" in Theorem 3.

Thus Theorem 3 means that we can obtain all the form-chains of a given Gauss-reduced form f by taking all the form-chains from f and from the two forms (3.5), (3.6). Since there is at least one Gauss-reduced form equivalent to any indefinite binary quadratic form g which does not represent zero, it follows that we can obtain all the form-chains of g by taking all the chains from at most three forms equivalent to g. This makes it possible to apply the methods of section 2 to sets of forms whose coefficients depend on a parameter in such a way that the number of equivalent I-reduced forms is unbounded, and in particular to the forms g_n, as well as to forms which have an infinite number of equivalent I-reduced forms (e. g. the form g given by (3.3)).

Further, it can be shown that, if g is proportional to a form with integral coefficients, then we can obtain all the I-reduced forms equivalent to g (though not usually all form-chains of g) by starting from just one Gauss-reduced form equivalent to g. This result, which is proved in my thesis, will not be needed in the discussion of the forms g_n and will therefore not be proved here.

Now we turn to the proof of Theorem 3.

First we note that if $f = (a, b, c)$ is *any* form given by (3.1) and if $F = (A, B, C)$ is any form equivalent to f under T, where T is given by (3.4), then

$$(3.7)\qquad \begin{aligned} A &= at^2+btv+cv^2,\\ B &= 2atu+b(tw+uv)+2cvw,\\ C &= au^2+buw+cw^2. \end{aligned}$$

If, further, the first and second roots of $F = (A, B, C)$ are denoted by

$$(3.8)\qquad R_1 = \frac{B+\Delta}{2C},\qquad R_2 = \frac{B+\Delta}{2A},$$

then it is easily deduced from (3.7) that

$$(3.9)\qquad R_1 = \frac{tr_1+v}{ur_1+w},$$

$$(3.10)\qquad R_2 = \frac{wr_2+u}{vr_2+t}.$$

We assume from now on that $f = (a, b, c)$ $(b > 0)$ is I-reduced (but not necessarily Gauss-reduced) so that (3.1), (3.2) hold, and that $F = (A, B, C)$ $(B > 0)$ is an I-reduced form equivalent to f under a non-trivial transformation T given by (3.4), so that relations (3.7) to (3.10) hold and

$$|R_1| > 1,\qquad |R_2| > 1.$$

We wish to know the types of matrix T for which this could be true; for this we require a number of lemmas.

The following result is easily proved by trivial case splitting.

LEMMA 3.1. *If t, u, v, w are integers such that $tw-uv = 1$, $w \geqslant 0$, and if it is not true that $t = w = 0$ or that $u = v = 0$, then exactly one of the following four sets of relations holds:*

$$\begin{aligned} &(3.11)\qquad &&\left\{\begin{array}{ll} w = |v| = 1, & |u| > |t|;\\ w > |v|, & |u| \geqslant |t|;\end{array}\right.\\ &(3.12) &&\\ &(3.13) &&\left\{\begin{array}{ll} w = |v| = 1, & |u| < |t|;\\ w < |v| & |u| \leqslant |t|.\end{array}\right.\\ &(3.14) && \end{aligned}$$

LEMMA 3.2. *If $F = fT$, then*

(i) $|u| > |t|$ *implies* $w \geqslant |u|$, $|v| \geqslant |t|$;
(ii) $|v| > w$ *implies* $|t| \geqslant |v|$, $|u| \geqslant w$.

Proof. This lemma depends on the fact that f and F are both I-reduced.

From (3.9) we have

$$r_1=\frac{-wR_1+v}{uR_1-t}=\frac{-w}{u}-\frac{1}{u(uR_1-t)}=\frac{-v}{t}-\frac{R_1}{t(uR_1-t)}. \tag{3.15}$$

If $t=0$, then $|u|=|v|=1$, and, since T is non-trivial, $w\geqslant 1$, so that (i) holds. If $|u|>|t|\geqslant 1$, then, since $|R_1|>1$,

$$|uR_1-t|\geqslant(|t|+1)|R_1|-|t|>|R_1|;$$

thus $|r_1|<1$ by (3.15) unless $w\geqslant|u|$, $|v|\geqslant|t|$. This proves (i) for all cases, and (ii) is proved similarly.

LEMMA 3.3. *If $F=fT$, and if* (3.11) *or* (3.12) *holds, then T must be one of the following matrices (where k is a positive integer):*

$$\begin{cases}\begin{bmatrix} t & u\\ v & w\end{bmatrix}\\ (|u|>|t|>0,\quad w>|v|,\\ \quad w>|u|,\quad |v|>|t|),\end{cases} \tag{3.16}$$

$$\begin{bmatrix} 0 & \mp 1\\ \pm 1 & 1\end{bmatrix}, \tag{3.17}$$

$$\begin{bmatrix} 1 & \pm 1\\ \pm(k-1) & k\end{bmatrix}\quad (k\geqslant 3), \tag{3.18}$$

$$\begin{bmatrix} 0 & \mp 1\\ \pm 1 & k\end{bmatrix}\quad (k\geqslant 2), \tag{3.19}$$

$$\begin{bmatrix} 1 & \pm(k-1)\\ \pm 1 & k\end{bmatrix}\quad (k\geqslant 2), \tag{3.20}$$

$$\begin{bmatrix} 1 & \pm 1\\ 0 & 1\end{bmatrix}. \tag{3.21}$$

This lemma follows from (i) of Lemma 3.2 by considering the special cases $|t|=|u|=1$ etc. which are not included in (3.16).

We now give some results on the chains from $F=fT$ when T il a matrix of one of the types (3.16) to (3.21); from these results we shals deduce Theorem 3.

The following lemma is an immediate consequence of the relation (3.9).

LEMMA 3.4. *If $F=fT$, and if*

$$R_2=\left[a_1,\frac{w_1r_2+u_1}{v_1r_2+t_1}\right],$$

where

$$\pm\begin{bmatrix} t_1 & u_1\\ v_1 & w_1\end{bmatrix}=\begin{bmatrix} a_1t-u & t\\ a_1v-w & v\end{bmatrix},$$

then

$$\frac{t_1r_1+v_1}{u_1r_1+w_1}=[a_1,R_1].$$

We shall say that the a-chain $\{a_n\}$ (or, equivalently, the corresponding orm-chain $\{f_n\}$) from F *leads forwards* to f, if, for some n,

$$R_2=[a_1,a_2,\ldots,a_n,r_2],$$
$$r_1=[a_n,a_{n-1},\ldots,a_1,R_1];$$

and we shall say that an a-chain from F *leads backwards* to f, if it leads forwards from f to F.

LEMMA 3.5. *If $F=fT$, where T is the matrix* (3.16), *then every a-chain from F leads forwards to a form fU, where U is one of the matrices* (3.18), (3.19), (3.20).

Proof. By (3.10),

$$R_2=\frac{wr_2+u}{vr_2+t}=\frac{u}{t}+\frac{r_2}{t(vr_2+t)}=\frac{w}{v}-\frac{1}{v(vr_2+t)},$$

so that

$$R_2=\frac{u}{t}+\frac{h}{t}=\frac{w}{v}+\frac{h'}{v}, \tag{3.22}$$

where, by (3.16), $|h|<1$, $|h'|<1$, and w/v is not integral.

