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ON THE CONVERSE OF THE BANACH "FIXED-POINT PRINCIPLE"

BY

C. BESSAGA (WARSAW)

Banach has proved the following "fixed-point principle":

If a mapping U of a complete metric space into itself satisfies the Lipschitz condition with a constant <1, then U has a unique fixed point.

Clearly, each iteration of U satisfies also the Lipschitz condition with a constant <1, whence the following statement holds true:

(S) *If a mapping U of a complete metric space into itself satisfies the Lipschitz condition with a constant <1, then each iteration of U has a unique fixed point.*

The statement (S) admits the following converse:

THEOREM 1. *Suppose U is a mapping of an abstract set X into itself such that each iteration U^n $(n=1,2,\ldots)$ of U has a unique fixed point. Let K be any number with $0<K<1$. Then there exists a complete metric ϱ for X such that U satisfies the Lipschitz condition with the constant K.*

Proof. Suppose that the assumptions of Theorem 1 are satisfied. Let us define in the set X two relations:

(1) $x\sim y$ if and only if either $x=y$ or for a positive integer p, $U^{p-1}(x) \neq U^p(x) = U^p(y) \neq U^{p-1}(y)$.

(2) $x\approx y$ if and only if there are positive integers p and q such that $U^p(x) = U^q(y)$.

(Here we assume $U^0(x) = x$ for every $x\in X$).

It is obvious that the two relations are of equivalence type and that $x\sim y$ implies $x\approx y$.

For arbitrary $x\in X$ let $[x] = \{y\in X: y\sim x\}$.

The set of all classes $[x]$, where $x\in X$, will be denoted by $[X]$.

For arbitrary $[x]\in[X]$ let

$$[[x]] = \{[y]\in[X]: \text{ there are } x_1\in[x], y_1\in[y] \text{ such that } x_1\approx y_1\}.$$

The family of all sets $[[x]]$, where $[x]\in[X]$, will be denoted by $\mathfrak{K}$. Let us define in $[X]$ the relation.

(3) $[x]\leqslant[y]$ if and only if there exist $x_1\in[x]$, $y_1\in[y]$ and a non-negative integer k such that $U^k(x_1)=y_1$.

It is easy to verify that the relation $\leqslant$ orders each of the sets $[[x]]$ in a type $\leqslant\omega^*+\omega$.

Applying the following particular case of the axiom of choice:

(Ch) *If $\mathfrak{K}$ is a family of disjoint ordered sets of types $\leqslant\omega^*+\omega$ then there exists a function $H(Z)$ defined on $\mathfrak{K}$ such that $H(Z)\in Z$ for every $Z\in\mathfrak{K}$,*

we can easily define on the set X an integer valued function $f(x)$ which satisfies conditions:

(4) if $x\sim y$, then $f(x)=f(y)$,

(5) if $x\in[y]^*$, then $f(y)=f(x)+1$

($[y]^*$ denotes the successor of $[y]$).

For every $x\in X$ let us put $U^\infty(x)=$ the fixed point of the mapping U.

The metric ϱ which fulfils the assertion of Theorem 1 can be defined by the formula:

$$\varrho(x,x')=\sum_{i=1}^{p}K^{f(x)+i}+\sum_{i=1}^{q}K^{f(x')+i},$$

where p, q are the smallest of the number $0,1,2,\ldots,\infty$ for which $U^p(x)=U^q(x')$ (we mean $\sum_{i=1}^{0}K^{f(x)+i}=0$).

Remark. *Theorem 1 and the axiom* (Ch) *are equivalent.*

Proof. We shall show that Theorem 1 implies (Ch). Note that the only order types less than $\omega^*+\omega$ are $1,2,\ldots,\omega,\omega^*$. Since to every set which is ordered in such a type one can effectively assign one of its elements, we may assume that $\mathfrak{K}$ consists of sets of type $\omega^*+\omega$.

Denote by X' the sum of all the sets belonging to the family $\mathfrak{K}$. Let $X=X'+\{a\}$, where a is an arbitrary object which does not belong to X'. It is easy to see that the mapping

$$U(x)=\begin{cases}x^* & \text{for} \quad x\in X',\\ a & \text{for} \quad x=a,\end{cases}$$

where x^* denotes the successor of x, fulfils the assumptions of Theorem 1. Let ϱ be the metric for X fulfilling the assertion of this Theorem.

The choice function $H(Z)$ can be defined as follows:

$$H(Z)=\max\{x\in Z:\varrho(x,a)>1\},$$

where the symbol max denotes the greatest element of a set ordered in the type ω^*.

MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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