Some theorems about two-dimensional polyhedra

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The main aim of this paper is to give a topological characterization of 2-dimensional polyhedra. This is done in § 3, theorem 3.1. Besides the topological notions from set-theoretical topology we also use the notion of the plane and of the 1-dimensional polyhedron. Since both can be defined in terms of the set-theoretical topology (see [7] and [13]) the characterization we give is "good" in the sense that all notions which serve to characterize 2-dimensional polyhedra in theorem 3.1 derive — directly or indirectly — from set-theoretical topology.

Another question is as to whether theorem 3.1 is "useful", i. e. what can be proved by using it. As such a justification we can quote the solution of the following problem of Borsuk: is A a polyhedron if $A\times B$ is a polyhedron? It can be proved by means of theorem 3.1 that the answer is positive if $\dim A \leqslant 2$ (see [9]). A recent example of Bing shows that without the condition $\dim A \leqslant 2$ the answer is no longer positive.

Theorem 3.1 also easily implies the well-known theorem stating that in dimension 2 local triangulability is equivalent to triangulability.

§ 1 contains all the auxiliary notions, lemmas and notations which will be used throughout the paper. § 2 is devoted to the study of 2-dimensional ANR-s satisfying the following condition: the set of points which do not possess a neighbourhood homeomorphic with the plane is of dimension $\leqslant 0$. It is proved that an ANR satisfying this condition is a polyhedron. This theorem does not follow from theorem 3.1. It characterizes only a special class of 2-dimensional polyhedra, but this class contains in particular all 2-dimensional closed pseudomanifolds. An explicit topological characterization of 2-dimensional closed pseudomanifolds is given in 2.4.

In the closing § 4 we establish a link between the notions of an r-point (see [10]) and of 2-dimensional polyhedra.

The main results of § 3 of this paper were announced without proof in [8].

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§ 1. The word "space" will always mean a metric and separable space. A space is a *continuum* if it is compact and connected and contains more than one point. A continuum which is locally connected will be called a *Peano continuum*.

We shall say that p *disconnects* a (connected) space K if $K-p$ is not connected. A connected and locally connected space which is not disconnected by any point will be called *cyclic*.

A point $p \epsilon K$ will be called a *local disconnecting point* of K if p disconnects an open and connected subset of K.

ANR will denote an absolute neighbourhood retract and AR — an absolute retract. By a *polyhedron* we shall mean a space homeomorphic with a finite Euclidean polyhedron in the sense of [1], p. 128-129 and 149. A *graph* will be a polyhedron of dimension $\leqslant 1$.

Q_2 will denote a 2-element, $I(Q_2)$ will be the interior of Q_2, i. e. an open 2-element; $F(Q_2) = Q_2 - I(Q_2)$.

If K is a space, then $\mathrm{reg}_n K$ means the subset of K consisting of those points which have neighbourhoods homeomorphic with the Euclidean n-space. If $p \epsilon \mathrm{reg}_n K$, we shall say that p is *n-regular*. We shall write $\mathrm{reg}\, K$ for $\mathrm{reg}_2 K$.

If K is compact and connected and $K = \mathrm{reg}_n K$, then K is a closed manifold. It is well-known that 2-manifolds are polyhedra ([14]).

We shall use the Čech homology theory with coefficients mod 2. Our standard reference in algebraic topology will be [6].

1.1. Identifications. Given a semi-continuous decomposition of a space K, there exists a continuous mapping $f: K \to K^*$ of the space K onto the space K^* such that the inverse-images of points of K are precisely the elements of the decomposition. If with the exception of the element T_0 all other elements of the decomposition are points we shall say that *K^* is obtained from K by the identification of T_0*. In this case the identification mapping $f: K \to K^*$ maps homeomorphically $K - T_0$ onto $K^* - f(T_0)$.

(1.1.1) *Let $B = \bar{B} \subset A$ and let f be a mapping of A onto C such that f is a homeomorphism on $A - B$. If A, B and $f(B)$ are locally contractible, then such is also $f(A)$.*

It is a special case of a theorem of Borsuk [4], theorem (T). It follows from (1.1.1) that if A is an ANR of finite dimension and $B = \bar{B} \subset A$ is an ANR, then the space obtained from A by the identification of B is an ANR.

