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THE EXCEPTIONAL VALUES OF MEROMORPHIC FUNCTIONS

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§ 1. The notion of cluster set was first formulated explicitly by Painlevé [17] in his well-known Stockholm lectures of 1895 on differential equations, where the cluster set was introduced as a descriptive device to characterise in an intuitive way the behaviour of an analytic function in the neighbourhood of a singularity in terms of the set of all limiting values of the function at the singularity. For a function $f(z)$ defined in a domain D we define the *cluster set* $C(f,P)$ of $f(z)$ at a point P of the frontier of D to be the set of all values a such that there exists a sequence $\{z_n\}$ of points of D with $f(z_n)\to a$ as $z_n\to P$. By its definition as an aggregate of limit points, the set $C(f,P)$ is closed, and if $f(z)$ is continuous, then $C(f,P)$ is also connected.

The concept of a cluster set is thus fairly general and is applicable to a wide range of topological mappings. Historically, the theory of cluster sets has developed within the theory of functions, to which it has added a number of theorems of a striking sort, and, for the most part, the approach to the theory of cluster sets has been function-theoretic, since the results were to be applied directly in the theory of functions. The present-day approach to the theory is more general, with the hypotheses less deeply rooted in the theory of functions, although the development has not yet reached the point of starting from the most general topological ideas and then particularising when a more concrete result is wanted.

In this lecture [1], we shall describe the application of some of the more recent topological ideas to a generalisation of the Gross-Iversen theorem. There are two results of topological character which are related to the form of the Gross-Iversen theorem to be discussed here. Collingwood [3] has shown that if a function $f(z)$ is meromorphic in $|z|<1$,

[1] Presented to the Second Conference on Analytic Functions, held in Lublin 2-6. IX. 1958.

then, for each point P on $|z| = 1$, there exists a curve γ in $|z| < 1$ terminating at P such that $C(f, P) = C_\gamma(f, P)$, where $C_\gamma(f, P)$ denotes the *curvilinear cluster set along* γ of $f(z)$ at P, which is defined in the same way as $C(f, P)$ with the added restriction that the sequence used in the definition of $C(f, P)$ lies on γ. As was pointed out in [11], the proof of Collingwood's theorem does not require any assumptions whatever concerning $f(z)$, so that the result is valid for an arbitrary function defined in $|z| < 1$. The curve γ depends on P, of course, but the extent to which a fixed form of curve can be prescribed, for example in the case that γ is a radius, is given by the following theorem of Collingwood [4]:

THEOREM 1. *If $f(z)$ is continuous in $|z| < 1$, then $C(f, e^{i\theta}) = C_\varrho(f, e^{i\theta})$ for a residual set of points $e^{i\theta}$ on $|z| = 1$, where $C_\varrho(f, e^{i})$ denotes the cluster set of $f(z)$ along the radius drawn to $e^{i\theta}$.*

Since the complement of a residual set is a set of the first category' it follows that the radial cluster set must coincide with the cluster set $C(f, P)$ "almost everywhere" in a topological sense. The study of curvilinear cluster sets has led to significant results in the theory of prime ends; we can do little more here than to refer the reader to Collingwood [5] for an account of these developments. The study of curvilinear cluster sets has led to the discovery of striking and useful results, one of which we shall describe before discussing the theorem of Gross and Iversen. The function

$$F(z) = \exp\frac{z+1}{z-1} \tag{1}$$

has the property that $|F(z)| = e^{-1}$ as $z \to 1$ along the curve $\gamma_1\colon |z-\frac{1}{2}| = \frac{1}{2}$ and $F(z) \to 0$ as $z \to 1$ along the curve γ_2: $\arg z = 0$, so that $C_{\gamma_1}(F, 1) \cap C_{\gamma_2}(F, 1) = \{|w| = e^{-1}\} \cap \{0\} = \Phi$. Given a function $f(z)$, defined in $|z| < 1$, we may ask how dense is the set of points P on $|z| = 1$ such that for each P there exists a pair of arcs γ_1 and γ_2 terminating at P with the property that

$$C_{\gamma_1}(f, P) \cap C_{\gamma_2}(f, P) = \Phi. \tag{2}$$

This problem has been solved recently by Bagemihl [1] who has shown that the set of points on $|z| = 1$ with the property (2) is at most denumerable.

§ 2. Before stating the theorem of Gross and Iversen, we must define an important subset of $C(f, P)$, the so-called *boundary cluster set* $C_B(f, P)$

of $f(z)$ at P, which is defined as follows. If $P = e^{i\theta_0}$, we form the set

$$C(f, 0 < |\theta-\theta_0| < \eta) = \bigcup_{0<|\theta-\theta_0|<\eta} C(f, e^{i\theta}) \tag{3}$$

for an arbitrary $\eta > 0$, and define $C_B(f, e^{i\theta_0})$ to be the intersection

$$C_B(f, e^{i\theta_0}) = \bigcap_{\eta>0} \bar{C}(f, 0 < |\theta-\theta_0| < \eta), \tag{4}$$

where $\bar{C}(f, 0 < |\theta-\theta_0| < \eta)$ denotes the closure of the set (3). The first significant result concerning the sets $C(f, P)$ and $C_B(f, P)$ was given by Iversen ([8], [9]):

THEOREM 2. *If $f(z)$ is meromorphic in $|z| < 1$, then for every point $e^{i\theta}$ on $|z| = 1$,*

$$\operatorname{Fr} C(f, e^{i\theta}) \subseteq C_B(f, e^{i\theta}), \tag{5}$$

where $\operatorname{Fr} C(f, e^{i\theta})$ denotes the frontier of $C(f, e^{i\theta})$.

