in the same way

$$nB(m,n) \equiv 0 \pmod{k}.$$

Hence if $k = mm_1 + nn_1$, we get

$$kB(m,n) \equiv 0 \pmod{k},$$

and therefore $B(m,n)$ is integral.

Reference

[1] S. Uchiyama, *Sur un problème posé par M. Paul Turán*, Acta Arithmetica 4 (1958), p. 240-246.

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Some cyclotomic matrices

by

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1. Introduction. In a recent paper [5] Lehmer remarks that for relatively few matrices M can one give explicit formulas for the determinant, characteristic roots and inverse of A as well as the general element of M^r. He then considers two classes of matrices whose elements involve the Legendre symbol for which these problems are solved explicitly.

Let $\chi(r)$ denote the Legendre symbol (r/p), where p is an odd prime. The first class of matrices is of the type

$$\big(a + b\chi(r) + c\chi(s) + d\chi(rs)\big) \quad (r, s = 1, \ldots, p-1), \tag{1.1}$$

where a, b, c, d are constants. The second is of the type

$$\big(c + \chi(a+r+s)\big) \quad (r, s = 1, \ldots, p-1), \tag{1.2}$$

where c is arbitrary but a is an integer.

In the present paper we consider some additional classes of matrices for which at least the characteristic roots can be computed. We discuss first the matrix

$$(\varepsilon^{rs}) \quad (r, s = 0, 1, \ldots, n-1), \tag{1.3}$$

where $\varepsilon = e^{2\pi i/n}$. This matrix is familiar in connection with Schur's derivation of the value of Gauss's sum ([4], vol. 1, p. 162). By means of his method it is easy to determine the characteristic roots of (1.3) for arbitrary n.

Next if $\chi(r)$ is an arbitrary character $(\mathrm{mod}\, n)$ we consider the matrix of order $\varphi(n)$

$$A = \big(a + b\chi(r) + c\bar{\chi}(s) + d\chi(r)\bar{\chi}(s)\big), \tag{1.4}$$

where r, s run through the numbers of a reduced residue system $(\mathrm{mod}\, n)$ in some prescribed order. This evidently generalizes (1.1). Similarly the matrix

$$\big(c + \chi(a+r+s)\big) \quad (r, s = 1, \ldots, p-1) \tag{1.5}$$

generalizes (1.2); however note that in (1.5) we confine ourselves to matrices of order $p-1$, where p is a prime. In each case the characteristic roots are determined, although for (1.5) the results (see Theorem 5) are not entirely explicit. The simpler matrix

$$(1.6)\qquad \big(c+\chi(r-s)\big) \quad (r, s = 1, \ldots, p-1)$$

is covered by Theorem 4.

The remainder of the paper is concerned with circulant and related matrices. Part of the difficulty encountered in dealing with (1.2) and (1.5) is due to the fact that the range of r, s is restricted. Thus, by contrast, we find for example that if $\chi(n)$ is a non-principle character $(\bmod n)$ then the characteristic roots of the matrix

$$\big(\chi(s-r)\big) \quad (r, s = 0, 1, \ldots, n-1)$$

are the numbers

$$\varepsilon^r \tau(\chi) \quad (r = 0, 1, \ldots, n-1)$$

while the characteristic polynomial of the matrix

$$\big(c+\chi(a+r+s)\big) \quad (r, s = 0, 1, \ldots, n-1)$$

is

$$(x-cn)\prod_{r=1}^{(n-1)/2}\{x^2-\bar{\chi}(-r^2)\tau^2(\chi)\} \quad (n \text{ odd}),$$

$$x(x-cn)\prod_{r=1}^{(n-2)/2}\{x^2-\bar{\chi}(-r^2)\tau^2(\chi)\} \quad (n \text{ even } >2),$$

where

$$\tau(\chi) = \sum_{s=0}^{n-1}\chi(s)e^{2\pi i s/n}.$$

All matrices occurring in the paper are square and will be denoted by capital italic letters. The elements of the matrices are complex number. For any matrix M, we denote by M' the *transpose*, by $\bar{M}$ the *complex conjugate*, and put $M^* = \bar{M}'$. We recall that M is normal provided $MM^* = M^*M$ and that a normal matrix is unitarily similar to a diagonal matrix.

2. The matrix

$$(2.1)\qquad E = (\varepsilon^{rs}) \quad (r, s = 0, \ldots, n-1),$$

where $\varepsilon = e^{2\pi i/n}$ and n is an arbitrary positive integer, satisfies the relation

$$(2.2)\qquad E^4 = n^2 I.$$

Hence the characteristic values of E are of the form

$$i^r n^{1/2} \quad (r = 0, 1, 2, 3).$$

Let m_r denote the multiplicity of the characteristic value $i^r n^{1/2}$. Then we have

$$(2.3)\qquad \begin{aligned} m_0+m_1+m_2+m_3 &= n,\\ m_0+im_1-m_2-im_3 &= S,\\ m_0-m_1+m_2-m_3 &= v,\\ m_0-im_1-m_2+im_3 &= \bar{S}, \end{aligned}$$

where $v = 1$ or 2 according as n is odd or even, and

$$S = \sum_{k=0}^{n-1}\varepsilon^{k^2}.$$

To find the m_r most rapidly, we make use of the well-known formula ([4], vol. 1, p. 153)

$$S = \begin{cases} (1+i)n^{1/2} & \text{for } n \equiv 0 \pmod 4,\\ n^{1/2} & \text{for } n \equiv 1 \pmod 4,\\ 0 & \text{for } n \equiv 2 \pmod 4,\\ in^{1/2} & \text{for } n \equiv 3 \pmod 4. \end{cases}$$