(1.1.2) *Let $B = \bar{B} \subset A$. Let C be a closed and connected subset of A such that $C - B \neq 0 \neq C \cap B$. Let $f: A \to A^*$ be the identification mapping induced by the identification of B.*

If $C \cap B$ is not connected, then $f(B)$ disconnects locally $f(C)$.

The proof is straightforward.

1.2. Strongly cyclic elements. The set of local disconnecting points in a space K will be denoted by L_K. If K is locally contractible, then a certain decomposition of K may be obtained by using the set L_K and this decomposition refines that of a space into cyclic elements.

Some properties of the set L_K and of the decomposition of the space into strongly cyclic elements will be used in the sequel. We list them here and the proofs will be published in [11]..

(1.2.1) *Let K be an* ANR *and let $A \subset K$ be a Peano continuum. Let $\mathfrak{B}$ be the family of those components of $K - A$ whose boundary contains more than one point but is of dimension zero.*

Then $\overline{\overline{\mathfrak{B}}} < \aleph_0$ and if $B \epsilon \mathfrak{B}$ then $\overline{\overline{\mathrm{Fr}(B)}} < \aleph_0$.

(1.2.2) *Let K be locally compact, connected and locally contractible. If C disconnects K, and $\dim C = 0$, then $C \cap L_K \neq 0$.*

Now let K be an ANR. The following subsets of K will be called *strongly cyclic elements* of K:

a. every point of L_K;

b. for every point $p \notin L_K$ the set of all such $x \epsilon K$ that no finite set disconnects K between p and x.

Strongly cyclic elements having more than one point will be called *true strongly cyclic elements*. We shall use the abbreviation t.s.c.e. and s.c.e.

(1.2.3) *Every* s.c.e. *is a retract of K.*

(1.2.4) *If E is a* t.s.c.e., *then L_E is a finite set not disconnecting E.*

(1.2.5) *With the exception of a finite number, all* s.c.e. *are contractible in themselves.*

(1.2.6) *If E_1, E_2 are two different* t.s.c.e. *then $E_1 \cap E_2$ is finite.*

1.3. Polyhedra. We include the following two elementary lemmas:

(1.3.1) *Let a compact space $K = K_1 \cup K_2 \cup \ldots \cup K_n$ where K_i are polyhedra and $K_i \cap K_j$ is a subcomplex of both K_i and K_j for all i, j. Then K is a polyhedron and $\bigcup_{i \neq j} K_i \cap K_j$ is a subpolyhedron of K.*

In particular K is a polyhedron if K_i are polyhedra and $K_i \cap K_j$ is finite for all i, j, $i \neq j$.

(1.3.2) *Let K be a compact space and let K^* be obtained from K by the identification of a finite subset. In order that K be a polyhedron it is necessary and sufficient that K^* be such.*

1.4. Manifolds with boundary imbeddable in the plane.

(1.4.1) *Let K be a compact 2-dimensional manifold with boundary F. In order that K be imbeddable in the plane it is necessary and sufficient that the inclusion homomorphism $H_1(F)\to H_1(K)$ be onto.*

Let K_1 be the closed manifold obtained from K by attaching to each component of F a closed 2-cell. We shall consider K as a subset of K_1. Obviously, K is imbeddable in the plane if and only if K_1 is a sphere.

Suppose that K_1 is a sphere. Then the exactness of the Mayer-Vietoris sequence ([6], § 15)

$$\text{(i)}\qquad \ldots\to H_1(F)\to H_1(K)+H_1(\overline{K_1-K})\to H_1(K_1)\to\ldots,$$

implies that $H_1(F)\to H_1(K)$ is onto. Thus the condition is necessary.

