Remarque 2. Les théorèmes 1 et 2 restent vrais dans le cas où (1) désigne le système de n équations différentielles $u_{xy}^{(i)}(x,y) = f^{(i)}(x, y, u^{(1)}, \ldots, u^{(n)}, u_x^{(1)}, \ldots, u_x^{(n)}, u_y^{(1)}, \ldots, u_y^{(n)})$ $(i = 1, \ldots, n)$ écrit sous la forme vectorielle [1].

Travaux cités

[1] E. Kamke, *Differentialgleichungen reeller Funktionen*, Leipzig 1930.

[2] H. Schaefer, *Eine Bemerkung über hyperbolische Systeme partieller Differentialgleichungen zweiter Ordnung*, Jahresbericht der Deutschen Mathematiker Vereinigung 58 (1955), p. 39-42.

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[1] La norme d'un vecteur $u = (u^{(1)}, \ldots, u^{(n)})$ étant donnée par la formule $|u| = \max_{1 \leqslant i \leqslant n} |u^{(i)}|$.

ANNALES
POLONICI MATHEMATICI
VI (1959)

Some properties of plane curves

by A. Sharma (Lucknow)

Recently Gołąb has generalized an old problem known since the time of Archimedes, viz. that if $\widetilde{AB}$ denotes an arbitrary arc of a parabola and $\overline{AB}$ the chord joining its extremeties and if p denotes the area of the segment and P the area of the rectangle with base AB circumscribing the parabola, then

$$\frac{p}{P} = \frac{2}{3}.$$

He is led in his investigations to a new formula of quadrature which is claimed to be an improvement on the trapezoidal formula.

In the same order of ideas we shall prove a few results partly generalizing Gołąb's results and partly of an analogous nature. We are further led to a result which in turn leads to a new quadrature formula. Part I deals with some situations analogous to those of Gołąb's while part II has been suggested by a generalization of Taylor's formula due to Kloosterman.

I. Let Γ be an arc of a curve given by the equation $y = f(x)$. If $f(x)$ possesses continuous derivatives of order k we say that the curve is of class C_k. We shall also make the following hypotheses about the curves we consider:

Hypothesis H_n. If $f(0) = f'(0) = \ldots = f^{(n+1)}(0) = 0$ we say that the curve satisfies the hypothesis H_n. The curve has then a contact of order at least $n+1$ with the x-axis at the origin O.

Hypothesis A. If the function $f(x)$ is increasing to the right of the origin in a neighbourhood of the origin, we say that the curve Γ satisfies the hypothesis A.

We can now prove

Theorem 1. *Let Γ be a curve of class C_1 satisfying H_0 and let it be infinitesimal of order $2+\alpha$ $(\alpha \geqslant 1)$ and let P and Q be two points on it with abscissae h and $\frac{1}{2}h$ respectively. Let the straight line through O drawn parallel* *to the tangent at Q meet the ordinate at P in R. If L_1 is the lentil between the chord $\overline{QP}$ and the arc $\overset{\frown}{QP}$ and L_0 the lentil between the chord $\overline{OQ}$ and the arc $\overset{\frown}{OQ}$, then*

$$\lim_{h\to 0}\frac{m(L_1)-m(L_0)}{m(T)}=\frac{2^a(a-1)+1}{(a+3)(2^{a+1}-a-2)} \tag{1}$$

where T denotes the triangle OPR and $m(T)$ denotes its area.

In order to prove this we observe that

$$m(T)=\tfrac{1}{2}h[f(h)-f(0)-hf'(\tfrac{1}{2}h)]$$

and

$$m(L_1)-m(L_0)=\frac{1}{2}\cdot\frac{1}{2}h\left[f\left(\frac{1}{2}h\right)+f(h)\right]-\int_{h/2}^{h}f(x)\,dx-$$

$$-\left[\frac{1}{2}\cdot\frac{1}{2}h\cdot f\left(\frac{h}{2}\right)-\int_0^{h/2}f(x)\,dx\right]=\frac{1}{4}hf(h)-\left[\int_{h/2}^{h}f(x)\,dx-\int_0^{h/2}f(x)\,dx\right].$$