Solving the system (2.3) we obtain the following results:

$$m_0 = \tfrac14 n+1, \quad m_1 = \tfrac14 n = m_2, \quad m_3 = \tfrac14 n-1 \quad \big(n \equiv 0 \pmod 4\big),$$
$$m_0 = \tfrac14(n+3), \quad m_1 = m_2 = m_3 = \tfrac14(n-1) \quad \big(n \equiv 1 \pmod 4\big),$$
$$m_0 = m_2 = \tfrac14(n+2), \quad m_1 = m_3 = \tfrac14(n-2) \quad \big(n \equiv 2 \pmod 4\big),$$
$$m_0 = m_1 = m_2 = \tfrac14(n+1), \quad m_3 = \tfrac14(n-3) \quad \big(n \equiv 3 \pmod 4\big).$$

We may accordingly state

THEOREM 1. *The characteristic polynomial of the matrix* (2.1) *is given by*

$$f(x) = (x-n^{1/2})^2(x-in^{1/2})(x+n^{1/2})(x^4-n^2)^{n/4-1} \quad \big(n \equiv 0 \pmod 4\big),$$
$$f(x) = (x-n^{1/2})(x^4-n^2)^{(n-1)/4} \quad \big(n \equiv 1 \pmod 4\big),$$
$$f(x) = (x^2-n)(x^4-n^2)^{(n-2)/4} \quad \big(n \equiv 2 \pmod 4\big),$$
$$f(x) = (x-in^{1/2})(x^2-n)(x^4-n^2)^{(n-3)/4} \quad \big(n \equiv 3 \pmod 4\big).$$

Since E is normal (indeed $n^{-1/2}E$ is unitary), it follows that E is unitarily similar to a diagonal matrix D; in each case the elements of D have been explicitly determined. It would be of interest to construct a particular unitary matrix U such that

$$U^{-1}EU = D, \tag{2.4}$$

however it is not clear how to do this.

3. Let n be an arbitrary positive integer and put $h = \varphi(n)$, the Euler φ-function; also let $\chi(r)$ denote any non-principal character $(\mathrm{mod}\, n)$. Corresponding to Lehmer's matrix of the first kind we consider

$$A = \big(a+b\chi(r)+c\bar{\chi}(s)+d\chi(r)\bar{\chi}(s)\big), \tag{3.1}$$

where r, s run through the numbers of a reduced residue system $(\mathrm{mod}\, n)$, say in ascending order. Thus A is of order h. The numbers a, b, c, d are arbitrary complex quantities.

Now if

$$A_1 = \big(a_1+b_1\chi(r)+c_1\bar{\chi}(s)+d_1\chi(r)\bar{\chi}(s)\big)$$

is a second matrix of the form (3.1) and we put $AA_1 = (a_{rs})$, then

$$a_{rs} = \sum_k \big(a+b\chi(r)+c\bar{\chi}(k)+d\bar{\chi}(r)\bar{\chi}(k)\big)\cdot\big(a_1+b_1\chi(k)+c_1\bar{\chi}(s)+d_1\chi(k)\bar{\chi}(s)\big).$$

Since

$$\sum_k \chi(k) = 0, \qquad \sum_k \chi(k)\bar{\chi}(k) = h,$$

it follows readily that

$$a_{rs} = h\big(a_2+b_2\chi(r)+c_2\bar{\chi}(s)+d_2\chi(r)\bar{\chi}(s)\big) \tag{3.2}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

As a corollary the coefficients of A^m, where $m \geqslant 1$, are determined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^m.$$

Note in particular that the special matrix

$$J = \big(1+\chi(r)\bar{\chi}(s)\big)$$

satisfies

$$AJ = JA = hA.$$

We may sum up the above by saying that we have established a ring isomorphism between the set of matrices $h^{-1}A$ and the corresponding second order matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus it is clear that

$$A^2-h(a+d)A+h(ad-bc)J = 0,$$

which implies

$$A^3-h(a+d)A^2+h^2(ad-bc)A = 0. \tag{3.3}$$

It follows that the minimum polynomial of A is a divisor of

$$x^3-h(a+d)x^2+h^2(ad-bc)x. \tag{3.4}$$

As for the characteristic polynomial we have

THEOREM 2. *The characteristic polynomial of the matrix* (3.1) *is determined by*

$$f(x) = x^{h-2}\{x^2-h(a+d)x+h^2(ad-bc)\}. \tag{3.5}$$

This theorem can be proved by the method used in proving Theorem 1 of Lehmer's paper. However the following method leads to somewhat more precise results.