Suppose now that $H_1(F)\to H_1(K)$ is onto. Then using again the sequence (i) we infer that $H_1(K)\to H_1(K_1)$ is trivial. But on the other hand, $H_1(K_1, K)$ being obviously zero, we infer from the exactness of the sequence

$$\ldots\to H_1(K)\to H_1(K_1)\to H_1(K_1, K)\to\ldots$$

that $H_1(K)\to H_1(K_1)$ is onto. It follows that $H_1(K_1)=0$. But there is only one 2-dimensional manifold K_1 satisfying this condition: it is a sphere.

(1.4.2) *Let K be a compact 2-manifold with boundary F, imbeddable in the plane, and let F_1 be a component of F. Let f_1 be a homeomorphism mapping F_1 onto $F(Q_2)$. Then there exists a homeomorphism f mapping K into Q_2, extending f_1 and such that the diameter of every component of $Q_2-f(K)$ is arbitrarily small.*

The trivial proof will be omitted.

§ 2. The set of irregular points in an ANR of dimension n, i. e. the set $K-\mathrm{reg}_n K$, may also be of dimension n; there exist even n-dimensional ANR's which have no n-regular points at all. The simplest example of this sort may be obtained as follows: Let K be a one-dimensional ANR such that the set of points which are of order $\neq 2$ is dense in K, let Q_n be a closed n-cell and let $K_n = K\times Q_{n-1}$. Were $(p, q)\in K_n$ n-regular, we should infer by theorem 3 in [5] that p in 1-regular, which contradicts our assumption on K. Nevertheless, if the dimension of the set of irregular points in a set K is inferior to the dimension of K, its structure is no longer quite arbitrary. In fact, we shall prove (theorem 2.3) that if the set of irregular points in an 2-dimensional ANR is of dimension $\leqslant 0$, then it is finite and the space is a polyhedron.

The proof of this theorem will be the main aim of this section. In 2.1 we prove some auxiliary lemmas concerning essentially compactifications of 2-dimensional open manifolds by the addition of one point. These lemmas may also be deduced from known theorems about 2-dimensional non-compact manifolds. The essential part of the proof of theorem 2.3 starts from 2.2.

In (2.2.1) we prove the theorem under the assumption that there is only one irregular point; in (2.2.2) it is proved that irregular points are contained in the closure of the set of local disconnecting points. The general theorem follows easily in 2.3.

In 2.4 we give a topological characterization of 2-dimensional closed pseudomanifolds.

(2.1.1) *Let K be a cyclic Peano continuum. Suppose that $K=A\cup p$ where A is a non-compact 2-dimensional manifold with a compact boundary not containing p. Then there exist arbitrarily small connected neighbourhoods U of p such that*

(a) *$K-U$ is a compact manifold with boundary,*

(b) *$\overline{U}-p$ has only a finite number of components $V_1, \ldots, V_n$,*

(c) *$\overline{V}_i\cap(K-U)$ is a simple closed curve, $i=1,2,\ldots,n$.*

We fix a triangulation T in A. By [15] there are arbitrarily small connected neighbourhoods of p with a connected complement. Let V be such a neighbourhood. We choose V so small that it is disjoint with the boundary of A. Let P be the sum of $K-V$ and of all simplexes which are not disjoint with $K-V$; P is then a connected polyhedron and $K-P$ a neighbourhood of p contained in V. Let W be the connected component of $K-P$ containing p. Let R be the sum of $K-W$ and of the stars of all locally disconnecting points of $K-W$ in a barycentric subdivision of T. Then R is a compact manifold with boundary, and putting $U=K-R$ we see that U is a connected neighbourhood of p satisfying (a). It satisfies also (b): since K is cyclic, every component of $\overline{U}-p$ intersects $K-U$, and therefore they are finite in number since K is locally connected.

Condition (c) in general is not satisfied. Suppose that $\overline{V}_i\cap(K-U)$ has more than one component. Since $\overline{V}_i$ satisfies the same conditions as K, there exists a neighbourhood V_i' of p in $\overline{V}_i$ satisfying (a) and (b). Then $\overline{V}_i-V_i'$ is a manifold with boundary and there exists a simple closed curve disconnecting $\overline{V}_i-V_i'$ into two components such that one of them, say V_i'', contains $(\overline{V}_i-V_i')\cap\overline{V}_i'$ and the second contains $\overline{V}_i\cap(K-U)$. Repeating this operation for every i and putting $U_i'=V_i'\cup V_i''$ and $U'=\bigcup U_i'$ we can easily see that U' satisfies (a)-(c).