Since $f(x)=x^{2+a}(g+\varepsilon(x))$ where $\varepsilon(x)\to 0$ as $x\to 0$, $g\neq 0$ and since $f(x)$ is of class C_1, we can easily see, as Gołąb has done, that $\varepsilon'(x)$ exists for $x>0$ and that $\lim\limits_{x\to 0}x\varepsilon'(x)=0$. Now

$$m(T)=\tfrac{1}{2}h\big[(g+\varepsilon(h))h^{a+2}-h\{(a+2)(\tfrac{1}{2}h)^{a+1}g+$$
$$+(\tfrac{1}{2}h)^{a+1}\varepsilon'(\tfrac{1}{2}h)+(a+1)(\tfrac{1}{2}h)^{a}\varepsilon(h)\}\big],$$

so that

$$\lim_{h\to 0}\frac{m(L_1)-m(L_0)}{m(T)}=\frac{\dfrac{1}{4}-\dfrac{1}{a+3}\left(1-\dfrac{1}{2^{a+2}}\right)}{\dfrac{1}{2}\left(1-\dfrac{a+2}{2^{a+1}}\right)},$$

which proves (1).

Denoting the right side of (1) by $\Phi(a)$, we have

$$\Phi(1)=\tfrac{1}{4}=\Phi(2),\qquad \lim_{a\to\infty}\Phi(a)=\tfrac{1}{2}.$$

Also

$$\Phi'(a)(a+3)^2(2^{a+1}-a-2)^2$$
$$=(2^{a+2}-1)(2^{a+1}-a-2)+(a+3)[2^a(a-1)+1]-(a+3)\log 2\,[2^a a(a+1)],$$

which shows that there exists a number a_0 such that for $a>a_0$, $\Phi(a)$ is increasing.

THEOREM 2. *Let Γ be a curve of class C_n satisfying H_{n-1} and let $f(x)$ be infinitesimal of order $n+a$ and $f^{(n)}(x)$ of order a $(a\geqslant 0)$ and let B be a point on it with abscissa h. If P is another point on the curve with abscissa Θh such that $f(h)=\dfrac{h^n}{n!}f^{(n)}(\Theta h)$, then*

$$\lim_{h\to 0}\frac{m(L)}{m(T)}=\frac{n+a-1}{n+a+1}\cdot\frac{\binom{n+a}{n}^{(n+a)/a}}{\binom{n+a}{n}^{(n+a-1)/a}-1}. \tag{2}$$

Since

$$m(L)=\frac{1}{2}hf(h)-\int_0^h f(x)\,dx$$
$$=\frac{1}{2}h^{n+a+1}(g+\varepsilon(h))-\int_0^h x^{n+a}(g+\varepsilon(x))\,dx$$
$$=\frac{1}{2}h^{n+a+1}(g+\varepsilon(h))-\frac{h^{n+a+1}}{n+a+1}g-h^{n+a+1}\vartheta^{n+a}\varepsilon(\vartheta h)$$

and the line OB being given by

$$yh=f(h)x$$

we have

$$m(T)=\tfrac{1}{2}|\Theta hf(h)-hf(\Theta h)|=\tfrac{1}{2}h^{n+a+1}|g(\Theta-\Theta^{n+a})+\Theta\varepsilon(h)-\varepsilon(\Theta h)|.$$

Since $\lim\limits_{h\to 0}\Theta=\binom{n+a}{n}^{-1/a}$, we get the required result.

Denoting the right side of (2) by $\Phi_n(a)$, and observing that

$$\lim_{a\to 0}\binom{n+a}{n}^{-1/a}=\exp\left(-1-\frac{1}{2}-\ldots-\frac{1}{n}\right),$$

we see that

$$\lim_{a\to 0}\Phi_n(a)=\frac{n-1}{n+1}\cdot\frac{\exp\left[n\left(1+\dfrac{1}{2}+\ldots+\dfrac{1}{n}\right)\right]}{\exp\left[(n-1)\left(1+\dfrac{1}{2}+\ldots+\dfrac{1}{n}\right)\right]-1}.$$

Also $\lim\limits_{a\to\infty}\Phi_n(a)=1$.

When $n=1$, $a=1$, the curve is a parabola and we get the classical case of Archimedes.

THEOREM 3. *Let the function $f(x)$ satisfy the condition H_0, A and also C_1. Let $f(x)$ be infinitesimal of order a and $f'(x)$ of order $a-1$ $(a \geqslant 1)$. Let P be any point on the curve Γ having abscissa h and let $OAPC$ be the rectangle with OP as diagonal. If Q is the point on the arc Γ where the tangent is parallel to OP and if R_1 denotes the rectangle with one side AC and the opposite side passing through Q, then*

$$\lim_{h\to 0}\frac{m(L)}{m(R_1)} = \frac{(a-1)a^{a/(a-1)}}{2(a+1)[a^{a/(a-1)}-a+1]}. \tag{3}$$

(If the curve is a parabola, we get the right side for $a=2$, to be $2/9$).