Let the h characters $(\mathrm{mod}\, n)$ be denoted by $\chi_r^{(s)}$ $(r = 0, 1, \ldots, h-1)$, where χ_0 is the principal character and $\chi_1 = \chi$. Define the matrix of order h

$$X = \big(\chi_s(c_r)\big) \qquad (r, s = 0, 1, \ldots, h-1), \tag{3.6}$$

where c_r runs through a reduced residue system $(\mathrm{mod}\, n)$. Then

$$X^*X = \big(\bar{\chi}_r(c_s)\big)\big(\chi_s(c_r)\big) = \Big(\sum_k \bar{\chi}_r(c_k)\chi_s(c_k)\Big).$$

Since

$$\sum_k \bar{\chi}_r(c_k)\chi_s(c_k) = h\delta_{r,s}, \tag{3.7}$$

it follows that

$$X^*X = hI; \tag{3.8}$$

in other words $h^{-1/2}X$ is unitary. Now consider the product

$$B = (b_{rs}) = AX,$$

where

$$b_{\;s} = \sum_k \big(a+b\chi(c_r)+c\bar{\chi}(c_r)+d\chi(c_r)\bar{\chi}(c_k)\big)\chi_s(c_k).$$

Using (3.7) this becomes

$$b_{rs}=\begin{cases} h\left(a+b\chi(c_r)\right) & \text{for} \quad s=0, \\ h\left(c+d\chi(c_r)\right) & \text{for} \quad s=1, \\ 0 & \text{for} \quad s\neq 0,1. \end{cases}$$

To find X^*B we examine

$$\sum_k \bar{\chi}_r(c_k)\, b_{ks}.$$

For $s=0$ this becomes

$$h\sum_k \bar{\chi}_r(c_k)\left(a+b\chi(c_k)\right)=\begin{cases} h^2a & \text{for} \quad r=0, \\ h^2b & \text{for} \quad r=1, \\ 0 & \text{for} \quad r\neq 0,1; \end{cases}$$

for $s=1$ we get

$$h\sum_k \bar{\chi}_r(c_k)\left(c+d\chi(c_k)\right)=\begin{cases} h^2c & \text{for} \quad r=0, \\ h^2d & \text{for} \quad r=1, \\ 0 & \text{for} \quad r\neq 0,1; \end{cases}$$

for $s\neq 0,1$ the sum $=0$. Consequently we have

$$X^*AX=h^2\begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_0=\begin{bmatrix} a & c \\ b & d \end{bmatrix}. \tag{3.9}$$

Clearly (3.9) implies Theorem 2 and indeed the following

THEOREM 3. *The matrix* (3.1) *satisfies*

$$U^*AU=h\begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_0=\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \tag{3.10}$$

where X is defined by (3.6) *and $U=h^{-1/2}X$ is unitary. Moreover A is normal if and only if A_0 is normal.*

The last part of the theorem is a direct consequence of (3.2); indeed it follows from (3.2) that AA^* corresponds to $A_0'\bar{A}_0$, while A^*A corresponds to $\bar{A}A'$.

4. As Lehmer points out, matrices of the second kind are more difficult to deal with. We shall now limit our discussion to the case n prime and define

$$A=\left(c+\chi(a+r+s)\right) \qquad (r,s=1,2,\ldots,p-1), \tag{4.1}$$

where a is an integer while c is an arbitrary complex number; also χ is an arbitrary non-principal character $(\bmod p)$. We shall not attempt to compute A^m, which is presumably quite complicated. However it is not difficult to verify that

$$AB=p\left(1+cp\bar{\chi}(a)\right)I \tag{4.2}$$

where (compare [5], formula (18))

$$B=\left(\bar{\chi}(a)-\bar{\chi}(a+r)-\bar{\chi}(a+s)-cp\bar{\chi}(a+r)\bar{\chi}(a+s)+\left(1+cp\bar{\chi}(a)\right)\bar{\chi}(a+r+s)\right).$$

Indeed if $AB=(b_{rs})$, then we have

$$\begin{aligned} b_{rs} &= \sum_{k=0}^{p-1}\left(c+\chi(a+r+k)\right)\{\bar{\chi}(a)-\bar{\chi}(a+k)-\bar{\chi}(a+s)- \\ &\qquad\qquad -cp\bar{\chi}(a+k)\bar{\chi}(a+s)+\left(1+cp\,\bar{\chi}(a)\right)\bar{\chi}(a+k+s)\}- \\ &\quad -\left(c+\chi(a+r)\right)\{-\bar{\chi}(a+s)-cp\,\bar{\chi}(a)\bar{\chi}(a+s)+\left(1+cp\bar{\chi}(a)\right)\bar{\chi}(a+s)\} \\ &= ip\left(\bar{\chi}(a)-\bar{\chi}(a+s)\right)-\left(1+cp\,\bar{\chi}(a+s)\right)\sum_{k=0}^{p-1}\chi(a+r+k)\bar{\chi}(a+k)+ \\ &\qquad\qquad +\left(1+cp\,\bar{\chi}(a)\right)\sum_{k=0}^{p-1}\chi(a+r+k)\bar{\chi}(a+k+s). \end{aligned}$$