By the definition of semi-regular continued fractions, for *any* expansion $R_2=[a_1,a_2,a_3,\ldots]$, a_1 is an integer such that $|a_1|\geqslant 2$ and $|R_2-a_1|<1$. For any such a_1, we have, by (3.10),

$$R_2=\left[a_1,\frac{w_1r_2+u_1}{v_1r_2+t_1}\right],$$

where

$$\pm\begin{bmatrix} t_1 & u_1\\ v_1 & w_1\end{bmatrix}=\begin{bmatrix} a_1t-u & t\\ a_1v-w & v\end{bmatrix}.$$

Also it follows from (3.22) that

$$\left|a_1-\frac{u}{t}\right|<1,\qquad \left|a_1-\frac{w}{v}\right|\leqslant 1,$$

i. e.

$$|a_1t-u|<|t|,\qquad |a_1v-w|\leqslant|v|;$$

and without loss of generality $w_1>0$. Hence, by Lemma 3.4, every a-chain from F leads forwards to an I-reduced form fT_1, where

$$T_1=\begin{bmatrix} t_1 & u_1\\ v_1 & w_1\end{bmatrix}\qquad (t_1w_1-u_1v_1=1),$$

$$|u_1|\geqslant|t_1|,\qquad w_1>|v_1|,\qquad w_1>|u_1|,\qquad u_1\neq 0,\qquad v_1\neq 0,$$

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and

$$|t_1| \leqslant |t|, \quad |u_1| < |u|, \quad |v_1| < |v|, \quad w_1 < w.$$

Clearly T_1 satisfies the same conditions as the matrix T in Lemma 3.3, and since $w_1 > |v_1|$, $v_1 \neq 0$, T_1 cannot be either of the matrices (3.17), (3.21). Thus either T_1 is one of the matrices (3.18), (3.19), (3.20) or T_1 satisfies the conditions (3.16), when we can apply the same argument again.

Thus every a-chain from F must either lead forwards to a form fU, where U is one of the matrices (3.18), (3.19), (3.20) or determine a sequence of matrices

$$T_r = \begin{vmatrix} t_r & u_r \\ v_r & w_r \end{vmatrix}$$

such that the chain leads forwards from fT_{r-1} to fT_r and

$$|u_r| \geqslant |t_r|, \quad w_r > |v_r|, \quad w_r > |u_r|, \quad u_r \neq 0, \quad v_r \neq 0,$$

and

$$|t_r| \leqslant |t_{r-1}|, \quad |u_r| < |u_{r-1}|, \quad |v_r| < |v_{r-1}|, \quad w_r < w_{r-1}.$$

Since in such a chain $\{T_r\}$ we must eventually reach an r for which $|u_r| = |t_r|$ or $|v_r| = |t_r|$ or $t_r = 0$, so that T_r is one of the matrices (3.18), (3.19), (3.20), this completes the proof of the lemma.

We note that if k is any positive integer, and $|x| > 1$, $|y| > 1$, then $y = [2_k, x]$ if and only if

$$y = \frac{(k+1)x - k}{kx - (k-1)}.$$

By using this result, the equation (3.10) corresponding to the different cases, and Lemma 3.4, we can easily prove the following lemma.

Lemma 3.6. *Let $F = fT$. Then*

(i) *if T is the matrix* (3.17), *every a-chain from F leads forwards to the form fU, where U is the matrix*

$$\begin{vmatrix} 1 & 0 \\ \pm 1 & 1 \end{vmatrix}; \tag{3.23}$$

(ii) *if T is the matrix* (3.18), *every a-chain from F leads forwards to the form fU, where U is the matrix*

$$\begin{vmatrix} 1 & \pm 1 \\ \pm 1 & 2 \end{vmatrix}$$

(*i. e. where U is given by* (3.20) *with $k = 2$*);

(iii) *if T is the matrix* (3.19), *every a-chain from F leads forwards to f or to fU, where U is the matrix* (3.23);

(iv) *if T is the matrix* (3.20), *every a-chain from F leads forwards to fU, where U is either* (3.21) *or* (3.17).

It follows that if $F = fT$ and T is any of the matrices (3.17) *to* (3.20), *then every a-chain from F leads forwards either to f or to fU, where U is one of the matrices* (3.21), (3.23).

Proof of Theorem 3. Let the I-reduced forms f, F satisfy the conditions of Theorem 3, so that $r_1 < -1$, $r_2 > 1$, $|R_1| > 1$, $|R_2| > 1$. We first note that it follows from (3.9), (3.10) that none of the following forms is I-reduced:

$$f\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad f\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad f\begin{bmatrix} 1 & -1 \\ -(k-1) & k \end{bmatrix}, \quad f\begin{bmatrix} 1 & -(k-1) \\ -1 & k \end{bmatrix}$$

(where $k > 1$ is integral).

We first suppose that either (3.11) or (3.12) holds. If we exclude possibilities which would give non-I-reduced forms, and use Lemmas 3.3, 3.5, and 3.6, then we see that *either* every a-chain from F must lead forwards to f or to one of the forms (3.5), (3.6) *or* F is itself one of the forms (3.5), (3.6).

By Lemma 3.1, if (3.11), (3.12) do not hold, then (3.13) or (3.14) holds. In this case we consider the reverse of F, which is given by

$$F\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = f\begin{bmatrix} t & u \\ v & w \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = f\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} w & v \\ u & t \end{bmatrix},$$

where

$$f\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is the reverse of f. Without loss of generality we may take $t \geqslant 0$ instead of $w \geqslant 0$, so that (by (3.13) and (3.14)) either $t = |u| = 1$, $|v| > |w|$, or $t > |u|$, $|v| \geqslant |w|$. Then, by an argument exactly similar to that given above, it follows that *either* every a-chain from the reverse of F must lead forwards to the reverse of f or to one of the forms

$$\begin{gathered} f\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = f\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ f\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = f\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{gathered} \tag{3.24}$$

or the reverse of F is itself one of the forms (3.24), which are the reverses of the forms (3.5), (3.6). This is equivalent to saying that, if (3.11), (3.12) do not hold, then either every a-chain from F leads backwards to f or to one of the forms (3.5), (3.6) or F is itself one of these forms.

Thus the theorem holds in all cases.

4. The forms g_n

We now suppose that the forms g_n are given by (1.3) for $n \geqslant 1$ and derive some results which will be needed for the proof of Theorem 1 and for the discussion of the forms g_2, g_3.

