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ON A PROBLEM OF STEINHAUS ABOUT NORMAL NUMBERS

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§ 1. Introduction. Let ξ be a real number and b an integer >1. We say that ξ is *normal with respect to the integer b as base* if in the "decimal" expansion of ξ to the base b every digit occurs with the same asymptotic frequency [1]. Steinhaus in the "New Scottish Book" has raised the question as to how far the property of being normal with respect to different bases is independent [2]. In this note I shall prove a result which answers Steinhaus' particular question in the negative.

Let $\mathfrak{U}_3$ be the set of numbers in $0 \leqslant \xi < 1$ in whose expansion to the base 3 the digit 2 never occurs. We introduce a measure in $\mathfrak{U}_3$ as follows: To every number

$$\xi = \varepsilon_1 3^{-1} + \varepsilon_2 3^{-2} + \ldots + \varepsilon_j 3^{-j} + \ldots \quad (\varepsilon_j = 0 \text{ or } 1)$$

in $\mathfrak{U}_3$ corresponds the number

$$\Xi(\xi) = \varepsilon_1 2^{-1} + \varepsilon_2 2^{-2} + \ldots + \varepsilon_j 2^{-j} + \ldots$$

in $0 \leqslant \Xi(\xi) \leqslant 1$. Conversely every number Ξ in $0 \leqslant \Xi \leqslant 1$ is of the form $\Xi(\xi)$, and Ξ determines ξ uniquely except for the denumerably many Ξ such that $2^\tau \Xi$ is an integer for some integer τ. By the *μ-measure* of any subset $\mathfrak{V}$ of $\mathfrak{U}_3$ we shall understand the Lebesgue measure of the corresponding set of $\Xi(\xi)$, $\xi \in \mathfrak{V}$. We shall say that *μ-almost all ξ in $\mathfrak{U}_3$ have a certain property* if the set of ξ without that property has μ-measure 0.

We can now enunciate our theorem.

THEOREM. *μ-almost all ξ in $\mathfrak{U}_3$ are normal to every base b which is not a power of 3.*

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(1) G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, second edition, Oxford 1945.

(2) H. Steinhaus, *Problem 144*, The New Scottish Book, Wrocław 1946-1958, p. 14.

The numbers of $\mathfrak{U}_3$ are, of course, by definition not normal to base a power of 3. It will be clear that the method of proof is quite general and should probably enable one, for example, to prove the existence of numbers which are normal to every base b which is not a product of powers of any given set of primes $p_1, \ldots, p_t$, but not normal to the bases $p_1, \ldots, p_t$.

§ 2. A simplification. We first note that our theorem follows almost immediately from the following apparently much less general statement; where we write

$$e(x) = e^{2\pi i x}.$$

LEMMA 1. *Let h be any integer $\neq 0$ and let b be any integer > 1 which is not a power of 3. Then*

$$\sum_{0 \leqslant n < N} e(hb^n \xi) = o(N) \tag{1}$$

as $N \to \infty$, for μ-almost all $\xi \in \mathfrak{U}_3$.

We first deduce the theorem from Lemma 1. Since the union of a denumerable set of sets of μ-measure 0 also has μ-measure 0, it follows that for μ-almost all ξ the statement (1) is true for *every* $h \neq 0$ and *every* $b > 1$ which is not a power of 3. For such ξ and fixed b a well-known criterion of Weyl [3] shows that the sequence $b^n \xi$ $(n = 0, 1, 2, \ldots)$ is uniformly distributed (*gleichverteilt*) modulo 1. Hence ξ is normal to the base b; and this is true for all relevant b. Thus it remains only to prove Lemma 1.

§ 3. A lemma. The proof of Lemma 1 depends on the following result, which we first prove.

LEMMA 2. *Let b, h be as in the enunciation of Lemma 1. Then*

$$\sum_{0 \leqslant n < N} \prod_{j=1}^{\infty} |\cos(3^{-j} hb^n \pi)| < AN^{1-\delta}, \tag{1}$$

where A, δ are absolute constants.

Suppose that $b = 3^\tau b_1$, where b_1 is prime to 3. By hypothesis $b_1 > 1$, and so b_1 satisfies the conditions laid on b in the Lemma. Further, substituting b_1 for b increases the left-hand side of (1), since

$$\prod_{j=1}^{\infty} |\cos(3^{-j} hb^n \pi)| = \prod_{j=1-n\tau}^{\infty} |\cos(3^{-j} hb_1^n \pi)| \leqslant \prod_{j=1}^{\infty} |\cos(3^{-j} hb_1^n \pi)|.$$

Hence it is enough to prove the Lemma when b is prime to 3.

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[3] See H. Weyl, *Über die Gleichverteilung von Zahlen mod Eins*, Mathematische Annalen 77 (1916), p. 313-352, or also chapter IV of J. W. S. Cassels, *An Introduction to Diophantine Approximation*, Cambridge Tracts in Mathematics and Mathematical-Physics No. 45, Cambridge 1957.