But

$$\sum_{k=0}^{p-1}\chi(a+r+k)\bar{\chi}(a+k)=\sum_{k=1}^{p-1}\chi(r+k)\bar{\chi}(k)=\sum_{k=1}^{p-1}\chi(rk+1)=-1,$$

$$\sum_{k=0}^{p-1}\chi(a+r+k)\bar{\chi}(a+k+s)=\sum_{k=0}^{p-1}\chi(r-s+k)\bar{\chi}(k)=p\delta_{r,s}-1,$$

so that

$$\begin{aligned} b_{rs} &= cp\left(\bar{\chi}(a)-\bar{\chi}(a+s)\right)+\left(1+cp\,\bar{\chi}(a+s)\right)-\left(1+cp\,\bar{\chi}(a)\right)\left(p\delta_{r,s}-1\right) \\ &= p\left(1+cp\bar{\chi}(a)\right)\delta_{r,s}. \end{aligned}$$

It is clear from (4.2) that A is non-singular if and only if $1+cp\,\bar{\chi}(a)\neq 0$. Note in particular that the inverse of the matrix

$$\left(\chi(r+s)\right) \qquad (r,s=1,2,\ldots,p-1) \tag{4.3}$$

is furnished by

$$p^{-1}\left(\bar{\chi}(r+s)-\bar{\chi}(r)-\bar{\chi}(s)\right),$$

as can be checked very rapidly.

We remark that for complex χ, the matrix (4.1) is not normal. Indeed even the special case (4.3) is not normal, since if A stands for (4.3), then

$$AA^*=\left(p\delta_{r,s}-1-\chi(a+r)\bar{\chi}(a+s)\right),$$

$$A^*A=\left(p\delta_{r,s}-1-\bar{\chi}(a+r)\chi(a+s)\right).$$

Before finding the characteristic roots of the matrix (4.1) we consider the matrix

$$C = \big(c + \chi(s-r)\big) \qquad (r, s = 1, \ldots, p-1). \tag{4.4}$$

It will be convenient to enlarge C to a matrix of order p

$$C_1 = (c_{rs}) \qquad (r, s = 0, 1, \ldots, p-1),$$

where

$$c_{rs} = \begin{cases} 0 & \text{for} \quad r = 0, \\ c + \chi(s-r) & \text{for} \quad r = 1, \ldots, p-1. \end{cases}$$

Clearly the characteristic roots of C_1 are those of C together with the value 0. Now if

$$E = (\varepsilon^{rs}) \qquad (r, s = 0, 1, \ldots, p-1,\ \varepsilon = e^{2\pi i/p}), \tag{4.5}$$

and we put

$$D = p^{-1}E^* C_1 E = (d_{rs}) \qquad (r, s = 0, 1, \ldots, p-1),$$

then

$$pd_{rs} = \sum_{j=1}^{p-1} \sum_{k=0}^{p-1} \varepsilon^{-rj}\big(c + \chi(k-j)\big)\varepsilon^{ks} = p^2 c\delta_{r0}\delta_{s0} + p\bar{\chi}(s)\tau\delta_{rs} - \bar{\chi}(s)\tau - pc\delta_{s0},$$

where

$$\tau = \sum_{k=1}^{p-1} \chi(k)\varepsilon^k. \tag{4.6}$$

Now consider the characteristic polynomial of D, namely

$$f_1(x) = |x - d_{rs}| = |x - pc\delta_{r0}\delta_{s0} - \bar{\chi}(s)\tau\delta_{rs} + p^{-1}\bar{\chi}(s)\tau + c\delta_{s0}|.$$

If we subtract the first row of this determinant from each of the other rows, we get

$$\begin{vmatrix} x-(p-1)c & p^{-1}\bar{\chi}(1)\tau & \ldots & p^{-1}\bar{\chi}(p-1)\tau \\ -x+pc & x-\bar{\chi}(1)\tau & \ldots & \\ \ldots & \ldots & \ldots & \ldots \\ -x+pc & & \ldots & x-\bar{\chi}(1)\tau \end{vmatrix}. \tag{4.7}$$

Expanding the determinant, we get

$$f_1(x) = \big(x-(p-1)c\big)\varphi(x) + (x-pc)\varphi(x)\sum_{k=1}^{p-1}\frac{p^{-1}\bar{\chi}(k)\tau}{x-\bar{\chi}(k)\tau},$$

where

$$\varphi(x) = \prod_{k=1}^{p-1}\big(x - \bar{\chi}(k)\tau\big).$$

Now let f be the smallest positive integer such that $\chi^f = \chi_0$, the principal character. Then

$$\varphi(x) = (x^f - \tau^f)^t,$$

where $tf = p-1$. We have also

$$\sum_{k=1}^{p-1}\frac{\bar{\chi}(k)\tau}{x-\bar{\chi}(k)\tau} = 1-p+\sum_{k=1}^{p-1}\frac{x}{x-\bar{\chi}(k)\tau} = 1-p+\frac{tfx^f}{x^f-\tau^f} = \frac{(p-1)\tau^f}{x^f-\tau^f}$$

so that

$$f_1(x) = \big(x-(p-1)c\big)(x^f-\tau^f)^t + p^{-1}(p-1)\tau^f(x-pc)(x^f-x^f)^{t-1}$$

$$= (x^f-\tau^f)^{t-1}\Big(x^{f+1}-(p-1)cx^f-\frac{1}{p}\tau^f x\Big).$$

We recall that among the characteristic roots of C_1 the value 0 is superfluous. We may therefore state