The form g_n is Gauss-reduced; therefore, by Theorem 3, every form-chain of g_n must contain at least one of the forms

$$g_n = (u_{2n+3}, v_{2n+3}, -u_{2n+3}),$$

$$g_n\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = (u_{2n+3}, 2u_{2n+3}+v_{2n+3}, v_{2n+3}), \tag{4.1}$$

$$g_n\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} = (-v_{2n+3}, 2u_{2n+3}+v_{2n+3}, -u_{2n+3}). \tag{4.2}$$

If f is either of the forms (4.1), (4.2), then $\lambda(f) = u_{2n+3}$; it now follows from Lemmas 2.3 and 2.4 that, if $\{a_r\}$ is an a-chain from one of these forms and is *not even* (i. e. not all a_r are even), then for every corresponding ε-chain

$$M(P) = M(g_n; P) = M(\{a_r\}, \{\varepsilon_r\}) < \tfrac{1}{4}u_{2n+3}.$$

We shall next consider the simple and semi-regular continued fraction expansions of the roots of g_n for $n \geqslant 1$, and obtain the values of $M(\frac{1}{2}, \frac{1}{2})$, $M(0, \frac{1}{2})$, $M(\frac{1}{2}, 0)$ (corresponding to the even a-chains of g_n). Finally we shall show that, for $n \geqslant 11$, if $\{a_r\}$ is an a-chain from g_n which is not even, then

$$M(P) < \tfrac{1}{4}u_{2n+3}.$$

Hence we shall derive Theorem 1.

We denote the first and second roots of g_n by $R_1 = -S$, $R_2 = S$. Since the transformation

$$T = \begin{vmatrix}u_{2n+1} & u_{2n+3}\\ u_{2n+3} & u_{2n+5}\end{vmatrix}$$

is a proper automorph of g_n, it follows from (3.10) that

$$S = \frac{u_{2n+5}S + u_{2n+3}}{u_{2n+3}S + u_{2n+1}}. \tag{4.3}$$

We shall use

$$a = (a_1, a_2, a_3, \ldots)$$

to denote the simple continued fraction expension of any number a, and

$$a = [a_1, a_2, a_3, \ldots]$$

to denote a semi-regular continued fraction expansion of a.

Since

$$\frac{u_{2n+5}}{u_{2n+3}} = (2, 1_{2n+2}) = (2, 1_{2n}, 2), \qquad \frac{u_{2n+3}}{u_{2n+1}} = (2, 1_{2n}),$$

it follows from (4.3) that the simple continued fraction expansion of S is

$$S = (2, 1_{2n}, 2, S), \tag{4.4}$$

so that g_n is in fact a Markov form.

Lemma 4.1. *For all $n \geqslant 1$, we have*

(i) $$m(g_n) = u_{2n+3};$$

(ii) $$M(g_n; \tfrac{1}{2}, \tfrac{1}{2}) \geqslant M(g_n; \tfrac{1}{2}, 0) = M(g_n; 0, \tfrac{1}{2}) = \tfrac{1}{4}u_{2n+3};$$

(iii) *if $n \equiv 0 \pmod 3$, then*

$$M(g_n; \tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{4}(8u_{2n+3} - 3v_{2n+3}).$$

Proof. We have

$$(2, S) + (0, 1_{2n}, 2, S) = \Delta/u_{2n+3},$$

and

$$(1, 1, 2, S) + (0, 1_{2n-2}, 2, S) = \Delta/(8u_{2n+3} - 3v_{2n+3}).$$

Hence, by using Lagrange's Theorem (see Dickson [8], Ch. VII, p. 111), we can show that, for integral $(x, y) \neq (0,0)$, we have

$$|g_n(x, y)| \geqslant u_{2n+3};$$

and that, if in addition $|g_n(x, y)| \neq u_{2n+3}$, then

$$|g_n(x, y)| \geqslant 8u_{2n+3} - 3v_{2n+3}. \tag{4.5}$$

Since $g_n(1,0) = u_{2n+3}$, (i) follows immediately, and hence (ii) follows also.

We now suppose that $n \equiv 0 \pmod 3$. It is then easily shown that

$$4 \mid v_{2n+3}, \quad 2 \mid u_{2n+3}, \quad 4 \nmid u_{2n+3}.$$

Since

$$g_n(x, y) = u_{2n+3}(x^2-y^2)+v_{2n+3}xy,$$

and $2|(x^2-y^2)$ when x, y are both odd, it follows that in this case $4|g_n(x, y)$, and so

$$g_n(x, y) \neq \pm u_{2n+3}.$$

Hence (4.5) holds when x, y are both odd, and (iii) now follows, since $g_n(3,1) = 8u_{2n+3}-3v_{2n+3}$.

It follows form the first paragraph of this section that if $\{a_r\}$, $\{\varepsilon_r\}$ is a chain-pair of g_n, then

$$M(P) = M(\{a_r\}, \{\varepsilon_r\}) \leqslant \tfrac{1}{4}u_{2n+3},$$

except possibly when $\{a_r\}$ is an a-chain from g_n which does not lead backwards or forwards to either of the forms (4.1), (4.2) and which we shall call a *permissible* a-chain. By (3.9) and (3.10), the first and second roots of the forms (4.1) and (4.2) are, respectively,

$$(4.6) \qquad -S/(-S+1), \qquad S+1,$$

and

$$(4.7) \qquad -(S+1), \qquad S/(-S+1);$$

since g_n has integral coefficients, any form which has any of the numbers (4.6), (4.7) as a root must be one of the forms (4.1), (4.2). Hence permissible a-chains are determined by semi-regular continued fraction expansions of $R_1 = -S$, $R_2 = S$ which are not, for any r, of the form

$$[a_1, a_2, \ldots, a_r, Z],$$

where Z is any of the numbers (4.6), (4.7); we shall call such expansions of $\pm S$ *permissible* expansions.

Since

$$\frac{u_{2n+5}}{u_{2n+3}} = [3_{n+1}, 2] = [3_n, 2, -2], \qquad \frac{u_{2n+3}}{u_{2n+1}} = [3_n, 2],$$

we have, by (4.3),

$$(4.8) \qquad S = [3_n, 2, -2, S].$$

We note the following results:

$$(4.9) \qquad [-2, S] = [-3, S/(-S+1)], \qquad [2, -2, S] = [3, 2, S+1],$$

$$(4.10) \qquad [3, 2, -2, S] = [2, -2, -3, S] = [2, -2, -4, S/(-S+1)],$$

$$(4.11) \qquad [3, 3, 2, -2, S] = [2, -2, -3, -3, S] = [2, -2, -2, 2, 2, S+1].$$

Also, for any z such that $|z| > 1$, we have

$$[3, 3, z] = \left[2, -2, \frac{2z-1}{-z+1}\right];$$

and from this we deduce that, for $k \geqslant 0$,

$$(4.12) \qquad [3_{k+3}, 2, -2, S] = [2, -2, -3_{k+3}, S],$$

$$(4.13) \qquad [-3_{k+3}, S] = [-2, 2, 3_k, 2, -2, S],$$

$$(4.14) \qquad [3_{k+3}, 2, -2, S] = [2, -2, -2, 2, 3_k, 2, -2, S].$$

We can now prove

LEMMA 4.2. *If $n \not\equiv 0 \pmod 3$ and $n \geqslant 1$, then*

$$M(g_n; \tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{4}u_{2n+3}.$$

Proof. By (4.14) and (4.10), if $n \equiv 1 \pmod 3$, the even a-chain from g_n is determined by the expansion

$$S = [[2, -2, -2, 2]_k, 2, -2, -4, S/(-S+1)],$$

and so is not permissible. Similarly, by (4.14) and (4.11), if $n \equiv 2 \pmod 3$, the even chain from g_n is not permissible. The lemma now follows from Lemma 4.1. (We note that, by (4.14), if $n \equiv 0 \pmod 3$, the even a-chain from g_n is permissible, which explains why $M(g_n; \frac{1}{2}, \frac{1}{2})$ is large in this case.)