The theorem of Gross ([6], [7]) and Iversen ([8]-[10]) yields a relationship between (5) and Picard's theorem:

THEOREM 3. *Under the hypotheses of Theorem 2, every value of $C(f, e^{i\theta}) - C_B(f, e^{i\theta})$ is assumed infinitely often in every neighbourhood of $e^{i\theta}$, with two possible exceptions. If there are any exceptional values, they are asymptotic values of $f(z)$ at $e^{i\theta}$.*

We remark first that Theorems 2 and 3 are meaningless unless the point $e^{i\theta}$ is a singularity of $f(z)$, for otherwise the sets $C(f, e^{i\theta})$ and $C_B(f, e^{i\theta})$ are identical and consist of a single point. We illustrate these theorems by means of the function (1); here $C(F, 1) = \{|w| \leqslant 1\}$ and $C_B(F, 1) = \{|w| = 1\}$, and the fact that every value of $C(F, 1) - C_B(F, 1)$, except for $w = 0$, is assumed infinitely often in every neighbourhood of $z = 1$ follows at once from the classical form of Picard's theorem together with the observation that $|F(z)| < 1$ in $|z| < 1$.

Theorem 3 is limited essentially to the case of an isolated singularity, as the following example shows. Let $w = b(z)$ be a Blaschke product with the property that every point of $|z| = 1$ is a limit point of zeros of $b(z)$; thus every point $e^{i\theta}$ of $|z| = 1$ is a singularity of $b(z)$, so that, by a theorem of Seidel [18], the cluster set $C(b, e^{i\theta})$ is the closed circle $|w| \leqslant 1$ for each $e^{i\theta}$. Consequently, the set $C_B(b, e^{i\theta})$ is also the closed circle $w \leqslant 1$ for each $e^{i\theta}$, so that the set $C(b, e^{i\theta}) - C_B(b, e^{i\theta})$ appearing in Theorem 3 is empty for each $e^{i\theta}$. Attempts have been made during the past twenty years to modify the set $C_B(f, P)$ by excluding from the union (3) various sets of points on $|z| = 1$, but as the example $b(z)$ shows, the results obtained are applicable only to rather restricted types of singularities; for an account of these difficulties the reader is referred to [11].

Having defined the radial cluster set $C_\varrho(f, e^{i\theta})$ in § 1, we proceed next to form the union

$$C_\varrho(f, 0<|\theta-\theta_0|<\eta; E) = \bigcup_{\substack{0<|\theta-\theta_0|<\eta \\ e^{i\theta}\notin E}} C_\varrho(f, e^{i\theta}) \tag{6}$$

for a fixed $\eta > 0$ and a given set E on $|z| = 1$, and then define the *radial boundary cluster set of $f(z)$, modulo E, at $e^{i\theta_0}$* as the intersection

$$C_{R-E}(f, e^{i\theta_0}) = \bigcap_{\eta>0} \overline{C}_\varrho(f, 0<|\theta-\theta_0|<\eta; E). \tag{7}$$

With these concepts, it follows easily from a theorem of Carathéodory [2] that, *if $f(z)$ is meromorphic in $|z|<1$, and if E is an arbitrary set of measure zero on $|z|=1$, then*

$$\operatorname{Fr} C(f, e^{i\theta}) \subseteq C_{R-E}(f, e^{i\theta}) \tag{8}$$

at every point $e^{i\theta}$ on $|z|=1$. If, in the case of the Blaschke product $b(z)$ considered above, we take E to be the set (of measure zero) where the radial limits of $b(z)$ either fail to exist or, if they exist, are of modulus less than 1, we have that $C(b, e^{i\theta}) = \{|w| \leqslant 1\}$ and $C_{R-E}(b, e^{i\theta}) = \{|w| = 1\}$ for each $e^{i\theta}$ on $|z| = 1$. We may now show that earlier methods of the author ([12]-[15]) yield the following result:

THEOREM 4. *If $f(z)$ is meromorphic in $|z|<1$, and if E is an arbitrary set of measure zero on $|z|=1$, then every value of $C(f, e^{i\theta}) - C_{R-E}(f, e^{i\theta})$ is assumed by $f(z)$ in every neighbourhood of $e^{i\theta}$, with the possible exception of a set of capacity zero. If, in addition, E is of logarithmic capacity zero, the exceptional set consists of at most two values. Any exceptional values are asymptotic values of $f(z)$, either at $e^{i\theta}$ or arbitrarily close to $e^{i\theta}$.*

Theorem 4 includes and extends some results recently announced by Noshiro [16], and the methods of proof consist essentially of refinements of the topological properties mentioned in § 1 together with an extended form of the maximum modulus principle. We remark that, by Theorem 1, the sets $C(f, e^{i\theta})$ and $C_\varrho(f, e^{i\theta})$ are the same for all points $e^{i\theta}$ belonging to a residual set on $|z| = 1$, and the nature of the proof of Theorem 1 and various of its extensions indicates that little improvement can be expected by replacing $C_\varrho(f, e^{i\theta})$ by another type of cluster set. The recent interest is therefore centred in the conditions under which a set E can be chosen so that $C(f, e^{i\theta}) - C_{R-E}(f, e^{i\theta})$ is not empty, and certain uniqueness theorems involving the concept of category now come to the fore.

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