Theorem 4. *The characteristic polynomial of the matrix* (4.4) *is given by*

$$f(x) = (x^f-\tau^f)^{t-1}\Big(x^f-(p-1)cx^{f-1}-\frac{1}{p}\tau^f\Big), \tag{4.8}$$

where f is the smallest positive integer such that $\chi^f = \chi_0$, $ft = p-1$, and τ is defined by (4.6). *In particular, when χ is the Legendre symbol,* (4.8) *reduces to*

$$f(x) = \big(x^2-(-1)^{(p-1)/2}p\big)^{(p-3)/2}\big(x^2-(p-1)c-(-1)^{(p-1)/2}\big). \tag{4.9}$$

We shall now determine the characteristic polynomial of the matrix (4.1). We first enlarge A to a matrix of order p

$$A_1 = (a_{rs}) \qquad (r, s = 0, 1, \ldots, p-1),$$

where

$$a_{rs} = \begin{cases} 0 & \text{for} \quad r = 0, \\ c + \chi(a+r+s) & \text{for} \quad r = 1, \ldots, p-1. \end{cases}$$

We next construct

$$B = p^{-1}E^* A_1 E = (b_{rs}),$$

where E is defined by (4.5); we find that

$$pb_{rs} = p^2c\delta_{r0}\delta_{s0} - pc\delta_{s0} + p\varepsilon^{-as}\bar{\chi}(s)\tau\delta_{r,n-s} - \varepsilon^{-as}\tau.$$

Thus the characteristic polynomial of B is equal to

$$|x-d_{rs}| = |x-pc\delta_{r0}\delta_{s0}+c\delta_{s0}-\varepsilon^{-as}\bar{\chi}(s)\tau\delta_{r,n-s}+p^{-1}\varepsilon^{-as}\bar{\chi}(s)\tau|.$$

On subtracting the first row from each of the other rows, this becomes

$$\begin{vmatrix} x-(p-1)c & p^{-1}\varepsilon^{-a}\bar{\chi}(1)\tau & \dots & p^{-1}\varepsilon^{-(p-1)a}\bar{\chi}(p-1)\tau \\ -x+pc & x & \dots & -\varepsilon^{-(p-1)a}\bar{\chi}(p-1)\tau \\ \dots & \dots & \dots & \dots \\ -x+pc & -\varepsilon^{-a}\bar{\chi}(1)\tau & \dots & x \end{vmatrix}. \tag{4.10}$$

Put

$$\varphi(x) = \prod_{k=1}^{(p-1)/2}\left(x^2-\bar{\chi}(k^2)\omega^2\right), \qquad \omega = \begin{cases} \tau & \text{for} \quad \chi(-1)=1, \\ -i\tau & \text{for} \quad \chi(-1)=-1. \end{cases}$$

Then expansion of (4.10) yields

$$\begin{aligned}(x-(p-1)c)\varphi(x)+p^{-1}\tau x(x-pc)\varphi(x)\sum_{k=1}^{p-1}\frac{\varepsilon^{-ka}\bar{\chi}(k)}{x^2-\bar{\chi}(k^2)\omega^2}+ \\ +p^{-1}\omega^2(x-pc)\varphi(x)\sum_{k=1}^{p-1}\frac{\bar{\chi}(k^2)}{x^2-\bar{\chi}(k^2)\omega^2}.\end{aligned} \tag{4.11}$$

Let f be the least positive integer such that $\chi^f=\chi_0$ and let g denote a fixed primitive root $(\bmod p)$. Define

$$\eta_r = \sum_{s=0}^{(p-1)/f}\varepsilon^{g^{fs+r}} \tag{4.12}$$

and put

$$-a \equiv g^z \pmod{p}. \tag{4.13}$$

A detailed examination of (4.11) now leads to the following

THEOREM 5. *The characteristic polynomial of the matrix* (4.1) *is equal to*

$$\frac{(x^{2f}-\omega^{2f})^{(p-1)/2f}}{x^f-\omega^f}\left(x^f-(p-1)cx^{f-1}-\frac{1}{p}\omega^f\right) \qquad (a=0,\ f \text{ odd}), \tag{4.14}$$

$$(x^f-\omega^f)^{(p-1)/f-1}\left(x^f-(p-1)cx^{f-1}-\frac{1}{p}\omega^f\right) \qquad (a=0,\ f \text{ even}), \tag{4.15}$$

$$(x^{2f}-\omega^{2f})^{(p-1)/2f-1}\left(x^{2f}-(p-1)cx^{2f-1}-\frac{1}{p}\omega^{2f}+(x-pc)R_1(x^2)\right) \qquad (a\neq 0,\ f \text{ odd}), \tag{4.16}$$