If the function $f(x)$ is infinitesimal of order a near the origin, then for $x \geqslant 0$, $f(x) = x^a\big(g+\varepsilon(x)\big)$, where $g \neq 0$, $\lim\limits_{x\to 0}\varepsilon(x) = 0$. P is the point $(\Theta h, f(\Theta h))$. The line AC is given by the equation

$$-hy = f(h)(x-h)$$

and so

$$m(R_1) = |hf(\Theta h) - hf(h)(1-\Theta)|.$$

Hence

$$\lim_{h\to 0}\frac{m(L)}{m(R_1)} = \lim_{h\to 0}\frac{\frac{1}{2}hf(h) - \int\limits_0^h f(x)\,dx}{\Theta f(\Theta h) - hf(h)(1-\Theta)}$$

$$= \lim_{h\to 0}\frac{\frac{1}{2}g - g/(a+1) + \frac{1}{2}\varepsilon(h) - \vartheta^a\varepsilon(\vartheta h)}{\Theta^a\{g+\varepsilon(\Theta h)\} + \{g+\varepsilon(h)\}(1-\Theta)},$$

which at once leads to the desired result since $\lim\limits_{h\to 0}\Theta = a^{-1/(a-1)}$. When $a\to 1$, the right side of (3), call it $\Phi(a)$, tends to 0 and for $a\to\infty$, $\Phi(a)\to\infty$.

THEOREM 4. *Let $f(x)$ satisfy the hypothesis H_0 and A and C_1. Let $f(x)$ be of the order of smallness a and let $f'(x)$ be of the order of smallness $a-1$ near the origin. Let P be a point on Γ having abscissa h and let the tangent to Γ at P meet the tangent at the origin at P'. If C divides PP' so that $PC/CP' = a$ and if T denotes the triangle OPC, then*

$$\lim_{h\to 0}\frac{m(L)}{m(T)} = 1. \tag{4}$$

We know $m(L)$ from the previous theorem. The co-ordinates of P' are obtained from

$$y - f(h) = (x-h)f'(h)$$

on putting $y=0$. Then P' is $\big((hf'(h)-f(h))/f'(h), 0\big)$, so that C is given by (x_1, y_1) where

$$x_1 = \frac{(a+1)hf'(h) - af(h)}{(a+1)f'(h)}, \qquad y_1 = \frac{f(h)}{a+1}.$$

Again, the line OP has the equation

$$hy = xf(h),$$

so that

$$m(T) = \tfrac{1}{2}[x_1 f(h) - hy_1]$$

$$= \tfrac{1}{2}\frac{f(h)[(a+1)hf'(h) - af(h)] - hf(h)f'(h)}{(a+1)f'(h)}.$$

Hence

$$\frac{m(L)}{m(T)} = \frac{2h^{a+1}(a+1)f'(h)[\frac{1}{2}g - g/(a+1) + \frac{1}{2}\varepsilon(h) - \Theta^a\varepsilon(\Theta h)]}{a[hf'(h)-f(h)]}$$

$$\to \frac{2(a+1)\big(\frac{1}{2} - 1/(a+1)\big)a}{a(a-1)} = 1 \quad \text{as} \quad h\to 0.$$

This theorem can lead to a new formula of quadrature as the corresponding theorem of Gołąb has done.

II. In this we propose to extend a theorem of Gołąb by using n^{th} differences. The starting point of these results is a theorem of Kloosterman, which may be treated as a generalization of Taylor's formula.

Denote

$$\Delta_h^r f(x) = \sum_{i=0}^{r-1}\binom{r}{i}(-1)^{r-i}f(x+ih).$$

Then if

$$\left(\frac{e^x-1}{x}\right)^r = \sum_{n=0}^{\infty}P_n(r)x^n,$$

Kloosterman's formula can be written thus:

