et a une solution périodique de période $T = 2\pi/\omega$ donnée par la formule $x = B\cos\omega t$. En appliquant à l'équation (9.6) la transformation $\Pi(t, x)$: $y = x - B\cos\omega t$, on obtient l'équation

$$(9.8)\quad y'' = -A^2\sigma(\lambda)(y+B\cos\omega t) - \cos\big(\omega t - \lambda(y+B\cos\omega t)\big) + B\omega^2\cos\omega t,$$

pour $\lambda = 0$ on a

$$y'' = -A^2\sigma(0)(y+B\cos\omega t) - \cos\omega t + B\omega^2\cos\omega t.$$

Comme

$$-A^2B\cos\omega t - \cos\omega t = -B\omega^2\cos\omega t,$$

on a donc pour $\lambda = 0$

$$(9.9)\qquad y'' = -A^2 y.$$

De même que dans les exemples précédents, on a $m = TA^2$, $|f'_y(t, 0, 0, 0)| = A^2$, $|f'_\lambda(t, 0, 0, 0)| = 0$, et enfin

$$|f'_\lambda(t, 0, 0, 0)| \leqslant A^2B|\sigma'(0)| + B.$$

Par conséquent, si $M = \max\big(A^2B|\sigma'(0)|+B, A^2\big)$, et les constantes A, T, M satisfont à l'inégalité (10.4), il existe (pour $|\lambda| \leqslant L$, où $L > 0$ est convenablement choisi) une solution périodique $\varphi(t, \lambda)$ de l'équation (9.8) de période $T = 2\pi/\omega$, telle que $\varphi(t, 0) \equiv 0$. En raison de la forme de la transformation $\Pi(t, x)$, il en résulte immédiatement que l'équation (9.6) a une solution périodique de période $T = 2\pi/\omega$, $x = \tilde{\varphi}(t, \lambda)$ (valable pour $|\lambda| \leqslant L$), telle que

$$\tilde{\varphi}(t, 0) \equiv B\cos\omega t.$$

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A note on Fourier series of functions of an infinite number of variables

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1. We consider the *torus space* Q_ω of all sequences of real numbers $x = (x_1, x_2, \ldots)$, with all coordinates reduced mod 1. We denote by

$$\int_{Q_\omega} f(x)\,dw_\omega$$

the integral of a measurable function $f(x)$, defined in Q_ω, over the whole space Q_ω, and by

$$\int_{Q_H} f(x)\,dw_H$$

where $H = (k_1, k_2, \ldots)$ is a non-empty sequence of indices, the integral of $f(x)$ over the space of subsequences $(x_{k_1}, x_{k_2}, \ldots)$ (see [1], p. 266). The set H may be finite or infinite.

2. We shall investigate a special orthonormal system, defined in Q_ω. Let $E = (1, 2, \ldots)$ be the set of all positive integers and A a subset of E. Then we indicate by $\bar{A}$ the complement of A with regard to E. Further let $m = (m_1, m_2, \ldots)$ be a sequence of non-negative integers such that $m_i = 0$ for sufficiently large i. By $n(m)$ we indicate the number of positive integers in the sequence $m = (m_1, m_2, \ldots)$. It is easily seen that the system of functions

$$(2.1)\qquad \varphi_m^A(x) = 2^{n(m)/2}\prod_{i\in A}\cos 2\pi m_i x_i \prod_{j\in\bar{A}}\sin 2\pi m_j x_j, \qquad \varphi_m^A(x) \not\equiv 0 \quad \text{in} \quad Q_\omega,$$

is an orthonormal one. We indicate by

$$a_m^A(f) = \int_{Q_\omega} f(x)\varphi_m^A(x)\,dw_\omega$$

the Fourier coefficients of a function $f \in L^2(Q_\omega)$ with regard to the system $\{\varphi_m^A(x)\}$. Continuing the investigations of [2] and [3] we establish for

$0 < \gamma < 2$ and arbitrary real $\varkappa$ some conditions sufficient for the convergence of the series

$$\sum_m \sum_{A \subset E} 2^{\varkappa\gamma n(m)} |a_m^A(f)|^\gamma. \tag{2.2}$$

We remember that the summation extends here over all sequences $m = (m_1, m_2, \ldots)$ of non-negative integers such that there exist only a finite number of $m_i \neq 0$.

Let us remark that the convergence of the series (2.2) with $\gamma = 1$ and $\varkappa = \frac{1}{2}$ implies absolute convergence of the Fourier series of function $f(x)$ with regard to the system $\{\varphi_m^A(x)\}$ ([1]).