As there are infinitely many semi-regular continued fraction expansions of any given number, we need a notation to indicate which particular expansion we are using; therefore we write

$$\alpha \equiv [a_0, a_1, \ldots, a_r, z]$$

when we mean that $\alpha = [a_0, a_1, \ldots, a_r, z]$ *and* that we are choosing expansions of α whose first $r+1$ partial quotients are $a_0, a_1, \ldots, a_r$. In order to examine the permissible a-chains from g_n we need the following lemma.

LEMMA 4.3. *Let $\{a_r\}$ be a permissible a-chain from g_n which is not even. Then $\{a_r\}$ (or its negative or its reverse or its negative reversed) contains a subchain determined by pairs of expansions of the roots $R_1 = -S$, $R_2 = S$ of g_n which begin in one of the following ways:*

(i) $-S$ *arbitrary,* $\quad S \equiv [3_k, 2, -2, y]$, *where* $\quad y = [-3_{n-k}, S]$, $3 \leqslant k \leqslant n$;

(ii) $-S$ *arbitrary,* $\quad S \equiv [3, 3, 2, -2, y]$, *where* $\quad y = [-3_{n-2}, S]$;

(iii) $-S \equiv [-3, -2, 2, -y]$, $S \equiv [3, 2, -2, y]$, *where* $y = [-3_{n-1}, S]$;

(iv) $-S \equiv [-2, 2, x]$, $S \equiv [3, 2, -2, y]$, *where* $x = [3_n, -S]$, $y = [-3_{n-1}, S]$;

(v) $-S \equiv [-2, 2, x]$, $S \equiv \big|[2, -2, -2, 2]_k, 2, -2, -3, y\big|$, *where* $x = [3_n, -S]$, $y = [-3_l, S]$, $3k+l+1 = n$, $k \geqslant 0$;

(vi) $-S \equiv [-2, 2, x]$, $S \equiv \big[2, -2, [-2, 2, 2, -2]_k, -2, 2, 3, y\big|$, *where* $x = [3_n, -S]$, $y = [3_l, 2, -2, S]$, $3(k+1)+l+1 = n$, $k \geqslant 0$.

Proof. By (4.9), $[-2, S]$ and $[2, -2, S]$ have no permissible alternative expansions; it now follows from (4.8) and equations (4.12) to (4.14) that any permissible expansion of S must begin in one of the following ways:

$$S \equiv [3_k, 2, -2, y] \quad \text{where} \quad y = [-3_{n-k}, S],\ 0 < k \leqslant n;$$

$$S \equiv \big|[2, -2, -2, 2]_k, 3, y\big|, \quad \text{where} \quad y = [3_l, 2, -2, S],\ 3k+l+1 = n,\ k > 0;$$

$$S \equiv \big|[2, -2, -2, 2]_k, 2, -2, -3, y\big|, \quad \text{where} \quad y = [-3_l, S],\ 3k+l+1 = n,\ k \geqslant 0;$$

$$S \equiv [2, -2, y], \quad \text{where} \quad y = [-3_n, S].$$

(We note that the last expansion includes the two previous ones as special cases, and that of course many expansions which are not permissible may begin in one of these ways also.) Lemma 4.3 now follows from the symmetry of g_n and the fact that the a-chains are assumed not to be even.

It is clear from the previous discussion that Theorem 1 will follow from Lemmas 4.1 and 4.2 and the following lemma.

LEMMA 4.4 *If $n \geqslant 11$ and $\{a_r\}$ is a permissible a-chain from g_n which is not even, then, for every corresponding ε-chain,*

$$M(P) = M(\{a_r\}, \{\varepsilon_r\}) < \tfrac{1}{4} u_{2n+3}.$$

In fact we show that in each case, for some r,

$$\pi_r < \Delta/3, \tag{4.15}$$

so that

$$M(P) \leqslant \tfrac{1}{4}\pi_r < \tfrac{1}{4}\Delta/3 = \tfrac{1}{4}\sqrt{(u_{2n+3}^2 - \tfrac{4}{9})} < \tfrac{1}{4} u_{2n+3}.$$

In Lemmas 4.5 to 4.13 we give a number of chain-pairs for which (4.15) holds for some r when $n \geqslant 11$. We then prove Lemma 4.4 by using Lemma 4.3 to show that the chain-pairs of Lemmas 4.5 to 4.13 include

all chain-pairs $\{a_r\}, \{\varepsilon_r\}$ of g_n for which $\{a_r\}$ is a permissible a-chain from g_n which is not even.

In Lemmas 4.5 to 4.13 we assume that $n \geqslant 11$ and use the following notation: by

$$\ldots, p, q, \underline{r}, s, t, u, \ldots$$
$$\ldots, a, b, \underline{c}, d, e, f, \ldots$$

we denote a chain-pair $\{a_r\}, \{\varepsilon_r\}$ such that

$$a_1 = r,\ a_2 = s,\ \ldots,\ a_0 = q,\ a_{-1} = p,\ \ldots,$$
$$\varepsilon_0 = c,\ \varepsilon_1 = d,\ \ldots,\ \varepsilon_{-1} = b,\ \varepsilon_{-2} = a,\ \ldots$$

If the values of one θ and one φ are given, then θ_0, φ_0 are determined and $\{a_r\}$ is an a-chain from f_0 which contains the subchain determined by the pair of expansions

$$\theta_0 \equiv [q, p, \theta_{-2}], \qquad \varphi_0 \equiv [r, s, t, u, \varphi_4],$$

and hence also $\theta_{-1}, \theta_{-2}, \varphi_1, \varphi_2, \ldots$ are determined. Sometimes, for the sake of clarity, the values of two θ's or two φ's are given, though only one of each is needed to determine the subchain.

Lemmas 4.5 to 4.13 can all be proved by the methods of section 2, and full numerical details of the proofs are given in my thesis [10]. The proofs are all fairly similar, and therefore I give here only the proof of Lemma 4.5. (This one is chosen because it is a good illustration of the use of the methods of section 2, and, in particular, of Lemma 2.2; as it involves case-splitting, it is a little more complicated than the other proofs.)

It is easily derived from (4.8) that, for $n \geqslant 4$ (and therefore certainly for $n \geqslant 11$),

$$2{,}61803 < S < 2{,}61804;$$

and that, if further $r \geqslant 5$, then

$$\begin{gathered} 2{,}6180 < [3_r, 2, -2, S] < 2{,}6181, \\ 2{,}6180 < [3_{r+1}, -S] < 2{,}6181. \end{gathered} \tag{4.16}$$

The reason for the assumption $n \geqslant 11$ is that, by using it with the inequalities (4.16) for $r \geqslant 5$, we can avoid a large amount of tedious case-splitting in the computations in the proofs of some of the lemmas (e. g. Lemma 4.8).