$$(x^f-\omega^f)^{(p-1)/f-1}\left(x^f-(p-1)cx^{f-1}-\frac{1}{p}\omega^f+(x-pc)R_0(x^2)\right) \qquad (a\neq 0,\ f \text{ even}), \tag{4.17}$$

where $R_1(y)$ is the unique polynomial of degree $<f$ such that for $\xi=\bar{\chi}(g)$

$$R_1(\xi^{2r}\omega^2) = f\xi^{-r}\eta_{r+z}\omega^{2f-2},$$

$R_0(y)$ is the unique polynomial of degree $<f'=\frac{1}{2}f$ such that

$$R_0(\xi^{2r}\omega^2) = f'\xi^{-r}(\eta_{r+z}-\eta_{r+z+f'})\omega^{2f'-2}.$$

In these formulas f is the least positive integer such that $\chi^f=\chi_0$, g is a fixed primitive root $(\bmod p)$, *η_r and z are defined by* (4.12) *and* (4.13), *respectively.*

In particular, when χ is the Legendre symbol, (4.15) and (4.17) reduce to

$$(x^2-p)^{(p-3)/2}(x^2-(p-1)cx-1),$$

$$(x^2-p)^{(p-3)/2}(x^2-(p-1)c-1+\chi(a)(x-pc)),$$

respectively; these results can be obtained at once from (4.11) and are in agreement with Theorem 2 of Lehmer's paper.

It follows from (4.8) that the determinant of (4.4) is equal to

$$\frac{1}{p}(-1)^{t-1}\tau^{p-1}, \tag{4.18}$$

while the determinant of (4.1) is

$$\begin{aligned}&\frac{1}{p}(-1)^{(p-1)/2f}\omega^{p-1}+(-1)^{(p-1)/2}\bar{\chi}\left((-1)^{(p+1)/2}a\right)\omega^{p-1} && (f \text{ odd}) \\ &\frac{1}{p}(-1)^{(p-1)/f}\omega^{p-1}+(-1)^{(p-1)/2}\bar{\chi}\left((-1)^{(p+1)/2}a\right)\omega^{p-1} && (f \text{ even}).\end{aligned} \tag{4.19}$$

For (4.18) compare [1].

5. If

$$C=(c_{s-r}) \qquad (r,s=0,1,\dots,n-1), \tag{5.1}$$

where $c_{r+n}=c_r$, is an arbitrary circulant matrix, then, as is well-known

$$n^{-1}\bar{E}CE=(d_r\delta_{rs}),$$

where E is defined by (2.1) and

$$d_r=\sum_{s=0}^{n-1}c_s\varepsilon^{rs} \qquad (r=0,1,\dots,n-1). \tag{5.2}$$

Thus C is unitarily similar to the diagonal matrix $(d_r\delta_{rs})$ and the d_r are the characteristic roots of C.

Next if

(5.3) $$K = (k_{r+s}) \quad (r, s = 0, 1, \ldots, n-1),$$

where $k_{r+n} = k_r$, then

$$n^{-1}\bar{E}KE = (l_{rs}),$$

where

$$l_{rs} = \begin{cases} l_0\delta_{0s} & \text{for} \quad r = 0, \\ l_r\delta_{r,n-s} & \text{for} \quad r = 1, \ldots, n-1, \end{cases}$$

(5.4) $$l_r = \sum_{s=0}^{n-1} k_s\varepsilon^{rs} \quad (r = 0, 1, \ldots, n-1).$$

It follows that the characteristic polynomial of K is equal to

(5.5) $$\begin{aligned} &(x-l_0)\prod_{r=1}^{(n-1)/2}(x^2-l_rl_{n-r}) \quad (n \text{ odd}), \\ &(x-l_0)(x-l_{n/2})\prod_{r=1}^{(n-2)/2}(x^2-l_rl_{n-r}) \quad (n \text{ even}). \end{aligned}$$

These general formulas can be specialized to yield results of arithmetic interest. In the first place, if $c_r = \chi(r)$, an arbitrary character $(\bmod n)$, then (5.2) becomes

$$\sum_{s=0}^{n-1}\chi(s)\varepsilon^{rs} = \bar{\chi}(r)\tau(\chi),$$

where

(5.6) $$\tau(\chi) = \sum_{s=0}^{n-1}\chi(s)\varepsilon^s.$$

Note that (5.6) coincides with (4.6) when $n = p$. Thus it follows that the characteristic roots of the matrix $(n > 2)$

(5.7) $$\big(\chi(s-r)\big) \quad (r, s = 0, 1, \ldots, n-1)$$

are the numbers

(5.8) $$\chi(n)\tau(\chi) \quad (r = 0, 1, \ldots, n-1).$$

Also the characteristic polynomial of the matrix (compare with (4.1))

(5.9) $$\big(c+\chi(a+r+s)\big) \quad (r, s = 0, 1, \ldots, n-1)$$

is furnished by

(5.10) $$\begin{aligned} &(x-cn)\prod_{r=1}^{(n-1)/2}\{x^2-\bar{\chi}(-r^2)\tau^2(\chi)\} && (n \text{ odd}) \\ &x(x-cn)\prod_{r=1}^{(n-2)/2}\{x^2-\bar{\chi}(-r^2)\tau^2(\chi)\} && (n \text{ even} > 2) \\ &(x-2c)\big(x-(-1)^a\big) && (n = 2) \end{aligned}$$