$$h^{-r}\Delta_h^r f(x) = \sum_{n=0}^{k-1}P_n(r)h^n f^{(r+n)}(x) + R_k \tag{5}$$

where r, k are positive integers and

$$R_k = P_k(r)h^k f^{(r+k)}(x+\Theta rh) \quad (0<\Theta<1).$$

We can then prove the following

THEOREM 5. *Let Γ be a curve satisfying H_n and A and let it be infinitesimal of order $n+a$ $(a \geqslant 1)$ near the origin. If $P_1, P_2, \ldots, P_{n+1}$ are $n+1$ points on it with abscissae $h, 2h, \ldots, (n+1)h$, L_{i-1} the lentil corresponding to arc $\overset{\frown}{P_{i-1}P_i}$ as shown in the figure 1, and R_{i-1} the rectangle with the chord $\overline{P_{i-1}P_i}$ as diagonal and with sides parallel to the axes, then*

$$\lim_{h\to 0}\frac{\Delta_1^n m(L_0)}{\Delta_1^n m(R_0)} = \frac{a}{(n+a+1)(n+1)}\cdot\frac{\Delta_1^n 1^{n+a}}{\Delta_1^n 1^{n+a-1}} \tag{6}$$

where

$$\Delta_1^n m(L_0) = \sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}m(L_\nu)$$

and $m(L_\nu)$ denotes the area of the lentil L_ν.

In order to prove this we observe that by hypothesis

$$f(x) = x^{n+a}\big(g+\varepsilon(x)\big)$$

where $\varepsilon(x)\to 0$ as $x\to 0$ and $g\neq 0$. Now

$$m(L_{i-1}) = hf(ih)-\int_{(i-1)h}^{ih} f(x)\,dx$$

and

$$m(R_{i-1}) = h\{f(ih)-f(ih-h)\},$$

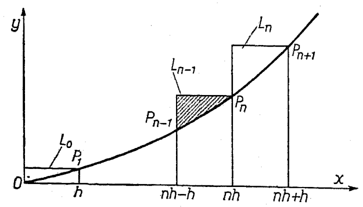

Fig. 1.

so that

$$\Delta_1^n m(L_0) = h\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}f(ih)-\Delta_h^n F(0)$$

where

$$F(x) = \int_0^x f(x)\,dx = \frac{gx^{n+a+1}}{n+a+1}+\int_0^x x^{n+a}\varepsilon(x)\,dx$$

$$= \frac{gx^{n+a+1}}{n+a+1}+x^{n+a+1}\vartheta^{n+a}\varepsilon(\vartheta x).$$

Also

$$\Delta_1^n m(R_0) = h\Delta_h^{n+1}f(0) = h^{n+a+1}\Big[g\Delta_1^{n+1}0^{n+a}+\sum_{i=0}^{n+1}(-1)^{n-i}\binom{n}{i}\varepsilon(ih)\Big].$$

Hence

$$\lim_{h\to 0}\frac{\Delta_1^n m(L_0)}{\Delta_1^n m(R_0)} = \frac{\Delta_1^n 1^{n+a}-\Delta_1^{n+1}0^{n+a+1}/(n+a+1)}{\Delta_1^{n+1}0^{n+a}}.$$

Since $\Delta_1^{n+1}0^{n+a+1} = (n+1)\Delta_1^n 1^{n+a}$, we get the result of the theorem.

For $n=0$, we get Gołąb's result. For $n=1$, we have

$$\lim_{h\to 0}\frac{m(L_1)-m(L_0)}{m(R_1)-m(R_0)} = \frac{(2^{a+1}-1)a}{2(a+2)(2^a-1)}. \tag{7}$$

For $a=1$, since $\Delta_1^{n+1}0^{n+1} = (n+1)!$, and $\Delta_1^{n+1}0^{n+2} = \frac{1}{2}(n+1)\cdot(n+2)!$ and $\Delta_1^n 1^{n+1} = \frac{1}{2}(n+2)\cdot(n+1)!$[1], we have the right side of (7) $= \frac{1}{2}$.

We can easily show that the right side of (7) is an increasing function of a for $a\geqslant 1$ and that it increases from $\frac{1}{2}$ to 1 as a varies from 1 to ∞. But in the general case we do not know if such a property is true.

A more general situation than that of theorem 1 above can be stated.

