$\alpha < \omega_\mu$. Suppose some $t \in \bigcap_{\beta<\alpha} T_{i_\beta}^{(\beta)}$. Then $t|\alpha \in S_\alpha$ and $t|\alpha = s|\alpha$ what implies $s|\alpha \in S_\alpha$. Now it follows from

$$m \bigcap_{\beta<\alpha} T_{i_\beta}^{(\beta)} \geqslant 1 - \sum_{\beta<\alpha} m(T^{(\beta)} \cup T_{1-i_\beta}^{(\beta)}) = 1$$

that $\bigcap_{\beta<\alpha} T_{i_\beta}^{(\beta)}$ is not empty for $\alpha < \omega_\mu$.

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ON THE REPRESENTATION OF FIELDS AS FINITE UNIONS OF SUBFIELDS

BY

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The purpose of this paper is to prove the following theorem:

THEOREM. *An algebraic field cannot be represented as a union (in the sense of the theory of sets) of a finite number of proper subfields.*

LEMMA 1. *If $G, G_1, G_2, \ldots, G_n$ are groups, $G = \bigcup_{i=1}^{n} G_i$,*

(1) $$G \neq \bigcup_{i=1}^{j-1} G_i \cup \bigcup_{i=j+1}^{n} G_i \quad \text{for} \quad 1 \leqslant j \leqslant n$$

and G is an infinite set, then $\bigcap_{i=1}^{n} G_i$ is infinite.

Proof. We shall prove by induction that for each $k \leqslant n$ there exists such a sequence $i_1, i_2, \ldots, i_k$ of different natural numbers $\leqslant n$ that

(2) $$\bigcap_{j=1}^{k} G_{i_j} \text{ is infinite.}$$

For $k=1$, (2) follows from the fact that $\bigcup_{i=1}^{n} G_i$ is infinite. Suppose that (2) holds for $k<n$ and let $\{a_n\}$ be an infinite sequence of different elements of the group $\bigcap_{j=1}^{k} G_{i_j}$. By (1) $G \neq \bigcup_{j=1}^{k} G_{i_j}$ and so there exists a $b \in G - \bigcup_{j=1}^{k} G_{i_j}$.

Consequently $a_n b \notin \bigcup_{j=1}^{k} G_{i_j}$ and $a_n b \in \bigcup_{i \neq i_1, i_2, \ldots, i_k} G_i$. Hence there exists a number $i_{k+1} \neq i_1, i_2, \ldots, i_k$ such that infinitely many elements of the sequence $\{a_n b\}$ belong to $G_{i_{k+1}}$. Let $a_{m_n} b \in G_{i_{k+1}}$ $(n=1,2,\ldots)$. Then $a_{m_n} a_{m_1}^{-1} = (a_{m_n} b)(a_{m_1} b)^{-1} \in G_{i_{k+1}}$ and by the definition of $\{a_n\}$: $a_{m_n} a_{m_1}^{-1} \in \bigcap_{j=1}^{k} G_{i_j}$. Then $a_{m_n} a_{m_1}^{-1} \in \bigcap_{j=1}^{k+1} G_{i_j}$, and $\bigcap_{j=1}^{k+1} G_{i_j}$ is an infinite set, which completes our inductive proof.

For $k = n$ we obtain Lemma 1.

LEMMA 2. *If* $L, L_1, L_2, \ldots, L_n$ *are linear spaces over a field* K, $L = \bigcup_{i=1}^{n} L_i$, $L \neq \bigcup_{i=1}^{j-1} L_i \cup \bigcup_{i=j+1}^{n} L_i$ $(1 \leqslant j \leqslant n, n \geqslant 2)$, *then* K *contains less than* n *elements.*

Proof. Let us suppose that there exist different elements $a_1, a_2, \ldots, a_n \in K$. Let $a \in L_1 - \bigcup_{i=2}^{n} L_i$, $b = L_2 - L_1$. Then $a_i a + b \notin L_1$, because otherwise it would be $b \in L_1$. One has $a_i a + b \neq a_j a + b$ for $i \neq j$, because $a \neq 0$. Among n elements $a_i a + b$ two at least must be contained in the same subspace L_k $(k > 1)$. E. g. let $a_{i_1} a + b \in L_k$, $a_{i_2} a + b \in L_k$. Consequently $(a_{i_1} - a_{i_2}) a \in L_k$, and $a \in L_k$, because $a_{i_1} \neq a_{i_2}$. This contradicts the definition of element a.