LEMMA 4.5. *For the chain-pairs*

$$\begin{matrix} \ldots, 3, 3, \underline{3}, 2, -2, \ldots \\ \ldots, 1, 1, \underline{1}, 0, \quad 0, \ldots \end{matrix} \quad \textit{and} \quad \begin{matrix} \ldots, \quad 3, \quad 3, \underline{3}, 2, -2, \ldots \\ \ldots, \pm 1, -1, \underline{1}, 0, \quad 0, \ldots \end{matrix},$$

where

$$\theta_{-2} = [3_m, -S] \quad (m \geqslant 0), \qquad \varphi_3 = [-3_k, S] \quad (k \geqslant 0),$$

we have

$$3\pi_0/\Delta < 0{,}997,$$

$$\pi_0 < \Delta/3.$$

Proof. By Lemma 2.1,

$$\tau_0 = 1 + \left\| \frac{1}{\varphi_1\varphi_2}\left(1 - \frac{1}{|\varphi_3|}\right)\right\|.$$

For $k = 0$,

$$\tau_0 = 1 + \|0{,}414 \times 0{,}420 \times 0{,}619\|.$$

For $k \geqslant 2$,

$$\tau_0 = 1 + \|0{,}3827 \times 0{,}6181 \times 0{,}6303\|.$$

For $k = 1$, we must have $\varphi_3 = [-3, S]$, as no other expansion of φ_3 is permissible, so that φ_3, φ_4 are opposite in sign, and, by Lemma 2.2,

$$\tau_0 = 1 + \left\| \frac{1}{\varphi_1\varphi_2}\left(1 - \frac{1}{|\varphi_3|} - \frac{2}{|\varphi_3\varphi_4|}\right)\right\| \tag{4.17}$$

$$= 1 + \|0{,}387 \times 0{,}587 \times (1 - 0{,}295 - 0{,}295 \times 0{,}762)\|.$$

Hence for all k

$$\tau_0 = 1 + \|0{,}1491\| > 0{,}8509.$$

Also

$$2{,}586 < \varphi_0 < 2{,}6181.$$

Hence

$$|1 - \varphi_0 + \tau_0| < 0{,}7672. \tag{4.18}$$

If $\varepsilon_{-1} = -1$, $\sigma_0 < 0$, and if $\varepsilon_{-1} = \varepsilon_{-2} = 1$, then

$$\sigma_0 = 1 - \frac{1}{\theta_{-1}} + \left\| \frac{1}{\theta_{-1}}\left(1 - \frac{1}{|\theta_{-2}|}\right)\right\|.$$

Hence, in either case,

$$\sigma_0 < 1 - \frac{1}{\theta_{-1}} + \left| \frac{1}{\theta_{-1}}\left(1 - \frac{1}{|\theta_{-2}|}\right)\right|.$$

For $m = 0$,

$$\sigma_0 < 1 - 0{,}295 + 0{,}296 \times 0{,}6181,$$

$$2{,}704 < \theta_0 < 2{,}705.$$

Therefore

$$|-1 + \theta_0 + \sigma_0| < 2{,}593, \tag{4.19}$$

$$3/|\theta_0\varphi_0 - 1| < 3/(2{,}704 \times 2{,}586 - 1) < 0{,}501.$$

For $m \geqslant 2$,

$$\sigma_0 < 1 - 0{,}380 + 0{,}382 \times 0{,}631,$$

$$2{,}618 < \theta_0 < 2{,}620.$$

Therefore

$$|-1 + \theta_0 + \sigma_0| < 2{,}482, \tag{4.20}$$

$$3/|\theta_0\varphi_0 - 1| < 3/(2{,}618 \times 2{,}586 - 1) < 0{,}520.$$

For $m = 1$, by the same type of argument as that used to get (4.17), we have

$$\sigma_0 < 1 - \frac{1}{\theta_{-1}} + \left| \frac{1}{\theta_{-1}}\left(1 - \frac{1}{|\theta_{-2}|} - \frac{2}{|\theta_{-2}\theta_{-3}|}\right)\right| < 1 - 0{,}369 + 0{,}370 \times 0{,}481,$$

$$2{,}630 < \theta_0 < 2{,}631.$$

Therefore

$$|-1 + \theta_0 + \sigma_0| < 2{,}440, \tag{4.21}$$

$$3/|\theta_0\varphi_0 - 1| < 0{,}520.$$

From (4.18), (4.19), (4.20), and (4.21), it now follows that, for all k, m,

$$3\pi_0/\Delta < 0{,}7672 \times 1{,}2991 < 0{,}997 < 1.$$

LEMMA 4.6. *For the chain-pair*

$$\begin{matrix} \ldots, & 3, 3, \underline{3}, 2, -2, \ldots \\ \ldots, & -1, 1, \underline{1}, 0, \quad 0, \ldots \end{matrix},$$

where

$$\theta_{-2} = [3_m, -S] \quad (m \geqslant 0), \qquad \varphi_3 = [-3_k, S] \quad (k \geqslant 0),$$

we have

$$3\pi_{-1}/\Delta < 0{,}815,$$

$$\pi_{-1} < \Delta/3.$$

LEMMA 4.7. *For the chain-pairs*

(i)
$$\begin{matrix} \ldots, & 3, \underline{3}, 2, -2, -2, 2, \ldots \\ \ldots, & \pm 1, \underline{1}, 0, \quad 0, \quad 0, 0, \ldots \end{matrix},$$

where

$$\theta_{-1} = -S, \qquad \varphi_{-1} = S, \qquad \varphi_5 = [3_k, 2, -2, S] \quad (k \geqslant 6),$$

and

(ii)
$$\begin{matrix} \ldots, & 3, \underline{3}, 2, -2, -3, -3, \ldots \\ \ldots, & \pm 1, \underline{1}, 0, \quad 0, -1, \pm 1, \ldots \end{matrix},$$

where
$$\theta_{-1} = -S,\quad \varphi_{-1} = S,\quad \varphi_5 = [-3_m, S]\quad (m \geqslant 7),$$
we have
$$3\pi_0/\Delta < 0{,}998,$$
$$\pi_0 < \Delta/3.$$

LEMMA 4.8. *For the chain-pair*
$$\begin{matrix} \ldots, & 3, 2, & -2, & -\underline{3}, & -2, 2, \ldots \\ \ldots, & \pm 1, 0, & 0, & \underline{1}, & 0, 0, \ldots \end{matrix},$$
where
$$\theta_{-3} = -S \text{ or } [3, -S],\quad \varphi_{-3} = S \text{ or } [3_{n-1}, 2, -2, S],$$
$$\varphi_3 = [3_k, 2, -2, S]\quad (k \geqslant 5)\quad (\text{using (4.13) and } n \geqslant 11),$$
we have
$$3\pi_0/\Delta < 0{,}817,$$
$$\pi_0 < \Delta/3.$$