In view of the above it is of interest to know when $\tau(\chi) \neq 0$. The following result may be cited ([4], vol. 3, p. 333). Corresponding to each character $(\bmod n)$ there is a smallest positive integer n_0, $n_0|n$, such that if $r \equiv 1 (\bmod n_0)$ then $\chi(r) = 1$. Then we have $\tau(\chi) = 0$ if there exists a prime p such that $p|(n/n_0)$, $p^2|n$; otherwise (that is when each $p|(n/n_0)$ occurs only once in n)

$$|\tau(\chi)| = k^{1/2}.$$

In particular if $n = p$ then it is familiar that $|\tau| = p^{1/2}$ so that (5.7) has in this case just one zero characteristic root.

Specializing further, if $n = p$ and χ is the Legendre symbol, then the non-vanishing characteristic roots of (5.7) are the numbers

$$i^{(p-1)^2/4}\varepsilon^r p^{1/2} \quad (r = 1, \ldots, p-1).$$

Also the characteristic polynomial of (5.9) reduces to

(5.11) $$(x-cp)(x^2-p)^{(p-1)/2},$$

which may be compared with Theorem 5.

If $n = p$ and χ is any non-principal character $(\bmod p)$, we may put

$$\chi(r) = \begin{cases} a^{\operatorname{ind} r} & \text{for} \quad p \nmid r, \\ 0 & \text{for} \quad p|r, \end{cases}$$

where a is a primitive fth root of unity. Thus

$$\tau = \sum_{r=1}^{p-1} a^{\operatorname{ind} r}\varepsilon^r = (a, \varepsilon),$$

the so-called Lagrange resolvent ([2], p. 429). This function has the following properties:

$$(a, \varepsilon)(a^{-1}, \varepsilon) = (-1)^t p \quad (ft = p-1),$$
$$(a, \varepsilon)(a^r, \varepsilon) = \psi_r(a)(a^{r+1}, \varepsilon) \quad (p-1 \nmid (r+1)t),$$

where

(5.12) $$\psi_r(a) = \sum_{s=1}^{p-2} a^{\operatorname{ind} s-(r+1)\operatorname{ind}(s+1)} \quad (r = 1, \ldots, p-2).$$

If we assume in addition that $f = p-1$, then the matrix

$$U = (a^{-s\operatorname{ind}r}) = \big(\bar{\chi}(r^s)\big) \qquad (r, s = 1, \ldots, p-1)$$

satisfies

$$U^* U = \Big(\sum_{k=1}^{p-1} \chi(k^r)\,\bar{\chi}(k^s)\Big) = \big((p-1)\,\delta_{rs}\big),$$

so that $(p-1)^{-1/2}\,U$ is unitary. In the next place, if we put

(5.13) $$M = (a^{\operatorname{ind}(s-r)-\operatorname{ind}s}) = \big(\chi(1-rs^{-1})\big) \qquad (r, s = 1, \ldots, p-1),$$

then

$$MU = \sum_{k=1}^{p-1} \chi\big((k-r)\,k^{-s-1}\big) = \sum_{k} \chi\big(k\,(k+r)^{-s-1}\big) = \Big(\bar{\chi}(r^s) \sum_{k} \chi\big(k\,(k+1)^{-s-1}\big)\Big),$$

which is the same as

$$MU = U\big(\psi_s(a)\,\delta_{rs}\big),$$

where $\psi_s(a)$ is defined by (5.12) for $1 \leqslant s \leqslant p-2$, and we set $\psi_{p-1}(a) = +1$. It therefore follows that the characteristic roots of the matrix (5.13) are the numbers

$$\psi_r(a) \qquad (r = 1, \ldots, p-1).$$

It should be observed that in proving this result we have assumed that a is a primitive $(p-1)$th root of unity.

In the present connection we also note that the matrix

$$(\eta_{s-r}) \qquad (r, s = 0, 1, \ldots, f-1),$$

where η_r is defined by (4.12), has the characteristic roots

(5.14) $$(a^r, \eta) = \sum_{s=0}^{t-1} a^{rs}\eta_s \qquad (r = 0, 1, \ldots, f-1),$$

where $ft = p-1$ and a is a primitive fth root of unity. In this case $(1, \eta) = -1$,

(5.15) $$(a^r, \eta)(a^{-r}, \eta) = (-1)^{rt} p$$

so that all of the numbers (5.14) are different from a zero. It follows readily from (5.15) that the determinant (compare [1])

$$|\eta_{s-r}| = \begin{cases} -p^{(f-1)/2} & \text{for } f \text{ odd}, \\ -i^{(p-1)(p+f-3)/4}\,p^{(f-1)/2} & \text{for } f \text{ even}. \end{cases}$$

6. In conclusion we mention a few more special results of a different kind. If we put

$$\frac{xe^{xt}}{e^t-1} = \sum_{m=0}^{\infty} B_m(x)\,\frac{t^m}{m!}, \qquad \frac{1-\lambda}{1-\lambda e^t}\,e^{xt} = \sum_{m=0}^{\infty} \varphi_m(x, \lambda)\,\frac{t^m}{m!},$$

then it is easy to verify the formula (see [3], p. 825)