Proof of the theorem. For finite fields it follows from the fact that all such fields are generated by one element.

For infinite fields we give an inductive proof with respect to the number n of subfields. For $n = 1$ the theorem is trivial. Suppose that it holds for some $n-1$ and suppose that $K = \bigcup_{i=1}^{n} K_i$ is a decomposition of the field K. By the induction hypothesis $K \neq \bigcup_{i=1}^{j-1} K_i \cup \bigcup_{i=j+1}^{n} K_i$ $(1 \leqslant j \leqslant n)$. From lemma 1 it follows that $\bigcap_{i=1}^{n} K_i$ is an infinite set. But the fields K and K_i $(1 \leqslant i \leqslant n)$ are linear spaces over the field $\bigcap_{i=1}^{n} K_i$; thus we obtain a contradiction of lemma 2, which implies that $\bigcap_{i=1}^{n} K_i$ has less than n elements.

Remark. A ring of real numbers with the unity can be represented as a union of three proper subrings with the unity:

$$P = \{a + b\sqrt[3]{D} + c\sqrt[3]{D^2}:\ a \text{ is an integer, } b \equiv c \equiv 0 \pmod 2\},$$

$$P_1 = \{a + b\sqrt[3]{D} + c\sqrt[3]{D^2}:\ a \text{ is an integer, } b \equiv 0 \pmod 4,\ c \equiv 0 \pmod 2\},$$

$$P_2 = \{a + b\sqrt[3]{D} + c\sqrt[3]{D^2}:\ a \text{ is an integer, } b \equiv 0 \pmod 2,\ c \equiv 0 \pmod 4\},$$

$$P_3 = \{a + b\sqrt[3]{D} + c\sqrt[3]{D^2}:\ a \text{ is an integer, } b \equiv c \pmod 4,\ c \equiv 0 \pmod 2\},$$

where D is not a cube of an integer and $64 \nmid D$. It is easy to verify that P_i $(i = 1, 2, 3)$ are proper subrings of the ring P and $P = P_1 \cup P_2 \cup P_3$.

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ON THE FIRST COUNTABILITY AXIOM FOR LOCALLY COMPACT HAUSDORFF SPACES

BY

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Recently A. Hulanicki [1 and 2] has shown that:

Every locally compact topological group of $\aleph_1$ elements is metric.

This theorem also follows from the Birkhoff-Kakutani theorem on the metrizability of groups, and the following lemma. Hulanicki's methods may be modified to prove this lemma, but the proof I outline below is somewhat more elementary.

If U is an open subset of a compact (= bicompact) Hausdorff space S of $\aleph_1$ elements, then the first countability axiom holds true at some point of U.

Indication of proof. Let $a = \{a_z\}_{z<\omega_1}$ be a well-ordering of S. If the first countability axiom is false at every point of U, there exists a well-ordered sequence $\beta = \{U_z\}_{z<\omega_1}$ of non-vacuous open subsets of S such that

(1) $\overline{U}_1 \subset U - a_1$,

(2) if $z < \omega_1$ and $z-1$ exists, then $\overline{U}_z \subset U_{z-1}$ and U_z contains a point of $\bigcap\{\overline{U}_y\}_{y<z}$ but $\overline{U}_z$ does not contain a_z,

and

(3) if $z < \omega_1$ and $z-1$ does not exist, then U_z contains a point of $\bigcap\{\overline{U}_y\}_{y<z}$ but $\overline{U}_z$ does not contain a_z.

Now S is regular; so β exists because if $\bigcap\{\overline{U}_y\}_{y<z}$ contains only a_z, then, as is easily proved, $a_z = \bigcap\{U_y\}_{y<z}$ and the collection of intersections of finite subsequences of $\{U_y\}_{y<z}$ forms a countable basis for S at a_z. But $\bigcap\{\overline{U}_z\}_{z<\omega_1}$ is vacuous which contradicts the existence of β.