3. Now we shall introduce some symbols generalizing those used in [2] and [3] for n variables. Let $H = (k_1, \ldots, k_s)$ be a non-empty and finite subset of the set $E = (1, 2, \ldots)$ of all positive integers, with elements $k_1 < k_2 < \ldots < k_s$. Further let us write $x = (x_1, x_2, \ldots)$, $h = (h_1, h_2, \ldots)$. Then we write for $H = (k_1)$,

$$F^H(f; x; h) = f(x_1, \ldots, x_{k_1-1}, x_{k_1} + h_{k_1}, x_{k_1+1}, \ldots) - f(x_1, \ldots, x_{k_1-1}, x_{k_1} - h_{k_1}, x_{k_1+1}, \ldots)\ ([2])$$

and for $H = (k_1, \ldots, k_s)$ $(s > 1)$,

$$F^H(f; x; h) = F^{(k_s)}[F^{H-(k_s)}; x; h]\ ([3]).$$

Moreover, we write

$$\omega^H(f; x; h) = \sup_{x_i, |\delta_i| \leqslant h_i, i \in H} |F^H(f; x; \delta)|.$$

Let Π be a partition of Q_ω:

$$0 = x_i^{(0)} < x_i^{(1)} < \ldots < x_i^{(N_i)} = 1, \qquad i = 1, 2, \ldots$$

Then, given a non-empty and finite set of integers $H = (k_1, \ldots, k_s)$ and a number $r \geqslant 1$, we call the value

$$V_r^H(f) = \Big\{ \sup_{x_i, i \in \overline{H}} \sup_{\Pi} \sum_{i_{k_1}=1}^{N_{k_1}} \cdots \sum_{i_{k_s}=1}^{N_{k_s}} |F^H[f; x_1, \ldots, x_{k_1-1}, \tfrac{1}{2}(x_{k_1}^{(i_{k_1})} + x_{k_1}^{(i_{k_1}-1)}), x_{k_1+1}, \ldots, x_{k_s-1}, \tfrac{1}{2}(x_{k_s}^{(i_{k_s})} + x_{k_s}^{(i_{k_s}-1)}), x_{k_s+1}, \ldots; \tfrac{1}{2}(x_1^{(i_1)} - x_1^{(i_1-1)}), \tfrac{1}{2}(x_2^{(i_2)} - x_2^{(i_2-1)}), \ldots]|^r \Big\}^{1/r}$$

the *r-th variation of order H* of the function $f(x)$.

([1]) The content of this paper was presented on January 17^{th}, 1957, to the Symposium of Functional Analysis organized by the Mathematical Institute of the Polish Academy of Sciences.

([2]) The values obtained by the addition and subtraction of the variables must be reduced mod 1.

([3]) We remark that $F^H(f; x; h)$ does not depend on h_n for $n \neq k_i$ $(i = 1, 2, \ldots, s)$.

4. Applying the above notation we formulate a lemma and two theorems on the convergence of the series (2.2). Here $\mathcal{K}$ will denote the class of all non-empty and finite sets of positive integers.

LEMMA. *Given real numbers $0 < \gamma < 2$, $\varkappa$ and $r_H \geqslant 1$, and $f \in L^2(Q_\omega)$, we suppose the series*

$$\sum_{H \in \mathcal{K}} 2^{[1+\gamma(\varkappa-1/2)]s} \sum_{\nu_{k_1}=1}^{\infty} \cdots \sum_{\nu_{k_s}=1}^{\infty} 2^{(1-\gamma/2)\sum_{i=1}^{s} \nu_{k_i}} \times \tag{4.1}$$

$$\times \Big\{ \int_{Q_{E-H}} \big[\omega^H(f; x; [2^{-\nu-1}])\big]^{2-r_H} \cdot \Big[\int_{Q_H} |F^H(f; x; [2^{-\nu-1}])|^{r_H} dw_H \Big] dw_{E-H} \Big\}^{\gamma/2},$$

where $H = (k_1, \ldots, k_s)$ and $[2^{-\nu-1}] = (2^{-\nu_1-1}, 2^{-\nu_2-1}, \ldots)$, to be convergent. Then the series (2.2) is also convergent.

For the proof it suffices to remark that in our case lemmas 3-5 of [3] remain true. Then, if we replace lemma 1 of [3] by the inequality

$$\sum_{i=1}^{N} |a_i|^\gamma \leqslant N^{1-\gamma/2} \Big(\sum_{i=1}^{N} |a_i|^2 \Big)^{\gamma/2},$$

our lemma may be deduced by applying the method used in the proof of theorem 1 in [3].

THEOREM 1. *Given $0 < \gamma < 2$, $\varkappa$ real and $f \in L^2(Q_\omega)$, we suppose that for every $H = (k_1, \ldots, k_s) \in \mathcal{K}$ we have*

$$\sqrt{\int_{Q_\omega} |F^H(f; x; h/4)|^2 dw_\omega} \leqslant K_{k_s} h_{k_1}^{a_1^H} \ldots h_{k_s}^{a_s^H}, \tag{4.2}$$

where

$$a_i^H \geqslant a_{k_s} > \frac{2-\gamma}{2\gamma}\ ([4]), \tag{4.3}$$

and the constants K_n satisfy the condition

$$\sum_{n=1}^{\infty} K_n^\gamma [2^{(1+2a_n)\gamma/2-1} - 1]^{-1} \{1 + 2^{1+\gamma(\varkappa-1/2)} [2^{(1+2a_n)\gamma/2-1} - 1]^{-1}\}^{n-1} < \infty. \tag{4.4}$$

Then the series (2.2) is convergent.