We note now that it is enough to prove Lemma 2 under the further restriction that

$$b \equiv 1 \pmod 3. \tag{2}$$

For denote the left-hand side of (1) by $S(N, h, b)$. Clearly

$$S(N, h, b) = S([\tfrac{1}{2}N], h, b^2) + S(N - [\tfrac{1}{2}N], hb, b^2). \tag{3}$$

In any case, $b^2 \equiv 1 \pmod 3$. Thus if the required inequality (1) has been proved under the assumption (2) for some particular A and δ then it is true in general on replacing the value found for A by $2A$.

We first prove Lemma 2 in the special case $N = 3^r$ for some integer $r \geqslant 0$. Let the integer l be defined by

$$\begin{cases} b \equiv 1 \pmod{3^l}, \\ b \not\equiv 1 \pmod{3^{l+1}}. \end{cases} \tag{4}$$

Then, as is well known, b^n $(0 \leqslant n < 3^r)$ runs modulo 3^{l+r} through all residue classes which are congruent to 1 modulo 3^l. Let $h = 3^m h'$, where h' is prime to 3. Then clearly $hb^n (0 \leqslant n < 3^r)$ runs modulo 3^{l+m+r} through all residue classes which are congruent to h modulo 3^{l+m}. That is, if we have the expansion

$$hb^n = \sum_{k \geqslant 0} \varepsilon_k(n) 3^k, \quad \text{where} \quad \varepsilon_k(n) = 0, 1 \text{ or } 2,$$

then

$$\varepsilon_{l+m}(n), \varepsilon_{l+m+1}(n), \ldots, \varepsilon_{l+m+r-1}(n) \tag{5}$$

take precisely once every one of the 3^r possible sets of values as n runs from 0 to $3^r - 1$.

We now divide the integers n in $0 \leqslant n < 3^r$ into two classes which we treat differently. Let (I) be the set of n such that the digit 1 occurs more than $r/6$ times in the set (5) and let (II) denote the remaining n.

If $n \in (I)$, then the fractional part of $3^{-j} hb^n$ lies between 1/3 and 2/3 for more than $r/6$ values of j, namely those j such that $\varepsilon_{j-1}(n) = 1$. Hence

$$\prod_{j=1}^{\infty} |\cos(3^{-j} hb^n \pi)| \leqslant (\cos \pi/3)^{r/6} \quad (n \in (I)). \tag{6}$$

The right-hand side of (6) is just $3^{-\delta' r}$ for some $\delta' > 0$. Since the set (I) contains trivially at most 3^r members, we have

$$\sum_{n \in (I)} \prod_{j=1}^{\infty} |\cos(3^{-j} hb^n \pi)| \leqslant 3^{(1-\delta')r}. \tag{7}$$

It remains to deal with the set (II). Here we use the fact that the number of sequences (5) having s members equal to 1 is at most

$$3^r \exp\{-K(s-\tfrac{1}{3}r)^2/r\}, \tag{8}$$

where $K>0$ is an absolute constant (see e. g. Hardy and Wright loc. cit.). For $n \in (II)$ we have $s \leqslant r/6$, and so $(s-\frac{1}{3}r)^2/r \geqslant r/36$. Hence on summing (8) over $0 \leqslant s \leqslant r/6$ we see that the number of n in (II) is at most

$$\{(r/6)+1\}3^r \exp\{-Kr/36\} \leqslant A'' 3^{(1-\delta'')r}$$

for some $A''>0$, $\delta''>0$ independent of r. Thus

$$\sum_{n\in(II)} \prod_{j=1}^{\infty} |\cos(3^{-j}hb^n\pi)| \leqslant A'' 3^{(1-\delta'')r}. \tag{9}$$

From (7) and (9) we have

$$\sum_{0\leqslant n<3^r} \prod_{j=1}^{\infty} |\cos(3^{-j}hb^n\pi)| \leqslant A''' 3^{(1-\delta''')r} \tag{10}$$

with

$$A''' = 1+A'', \qquad \delta''' = \min(\delta', \delta'').$$

Now let N be any positive integer, say

$$N = \sum_{0\leqslant r\leqslant R} \eta_r 3^r, \quad \text{where} \quad \eta_r = 0, 1 \text{ or } 2. \tag{11}$$

The range of summation $0 \leqslant n < N$ in (1) may be subdivided into $\sum\limits_{0\leqslant r\leqslant R} \eta_r$ intervals of the type

$$N_0 \leqslant n < N_0 + 3^r, \tag{12}$$

where there are precisely η_r intervals of length 3^r. But now

$$\sum_{N_0\leqslant n<N_0+3^r} \prod_{j=1}^{\infty} |\cos(3^{-j}hb^n\pi)| \tag{13}$$

is just a sum of the type in (10) with $b^{N_0}h$ instead of h. Hence the estimate (10) applies, and the sum (13) is at most

$$A'''(3^r)^{1-\delta'''} \leqslant A''' N^{1-\delta'''}.$$

But now

$$\sum_{0\leqslant r\leqslant R} \eta_r \leqslant K(\log N+1)$$

from (11) for some absolute constant $K>0$. Hence the left-hand side of (1) is at most

$$K(\log N+1)\cdot A''' N^{1-\delta'''} < AN^{1-\delta}$$

for some suitable $A>0$, $\delta>0$ independent of N.