THEOREM 6. *Let Γ be a curve satisfying H_{n-1}, let P be a point with abscissa h and let P_i have the abscissa $ih/(n+1)$ $(i=1,2,\ldots,n)$. If A is the point*

$$\Big\{h, \frac{h^n}{n!}f^{(n)}\Big(\frac{h}{n+1}\Big)\Big\},$$

if the lentils between the chords $\overline{P_iP_{i+1}}$ and the arc $\overset{\frown}{P_iP_{i+1}}$ are L_i $(i=0,1,\ldots,n)$ and if the triangle OAP is denoted by T, we have

$$\lim_{h\to 0}\frac{\Delta_1^n m(L_0)}{m(T)} \tag{8}$$

$$= \left(\frac{1}{n+1}\right)^{n+2+a}\frac{\left[\frac{1}{2}\Delta_1^n(0^{n+1+a}+1^{n+1+a})-\frac{1}{n+2+a}\Delta_1^n 0^{n+2+a}\right]}{\frac{1}{2}\left[1-\frac{(n+2+a)\ldots(2+a)}{n!(n+1)^{1+a}}\right]}.$$

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[1] These results can easily be seen from (5). Thus to show that $\Delta_1^{n+1}0^{n+2} = \frac{1}{2}(n+1)(n+2)!$, we put in (5), $f(x) = x^{n+2}$, $h=1$, and $n+1$ for $r, k = 1$. Then

$$\Delta_1^{n+1}0^{n+2} = P_1(n+1)\cdot(n+2)!,$$

But from $P_0(r) = 1, P_n(1) = 1/(n+1)!$ and the relation

$$P_n(r+1) = \sum_{i=0}^{n}P_{n-i}(r)P_i(1),$$

we get

$$P_1(n+1) = P_1(n)+\tfrac{1}{2}.$$

Hence the result $P_1(n+1) = \frac{1}{2}(n+1)$, which gives the desired results.

To prove this we observe that

$$m(L_i)=\frac{1}{2}\cdot\frac{h}{n+1}\left[f\left(\frac{ih}{n+1}\right)+f\left(\frac{i+1}{n+1}h\right)\right]-\int\limits_{ih/(n+1)}^{(i+1)h/(n+1)}f(x)\,dx,$$

so that

$$\Delta_1^n m(L_0)=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}m(L_i)$$

$$=\frac{1}{2}\cdot\frac{h}{n+1}\left[\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f\left(\frac{ih}{n+1}\right)+\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f\left(\frac{i+1}{n+1}h\right)\right]-$$
$$-\Delta_{h/(n+1)}^{n+1}F(0)$$

where $F(x)=\int\limits_0^x f(x)\,dx$. Also

$$m(T)=\frac{h}{2}\left[f(h)-\frac{h^n}{n!}f^{(n)}\left(\frac{h}{n+1}\right)\right].$$

Since $f(x)=x^{n+1+a}(g+\varepsilon(x))$, the result of the theorem is easily obtained.

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ANNALES
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Sur la limitation des solutions d'un système d'équations intégrales de Volterra

par Z. BUTLEWSKI (Poznań)

§ 1. M. T. Sato a publié récemment [1], sous le même titre que plus haut, un intéressant article dans lequel l'auteur considère un système d'équations intégrales linéaires

$$(1.1)\qquad u_j(x)=\sum_{k=1}^{n}\int\limits_a^x a_{jk}(x,t)u_k(t)\,dt+b_j(x)\qquad (j=1,2,\ldots,n),$$

où les fonctions $a_{jk}(x,t)$ $(j,k=1,2,\ldots,n)$ sont continues dans le domaine D: $a\leqslant t\leqslant x<\infty$ et $b_j(x)$ $(j=1,2,\ldots,n)$ bornées et continues dans J: $a\leqslant x<\infty$. En supposant que

$$(1.2)\qquad \max_{j,k}|a_{jk}(x,t)|\leqslant A(x,t),\qquad \sum_{j=1}^{n}|b_j(x)|\leqslant B,$$

où la fonction $A(x,t)$ et dérivée partielle $A'_x(x,t)$ sont continues et non négatives dans le domaine D, B est une constante dans l'intervalle J, T. Sato a démontré l'inégalité (¹)

$$(1.3)\qquad \sum_{j=1}^{n}|u_j(x)|\leqslant B\exp\Big(n\int\limits_a^x A(x,t)\,dt\Big)\qquad (x\geqslant a)$$

et il a remarqué aussi que l'inégalité

$$(1.4)\qquad \overline{\lim_{x\to\infty}}\int\limits_a^x A(x,t)\,dt<\infty$$

est, entre autres, une condition suffisante pour que la solution $u_1(x),\ldots,u_n(x)$ du système (1.1) soit bornée pour de grandes valeurs de la variable x.

(¹) Dans le travail de T. Sato [1] n ne figure pas (voir p. 274, (6)).