Remark. For $\gamma = 1$ and $\varkappa = \frac{1}{2}$, if we suppose that $K_n = O\{[(2a_n - 1)/16]^n\}$ then (4.4) is satisfied.

([4]) It should be remembered that k_s is the largest element of the set $H = (k_1, \ldots, k_s)$.

To prove theorem 1, we denote the sum of the series (4.1) with $r_H = 2$ by S. It suffices to prove that $S < \infty$. Since for $0 < h_{k_i} \leqslant 1$,

$$\sqrt{\int_{Q_\omega} |F^H(f; x; h/4)|^2 dw_\omega} \leqslant K_{k_s}(h_{k_1} \dots h_{k_s})^{a_{k_s}},$$

we have

$$S \leqslant \sum_{H \in \mathcal{K}} 2^{[1+\gamma(\varkappa-1/2)]s} K_{k_s}^{\gamma} \sum_{\nu_{k_1}=1}^{\infty} \dots \sum_{\nu_{k_s}=1}^{\infty} 2^{(1-\gamma/2-a_{k_s}\gamma)\sum_{i=1}^{s} \nu_{k_i}}.$$

However, (4.3) implies

$$\sum_{\nu_{k_1}=1}^{\infty} \dots \sum_{\nu_{k_s}=1}^{\infty} 2^{(1-\gamma/2-a_{k_s}\gamma)\sum_{i=1}^{s} \nu_{k_i}} = [2^{(1+2a_{k_s})\gamma/2-1} - 1]^{-s}.$$

Hence

$$S \leqslant \sum_{H \in \mathcal{K}} 2^{[1+\gamma(\varkappa-1/2)]s} K_{k_s}^{\gamma} [2^{(1+2a_{k_s})\gamma/2-1} - 1]^{-s}$$

$$= \sum_{n=1}^{\infty} \sum_{s=1}^{n} 2^{[1+\gamma(\varkappa-1/2)]s} K_n^{\gamma} \binom{n-1}{s-1} [2^{(1+2a_n)\gamma/2-1} - 1]^{-s}$$

$$= 2^{1+\gamma(\varkappa-1/2)} \sum_{n=1}^{\infty} K_n^{\gamma} [2^{(1+2a_n)\gamma/2-1} - 1]^{-1} \{1 + 2^{1+\gamma(\varkappa-1/2)} [2^{(1+2a_n)\gamma/2-1} - 1]^{-1}\}^{n-1}.$$

THEOREM 2. *Given $0 < \gamma < 2$, $1 \leqslant r_n \leqslant 2$ and $\varkappa$ real, we suppose that for a $f \in L^2(Q_\omega)$,*

$$\sup_{H \in \mathcal{K}} [V_{r_{k_s}}^H(f)]^{r_{k_s}} < \infty\,(^4)$$

and that for every $H = (k_1, \dots, k_s) \in \mathcal{K}$ we have

$$\omega^H(f; x; h/2) \leqslant K_H(x) h_{k_1}^{a_1^H} \dots h_{k_s}^{a_s^H}, \tag{4.5}$$

where $K_H(x)$ does not depend on $x_{k_1}, \dots, x_{k_s}$,

$$a_i^H \geqslant a_{k_s} > \frac{2(1-\gamma)}{(2-r_{k_s})\gamma} \quad \textit{for} \quad r_{k_s} < 2, \quad \gamma < 1 \quad \textit{for } r_{k_s} = 2 \tag{4.6}$$

and

$$\sum_{n=1}^{\infty} K_n^{\gamma} [2^{\gamma-1+a_n\gamma(2-r_n)/2} - 1]^{-1} \{1 + 2^{1+\varkappa\gamma} [2^{\gamma-1+a_n\gamma(2-r_n)/2} - 1]^{-1}\}^{n-1} < \infty \tag{4.7}$$

with

$$K_n = \max_{H, k_s = n} \sqrt{\int_{Q_{E-H}} |K_H(x)|^{2-r_n} dw_{E-H}}.$$

Then the series (2.2) is convergent.

Remark. For $\gamma = 1$ and $\varkappa = \frac{1}{2}$, if we suppose that $K_n = O[(a_n/16)^n]$, then (4.7) is satisfed.

We obtain the proof of theorem 2 analogically to that of theorem 1 if we apply lemma 2 from [3]:

$$\int_{Q_H} |F^H(f; x; h)|^r dw_H \leqslant 2^s [V_r^H(f)]^r \prod_{i \in H} h_i$$

and use our lemma with $r_H = r_{k_s}$.

5. To obtain non-trivial examples of our theorems, one may take e. g.

$$f(x) = \sum_{n=1}^{\infty} a_n f_n(x_n),$$

where $\sum_{n=1}^{\infty} |a_n| < \infty$, $|f_n(x_n)| \leqslant M$ and the sequence $\{a_n\}$ and functions $f_n(x_n)$ satisfy suitable conditions.

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