LEMMA 4.9. *For the chain-pair*
$$\begin{matrix} \ldots, 3, 2, & -2, & -3, & -\underline{3}, \ldots \\ \ldots, 1, 0, & 0, & 1, & \underline{\pm 1}, \ldots \end{matrix},$$
where
$$\theta_{-4} = -S \text{ or } [3, -S],\quad \varphi_{-4} = S \text{ or } [3_{n-1}, 2, -2, S],$$
$$\varphi_1 = [-3_k, S]\quad (k \geqslant 8),$$
we have
$$3\pi_0/\Delta < 0{,}992,$$
$$\pi_0 < \Delta/3.$$

LEMMA 4.10. *For the chain-pair*
$$\begin{matrix} \ldots, 2, & -2, & -3, & \underline{3}, 2, & -2, \ldots \\ \ldots, 0, & 0, & \pm 1, & \underline{1}, 0, & 0, \ldots \end{matrix},$$
where
$$-\theta_0 = \varphi_0 = S,\quad -\theta_{-3} = \varphi_3 = [-3_k, S]\quad (k \geqslant 10),$$
we have
$$3\pi_0/\Delta < 0{,}811,$$
$$\pi_0 < \Delta/3.$$

LEMMA 4.11. *For the chain-pairs*

(i)
$$\begin{matrix} \ldots, 2, & -2, \underline{3}, 2, & -2, & -2, 2, \ldots \\ \ldots, 0, & 0, \underline{1}, 0, & 0, & 0, 0, \ldots \end{matrix},$$

where
$$-\theta_0 = \varphi_0 = S,$$
$$\theta_{-2} = [3_m, -S]\quad (m \geqslant 11),\quad \varphi_5 = [3_k, 2, -2, S]\quad (k \geqslant 7);$$
and

(ii)
$$\begin{matrix} \ldots, 2, & -2, \underline{3}, 2, & -2, & -3, & -3, \ldots \\ \ldots, 0, & 0, \underline{1}, 0, & 0, & -1, & \pm 1, \ldots \end{matrix},$$
where
$$-\theta_0 = \varphi_0 = S,$$
$$\theta_{-2} = [3_m, -S]\quad (m \geqslant 11),\quad \varphi_5 = [-3_k, S]\quad (k \geqslant 8),$$
we have
$$3\pi_0/\Delta < 0{,}9983,$$
$$\pi_0 < \Delta/3.$$

LEMMA 4.12. *For the chain-pairs*

(i)
$$\begin{matrix} \ldots, 2, & -2, [2, & -2, & -2, 2]_k, 2, & -2, & -3, & \underline{-2}, 2, \ldots \\ \ldots, 0, & 0, [0, & 0, & 0, 0]_k, 0, & 0, & 1, & \underline{0}, 0, \ldots \end{matrix}\quad (k \geqslant 0),$$
where
$$-\theta_{-4k-3} = \varphi_{-4k-3} = S,$$
$$\theta_{-4k-5} = [3_n, -S],\quad \varphi_2 = [3_m, 2, -2, S]\quad (m \geqslant 0);$$
and

(ii)
$$\begin{matrix} \ldots, 2, & -2, [2, & -2, & -2, 2]_k, 2, & -2, & -3, & \underline{-3}, & -2, 2, \ldots \\ \ldots, 0, & 0, [0, & 0, & 0, 0]_k, 0, & 0, & 1, & \underline{\pm 1}, & 0, 0, \ldots \end{matrix}\quad (k \geqslant 0),$$
where
$$-\theta_{-4k-3} = \varphi_{-4k-3} = S,$$
$$\theta_{-4k-5} = [3_n, -S],\quad \varphi_3 = [3_m, 2, -2, S]\quad (m \geqslant 0),$$
we have
$$3\pi_0/\Delta < 0{,}945,$$
$$\pi_0 < \Delta/3.$$

LEMMA 4.13. *For the chain-pairs*

(i)
$$\begin{matrix} \ldots, 2, & -2, 2, & -2, [-2, 2, 2, & -2]_k, & -2, 2, 3, \underline{2}, & -2, \ldots \\ \ldots, 0, & 0, 0, & 0, [\ \ 0, 0, 0, & 0]_k, & 0, 0, 1, \underline{0}, & 0, \ldots \end{matrix}\quad (k \geqslant 0),$$
where
$$-\theta_{-4k-5} = \varphi_{-4k-5} = S,$$
$$\theta_{-4k-7} = [3_n, -S],\quad \varphi_2 = [-3_m, S]\quad (m \geqslant 3);$$

and

$$\text{(ii)}\quad \begin{matrix} \ldots, 2, & -2, 2, & -2, [-2, 2, 2, & -2]_k, & -2, 2, 3, & \underline{3}, 2, & -2, \ldots \\ \ldots, 0, & 0, 0, & 0, [\ \ 0, 0, 0, & 0]_k, & 0, 0, 1, & \underline{\pm 1}, 0, & 0, \ldots \end{matrix} \quad (k \geqslant 0),$$

where

$$-\theta_{-4k-5} = \varphi_{-4k-5} = S,$$

$$\theta_{-4k-7} = [3_n, -S], \quad \varphi_3 = [-3_m, S] \quad (m \geqslant 3)$$

we have

$$3\pi_0/\Delta < 0{,}993,$$

$$\pi_0 < \Delta/3.$$

Proof of Lemma 4.4. By Lemma 2.5 and the discussion following the statement of Lemma 4.4, it is sufficient to show that if $\{a_r\}$, $\{\varepsilon_r\}$ is a chain-pair of g_n and if $\{a_r\}$ is permissible and contains one of the subchains (i) to (vi) of Lemma 4.3, then $\{a_r\}$ or its negative or its reverse or its negative reversed is one of the a-chains considered in Lemmas 4.5 to 4.13 and $\{\varepsilon_r\}$ or its negative is one of the corresponding ε-chains considered in these lemmas.

Lemmas 4.5 and 4.6 cover all chain-pairs for which $\{a_r\}$ is given by (i).

If $y = [-3_k, S]$, then, by (4.13), any semi-regular continued fraction expansion of y must begin in one of the following ways:

$$(4.22)\qquad y \equiv [-2, 2, z), \qquad z = [3_{k-3}, 2, -2, S] \quad (k \geqslant 3);$$

$$(4.23)\qquad y \equiv [-3, -2, 2, z], \qquad z = [3_{k-4}, 2, -2, S] \quad (k \geqslant 4);$$

$$(4.24)\qquad y \equiv [-3, -3, z], \qquad z = [-3_{k-2}, S] \quad (k \geqslant 2).$$

Hence Lemmas 4.7, 4.8, and 4.9 cover all chain-pairs for which $\{a_r\}$ is given by (ii).

Lemma 4.10 covers all chain-pairs for which $\{a_r\}$ is given by (iii).

For the subchain (iv), y must be given by one of (4.22), (4.23), and (4.24). Hence Lemmas 4.8, 4.9, and 4.11 cover all chain-pairs for which $\{a_r\}$ is given by (iv).