(6.1) $$\varphi_{k-1}(nx, \varepsilon) = \frac{(\varepsilon-1)\,n^{k-1}}{k} \sum_{r=0}^{n-1} \varepsilon^r B_k\Big(x+\frac{r}{n}\Big),$$

where $\varepsilon^n = 1$, $\varepsilon \neq 1$. For $\varepsilon = 1$, on the other hand, (6.1) is replaced by the multiplication formula

(6.2) $$B_k(nx) = n^{k-1} \sum_{r=0}^{n-1} B_k\Big(x+\frac{r}{n}\Big).$$

If we replace $B_k(x)$ by the function $\bar{B}_k(x)$ which satisfies

$$\bar{B}_k(x) = B_k(x) \quad (0 \leqslant x < 1), \qquad \bar{B}_k(x+1) = \bar{B}_k(x),$$

and $\varphi_k(x, \varepsilon)$ by the corresponding function

$$\bar{\varphi}_k(x, \varepsilon) = \varphi_k(x, \varepsilon) \quad (0 \leqslant x < 1), \qquad \bar{\varphi}_k(x+1, \varepsilon) = \varepsilon^{-1}\bar{\varphi}_k(x, \varepsilon),$$

then $\bar{B}_k(x)$ and $\bar{\varphi}_{k-1}(x, \varepsilon)$ satisfy both (6.1) and (6.2).

Comparison with (5.2) now yields the following result. The matrix

(6.3) $$\Big(\bar{B}_k\Big(x+\frac{s-r}{n}\Big)\Big) \qquad (r, s = 0, 1, \ldots, n-1)$$

has the characteristic roots

(6.4) $$n^{1-k}\bar{B}_k(nx), \qquad kn^{1-k}\,\frac{\varphi_{k-1}(nx, \varepsilon^r)}{\varepsilon^r-1} \qquad (r = 1, \ldots, n-1),$$

where $\varepsilon = e^{2\pi i/n}$.

If we solve (6.1) and (6.2) for $\bar{B}_k(x+r/n)$, we find easily that ($\varepsilon = e^{2\pi i/n}$)

$$B_k(nx)+k \sum_{s=1}^{n-1} \frac{\bar{\varphi}_{k-1}(nx, \varepsilon^s)}{\varepsilon^s-1}\,\varepsilon^{rs} = n^k\bar{B}_k\Big(x-\frac{r}{n}\Big).$$

Consequently if we put

$$(6.5)\qquad F_r = \begin{cases} B_k(nx) & \text{for} \quad r = 0, \\ k\dfrac{\bar{\varphi}_{k-1}(n, x\,\varepsilon^s)}{\varepsilon^r - 1} & \text{for} \quad r = 1, \ldots, n-1, \end{cases}$$

it follows that the matrix

$$(6.6)\qquad (F_{s-r}) \quad (r, s = 0, 1, \ldots, n-1)$$

has the characteristic roots

$$(6.7)\qquad n^k \bar{B}_k\left(x - \frac{r}{n}\right) \quad (r = 0, 1, \ldots, n-1).$$

In particular we can evaluate the determinants of (6.3) and (6.5).

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A note on the real zeros of Dirichlet's *L*-functions

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1. For $s = \sigma + it$, the L-functions of Dirichlet belonging to a modulus k are defined for $\sigma > 1$ by

$$(1.1)\qquad L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $\chi(n)$ are the characters of the group of the reduced residue-classes $\bmod k$. It is well known that the study of zeros of these functions give the key to the distribution of primes in the arithmetical progressions $\bmod k$ and the essentially new difficulties, compared to those connected with the zeros of the Riemann zeta-function, are due to the appearance of real zeros. Concerning them we know(1) that for a suitable positive(2) c_1 at most one of the $L(s, \chi)$-functions $\bmod k$ can vanish in the interval

$$(1.2)\qquad 1 - \frac{c_1}{\log k} \leqslant s \leqslant 1$$

and, if such an exceptional $L(s, \chi)$ exists, it has here a single simple zero (called *exceptional zero* and denoted by β). The possibility of an exceptional zero gives a lot of trouble in the number-theory. A typical example is furnished by the formula (Page, [2]), valid for $\chi \neq \chi_0$

$$(1.3)\qquad \Big|\sum_{n \leqslant x} \Lambda(n)\chi(n)\Big| \leqslant \begin{cases} c_2(xe^{-c_3\sqrt{\log x}} + x^{\beta}) \\ c_2 xe^{-c_3\sqrt{\log x}}, \end{cases}$$

χ is an exceptional character or not, respectively; here $\Lambda(n)$ stands for the known Dirichlet symbol and $\varphi(k)$ is the usual Euler function.

(1) This is essentially due to E. Landau [1].

(2) In what follows, $c_1, c_2, \ldots$ stand for explicitly calculable positive numerical constants; as an exception $c_4 = c_4(\varepsilon)$ is not and depends on ε.