§ 4. Proof of Lemma 1. The first step in the proof of Lemma 1 is the following

LEMMA 3. *Let A, δ have the meanings they have in Lemma 2, where without loss of generality*

$$0<\delta<1<A.$$

Put $\delta = 3\Delta$.

Then for all integers $N \geqslant 1$ the set of $\xi \in \mathfrak{U}_3$ such that

$$\Big|\sum_{0\leqslant n<N} e(hb^n\xi)\Big| \geqslant N^{1-\Delta}$$

has μ-measure at most $4AN^{-\Delta}$, where b, h have the meanings they have in Lemma 1.

We shall use an averaging argument and must first introduce the integral

$$\int_{\xi\in\mathfrak{U}_3} f(\xi)\,d\mu \tag{1}$$

of a function $f(\xi)$ with respect to the measure μ introduced in § 1. If $f(x)$ is a function defined on the whole interval $0 \leqslant x \leqslant 1$ and continuous there, then clearly

$$\int_{\xi\in\mathfrak{U}_3} f(\xi)\,d\mu = \lim_{\tau\to\infty} 2^{-\tau} \sum_{M}{}' f(3^{-\tau}M), \tag{2}$$

where $\sum\limits_M{}'$ is taken over those non-negative integers $M<3^\tau$ which have no 2's in their expansion in the scale of 3. In particular, when

$$f(x) = e(\lambda x) = e^{2\pi i\lambda x} \tag{3}$$

for some λ, then

$$\int_{\xi\in\mathfrak{U}_3} e(\lambda\xi)\,d\mu = \lim_{\tau\to\infty} 2^{-\tau} \prod_{j<\tau} \{1+e(3^{-j}\lambda)\} = \prod_{j=0}^{\infty} [\tfrac{1}{2}\{1+e(3^{-j}\lambda)\}].$$

Hence

$$\Big|\int_{\xi\in\mathfrak{U}_3} e(\lambda\xi)\,d\mu\Big| = \prod_{j=0}^{\infty} |\cos(3^{-j}\lambda\pi)|.$$

In the notation of the enunciation of Lemma 3, we have now

$$(4)\qquad \int_{\xi\in\mathfrak{U}_3} \Big|\sum_{0\leqslant n<N} e(hb^n\xi)\Big|^2 d\mu = \sum_{0\leqslant m<N}\ \sum_{0\leqslant n<N}\ \int_{\xi\in\mathfrak{U}_3} e\{(hb^n-hb^m)\xi\}\,d\mu$$

$$\leqslant \sum_{0\leqslant m<N}\ \sum_{0\leqslant n<N}\ \prod_{j=0}^{\infty} |\cos\{3^{-j}h(b^n-b^m)\pi\}|.$$

In this sum put $s=\min(m,n)$ and $r=\max(m,n)-s$. Then certainly the right-hand side of (4) is not greater than

$$2\sum_{0\leqslant r<N}\ \sum_{0\leqslant s<N}\ \prod_{j=0}^{\infty} |\cos\{3^{-j}h(b^r-1)b^s\pi\}|.$$

When $r\neq 0$, the inner sum is at most $AN^{1-\delta}$ by Lemma 2. When $r=0$, the inner sum is trivially at most N. Hence

$$\int_{\xi\in\mathfrak{U}_3} \Big|\sum_{0\leqslant n<N} e(hb^n\xi)\Big|^2 d\mu \leqslant 2N+2AN^{2-\delta} < 4AN^{2-3\Delta},$$

since $0<3\Delta=\delta<1<A$.

The truth of Lemma 3 is now apparent.

The deduction of Lemma 1 from Lemma 3 now follows from Lemma 3, by a routine argument. Let

$$N_m = [\exp m^{1/2}].$$

(All that is really necessary is that N_m shall increase fairly fast but slower than a geometric progression). Then trivially

$$\sum_{m\geqslant 0} N_m^{-\Delta} < \infty.$$

Hence, by Lemma 3, for μ-almost all ξ there is an $m_0(\xi)$ such that

$$(5)\qquad \Big|\sum_{0\leqslant n<N_m} e(hb^n\xi)\Big| < N_m^{1-\Delta} \qquad (\text{all } m\geqslant m_0(\xi)).$$

On the other hand, if $m=m(N)$ is defined by the inequalities $N_m\leqslant N<N_{m+1}$, then clearly

$$(6)\qquad N-N_{m(N)} = o(N)$$

as $N\to\infty$. Since trivially

$$\Big|\sum_{0\leqslant n<N} e(hb^n\xi) - \sum_{0\leqslant n<N_m} e(hb^n\xi)\Big| \leqslant N-N_{m(N)},$$

it follows from (5) and (6) that

$$\sum_{0\leqslant n<N} e(hb^n\xi) = o(N) \qquad (N\to\infty)$$

for μ-almost all ξ.

This proves Lemma 1, and it was already shown in § 2 that this implies the truth of the Theorem.

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