For the subchain (v), y must be given by one of (4.22), (4.23), and (4.24). Lemma 4.12 covers all chain-pairs for which $\{a_r\}$ is given by (v) and y satisfies (4.22) or (4.23). Lemmas 4.5 and 4.6 cover all chain-pairs for which the reverse of the negative of $\{a_r\}$ is given by (v) and y satisfies (4.24).

For the subchain (vi), the semi-regular continued fraction expansion of y must begin in one of the following ways (see (4.12)):

$$(4.25)\qquad y \equiv [2, -2, z], \qquad z = [-3_m, S] \quad (m \geqslant 0),$$

$$(4.26)\qquad y \equiv [3, 2, -2, z], \qquad z = [-3_m, S] \quad (m \geqslant 0),$$

$$(4.27)\qquad y \equiv [3_p, 2, -2, z], \qquad z = [-3_m, S] \quad (p \geqslant 2,\ m \geqslant 0).$$

Lemma 4.13 covers all chain-pairs for which $\{a_r\}$ is given by (vi) and y satisfies (4.25) or (4.26) with $m \geqslant 3$. It follows from (4.9), (4.10), and (4.11) that if $\{a_r\}$ is a permissible a-chain given by (vi) and y satisfies (4.25) or (4.26) with $m = 0, 1, 2$, then $\{a_r\}$ must be the reverse of the negative of an a-chain containing one of the subchains (i) to (v). Lemmas 4.5 and 4.6 cover all chain-pairs for which $\{a_r\}$ is given by (vi) and (4.27) holds.

This completes the proof of Lemma 4.4 and therefore of Theorem 1.

5. The forms g_2 and g_3

We now show that Theorem 1 holds for $n = 2, 3$, that is, that the following theorems hold.

THEOREM 4. *For the form* $g_2 = (13, 29, -13) = F_3$, *we have*

$$M(g_2) = \tfrac{13}{4} = \tfrac{1}{4} m(g_2).$$

THEOREM 5. *For the form* $g_3 = (34, 76, -34) = F_4$, *we have*

$$M(g_3) = 11,$$

$$M_2(g_3) = \tfrac{17}{2} = \tfrac{1}{4} m(g_3).$$

All the results of section 4 up to and including Lemma 4.2 hold for $n \geqslant 1$. Hence, in order to prove Theorems 4 and 5, it is sufficient to show that, if $n = 2, 3$ and $\{a_r\}$ is a permissible a-chain from g_n which is not even, then, for every corresponding ε-chain,

$$M(P) = M(\{a_r\}, \{\varepsilon_r\}) < \tfrac{1}{4} u_{2n+3}.$$

We continue to use the notation of section 4.

For the form g_2 we have, by (4.8),

$$S = [3, 3, 2, -2, S].$$

Hence, by (4.10) and (4.11), the only permissible expansions of S are

$$S \equiv [3, 3, 2, -2, S],$$
$$S \equiv [3, 2, -2, -3, S],$$
$$S \equiv [2, -2, -3, -3, S].$$

It now follows from the symmetry of the form g_2, whose roots are $R_1 = -S$, $R_2 = S$, that any permissible a-chain from g_2 must be an arrangement of some or all of the three blocks of numbers

$$A = 3, 3, 2, -2,$$
$$B = 3, 2, -2, -3,$$
$$C = 2, -2, -3, -3.$$

In Lemmas 5.1 to 5.5 we show that, for every chain-pair such that $\{a_r\}$ contains AA, BB, AC, BAB, or $BCAB$, we have

$$M(P) < \tfrac{13}{4}.$$

Since C is the negative of A reversed and B is its own negative reversed, this result holds also for every chain-pair such that $\{a_r\}$ contains CC or BCB. Any arrangement of some or all of A, B, and C which does not contain BB must contain A or C; any arrangement which contains A but none of AA, BB, AC, CC must contain BAB or $BCAB$, and similarly any arrangement which contains C but none of AA, BB, AC, CC must contain BCB or $BCAB$. Thus the a-chains of Lemmas 5.1 to 5.5 include all permissible a-chains from g_2 or their negatives reversed.

The numerical details of the proofs of Lemmas 5.1 to 5.5 are given in my thesis [10] and are similar to those of Lemmas 4.5 to 4.13; here I merely give the statements of the lemmas.

LEMMA 5.1. *For the chain-pair*

$$\begin{matrix} \ldots, 3, & 3, 2, -2, & 3, \underline{3}, 2, -2, \ldots \\ \ldots, \pm 1, & \pm 1, 0, & 0, \pm 1, 1, 0, & 0, \ldots \end{matrix},$$

where $-\theta_{-5} = \varphi_3 = S$, *we have*

$$\pi_0 < 12{,}37 < 13.$$

LEMMA 5.2. *For the chain-pair*

$$\begin{matrix} \ldots, & 3, 2, -2, -3, \underline{3}, 2, -2, -3, \ldots \\ \ldots, & \pm 1, 0, \quad 0, \pm 1, \underline{1}, 0, \quad 0, \pm 1, \ldots \end{matrix},$$

where $-\theta_{-4} = \varphi_4 = S$, *we have*

$$\pi_0 < 10{,}8 < 13.$$

LEMMA 5.3. *For the chain-pair*

$$\begin{matrix} \ldots, & 3, \underline{3}, 2, -2, 2, -2, -3, -3, \ldots \\ \ldots, & \pm 1, \underline{1}, 0, \quad 0, 0, \quad 0, \pm 1, \pm 1, \ldots \end{matrix},$$

where $-\theta_{-1} = \varphi_7 = S$, $\varphi_5 = [-3, -3, S]$, *we have*

$$\pi_0 < 12{,}4 < 13.$$

LEMMA 5.4. *For the chain-pairs* (i):

$$\begin{matrix} \ldots, & 3, 2, -2, -3, & 3, & \underline{3}, 2, -2, & 3, 2, -2, -3, \ldots \\ \ldots, & \pm 1, 0, & 0, & 1, \pm 1, \underline{\pm 1}, 0, & 0, \pm 1, 0, & 0, \pm 1, \ldots \end{matrix},$$

and (ii):

$$\begin{matrix} \ldots, & 3, 2, -2, -3, 2, -2, -3, -3, 3, \underline{3}, 2, -2, 3, 2, -2, -3, \ldots \\ \ldots, & \pm 1, 0, \quad 0, \pm 1, 0, 0, -1, 1, \pm 1, \underline{\pm 1}, 0, \quad 0, \pm 1, 0, 0, \pm 1, \ldots \end{matrix},$$

where, in both cases, $-\theta_{-1} = \varphi_{-1} = S$, $\varphi_0 = [3, 2, -2, S]$, *we have*

$$\pi_0 < 12{,}9 < 13.$$

LEMMA 5.5. *For the chain-pair*

$$\begin{matrix} , 3, 2, -2, -3, 2, -2, -3, -3, & 3, & \underline{3}, 2, -2, & 3, 2, -2, -3, \\ , \pm 1, 0, & 0, \pm 1, 0, & 0, & 1, & 1, \pm 1, \underline{\pm 1}, 0, & 0, \pm 1, 0, & 0, \pm 1, \end{matrix}$$

where $-\theta_{-1} = \varphi_{-1} = S$, $\varphi_0 = [3, 2, -2, S]$, *we have*

$$\pi_{-2} < 7 < 13.$$

By the remarks preceding the lemmas, this completes the proof of Theorem 4.

For the form g_3, we have, by (4.8),

$$S = [3, 3, 3, 2, -2, S].$$

Using the relations (4.9) to (4.14) we deduce that the only permissible expansions of S are

$$S \equiv [3, 3, 3, 2, -2, S],$$
$$S \equiv [2, -2, -3, -3, -3, S],$$
$$S \equiv [3, 3, 2, -2, -3, S],$$
$$S \equiv [3, 2, -2, -3, -3, S],$$
$$S \equiv [2, -2, -2, 2, 2, -2, S].$$

It now follows from the symmetry of the form g_3, whose roots are $R_1 = -S$, $R_2 = S$, that any permissible a-chain from g_3 must be an arrangement of some or all of the five blocks of numbers:

$$A = 3, 3, 3, 2, -2,$$
$$B = 2, -2, -3, -3, -3,$$
$$C = 3, 3, 2, -2, -3,$$
$$D = 3, 2, -2, -3, -3,$$
$$E = 2, -2, -2, 2, 2, -2.$$

In Lemmas 5.6 and 5.7 we shall show that, for every chain-pair such that $\{a_r\}$ contains A,

$$M(P) < \tfrac{17}{2};$$

since B is the negative of A reversed, this is true also for chain-pairs such that $\{a_r\}$ contains B. In Lemma 5.8 we shall show that the same result holds when $\{a_r\}$ contains EC or DC; and we shall deduce from Lemmas 5.8 to 5.10 that it holds also for chains from g_3 containing CC. Any permissible a-chain from g_3 which is not even and does not contain A or B must contain C or D (which is the negative of C reversed) and must therefore contain one of EC, DC, CC or their negatives reversed. Thus, by the remarks following the statement of Theorem 5, the theorem will follow from Lemmas 5.6 to 5.10.

LEMMA 5.6. *For the chain-pairs*

$$\begin{matrix} \ldots, 3, 3, \underline{3}, 2, -2, \ldots \\ \ldots, 1, 1, \underline{1}, 0, \quad 0, \ldots \end{matrix} \quad \textit{and} \quad \begin{matrix} \ldots, \quad 3, \quad 3, \underline{3}, 2, -2, \ldots \\ \ldots, \pm 1, -1, \underline{1}, 0, \quad 0, \ldots \end{matrix},$$

where $-\theta_{-2} = \varphi_{-2} = S$, *we have*

$$3\pi_0/\Delta < 0{,}903,$$
$$\pi_0 < \Delta/3.$$

LEMMA 5.7. *For the chain-pair*

$$\begin{matrix} \ldots, \quad 3, 3, \underline{3}, 2, -2, \ldots \\ \ldots, -1, 1, \underline{1}, 0, \quad 0, \ldots \end{matrix},$$

where $-\theta_{-2} = \varphi_{-2} = S$, *we have*

$$3\pi_{-1}/\Delta < 0{,}8,$$
$$\pi_{-1} < \Delta/3.$$

LEMMA 5.8. *For the following chain-pairs*

(i) $$\begin{matrix} \ldots, \quad 3, \underline{3}, 2, -2, -3, \ldots \\ \ldots, -1, \underline{1}, 0, \quad 0, \pm 1, \ldots \end{matrix},$$

(ii) $$\begin{matrix} \ldots, 2, -2, -2, 2, 2, -2, 3, \underline{3}, 2, -2, -3, \ldots \\ \ldots, 0, \quad 0, \quad 0, 0, 0, \quad 0, 1, \underline{1}, 0, \quad 0, \pm 1, \ldots \end{matrix},$$

(iii) $$\begin{matrix} \ldots, \quad 3, 2, -2, -3, -3, 3, \underline{3}, 2, -2, -3, \ldots \\ \ldots, \pm 1, 0, \quad 0, \quad 1, -1, 1, \underline{1}, 0, \quad 0, \pm 1, \ldots \end{matrix},$$

where, in each case, $-\theta_{-1} = \varphi_{-1} = S$, $\varphi_0 = [3, 2, -2, -3, S]$, *we have*

$$3\pi_0/\Delta < 0{,}95,$$
$$\pi_0 < \Delta/3;$$

and for the chair-pair

(iv) $$\begin{matrix} \ldots, \quad 3, 2, -2, -3, -3, 3, \underline{3}, 2, -2, -3, \ldots \\ \ldots, \pm 1, 0, \quad 0, -1, \quad 1, 1, \underline{1}, 0, \quad 0, \pm 1, \ldots \end{matrix},$$

where $-\theta_{-1} = \varphi_{-1} = S$, $\varphi_0 = [3, 2, -2, -3, S]$, *we have*

$$3\pi_{-2}/\Delta < 0{,}95,$$
$$\pi_{-2} < \Delta/3.$$

Since $3M(P)/\Delta < 1$ for the chain-pair (i), it follows from Lemma 2.5 that $3M(P)/\Delta < 1$ also for the negative of its reverse:

(v) $$\begin{matrix} \ldots, \quad 3, 2, -2, \underline{-3}, -3, \ldots \\ \ldots, \pm 1, 0, \quad 0, \quad \underline{1}, \quad 1, \ldots \end{matrix},$$

where $-\theta_0 = \varphi_0 = S$. Thus in Lemma 5.8 we have considered every possible ε-chain (or its negative) corresponding to the a-chains of the pairs (ii) and (iii).

LEMMA 5.9. *For the chain-pair*

$$\begin{matrix} \ldots, 3, \underline{3}, 2, -2, -3, \ldots \\ \ldots, 1, \underline{1}, 0, \quad 0, -1, \ldots \end{matrix},$$

where $-\theta_{-1} = \varphi_{-1} = S$, $\varphi_0 = [3, 2, -2, -3, S]$, *we have*

$$3\pi_0/\Delta < 0{,}95,$$
$$\pi_0 < \Delta/3.$$

LEMMA 5.10. *For the chain-pair*

$$\begin{matrix} \ldots, 3, 3, 2, -2, -3, 3, \underline{3}, 2, -2, -3, \ldots \\ \ldots, 1, 1, 0, \quad 0, \quad 1, 1, \underline{1}, 0, \quad 0, \quad 1, \ldots \end{matrix},$$

where $-\theta_{-1} = \varphi_{-1} = S$, $\varphi_0 = [3, 2, -2, -3, S]$, *we have*

$$3\pi_0/\Delta < 0{,}993,$$

$$\pi_0 < \Delta/3.$$

The ε-chains of the chain-pairs (i) and (v) of Lemma 5.8 and of the chain-pairs of Lemmas 5.9 and 5.10 include every possible ε-chain (or its negative) corresponding to an α-chain from g_3 which contains CC, where C is the block of numbers

$$C = 3, 3, 2, -2, -3.$$

By the argument preceding Lemma 5.6, this completes the proof of Theorem 5.

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