(2.1.2) *Let K be a cyclic Peano continuum, $K = A \cup p$, $A \subset \operatorname{reg} K$. There exists a sequence $\{U_n\}$, $n = 0, 1, 2, \ldots$, of connected neighbourhoods of the point p having the following properties:*

(a) $\overline{U}_{n+1} \subset U_n$, $\delta(U_n) < 1/n$;

(b) *$\overline{U}_n - U_{n+1}$ is a sum of a finite number of disjoint compact manifolds with boundary, $W_n^1, W_n^2, \ldots, W_n^{i_n}$. Two different W_n^i belong to different components of $\overline{U}_n - p$;*

(c) *$W_n^i \cap (K - U_n)$ is a simple closed curve for $n \geqslant 1$, $i = 1, \ldots, i_n$;*

(d) *$W_n^i \cap \overline{U}_{n+1}$ is non-empty and is a sum of a finite number of disjoint simple closed curves, $i = 1, \ldots, i_n$.*

The construction of the sequence U_n is as follows. We put $U_0 = K$ and let U_1 be the neighbourhood of p in K with the diameter < 1 and satisfying (2.1.1). Let $W_0^1 = \overline{U}_0 - U_1$. Obviously W_0^1 satisfies (b)-(d). Moreover, by (2.1.1),

(i) if $V_1, \ldots, V_{n_1}$ are all components of $\overline{U}_1 - p$, then $\overline{V}_i \cap (K - U_1)$ is a simple closed curve and every $\overline{V}_i$ satisfies the same conditions as the set K in (2.1.1).

Let V_i' be a compact neighbourhood of p in V_i, $\delta(V_i') < \frac{1}{4}$, satisfying (2.1.1) and closure-disjoint with $\overline{V}_i \cap (K - U_1)$. Put $W_1^i = \overline{V}_i - V_i'$, $i = 1, \ldots, n_1$, and $U_2 = \bigcup V_i'$. Then $\overline{U}_2 \subset U_1$ and $\delta(U_2) < \frac{1}{2}$, i. e. U_2 satisfies (a). It satisfies (b) obviously, (c) follows from (i) and from $W_1^i \cap (K - U_1) = \overline{V}_i \cap (K - U_1)$. Since $\overline{V}_i$ is connected, then $W_1^i \cap \overline{U}_2 = W_1 \cap \overline{V}_i' = (\overline{V}_i - V_i') \cap \cap \overline{V}_i' \neq 0$, which proves (d).

Now, since $\overline{U}_2 - p$ satisfies again (i), the iteration of the above construction yields the sequence $\{U_n\}$.

We shall introduce some definitions. The set K satisfying the conditions of (2.1.2) will be called an *infinite manifold*; the decomposition of K by aid of the sequence U_n satisfying (2.1.2) will be called a *decomposition* of an infinite manifold, the sets W_j^i will be called the *elements* of the decomposition.

An element W_n^k will be called a *predecessor* of an element W_{n+1}^i if they are not disjoint. If W_n^k is a predecessor of W_{n+1}^i, then we shall say also that W_{n+1}^i is a *successor* of W_n^k.

Let $W_n^{i_0}, W_{n+1}^{i_1}, \ldots$ be an infinite sequence of elements of a decomposition of an infinite manifold, such that every element of this sequence is a predecessor of the element which comes next in the sequence. In that case the sum $\bigcup\limits_{j=0}^{\infty} W_{n+j}^{i_j}$ will be called a *branch emering from $W_n^{i_0}$*; the elements of the sequence will be called the elements of the branch. Since by (2.1.2) (d) every element has at least one successor we infer that

(2.1.3) *From every W_n^i emerges a branch. If all W_j^i, $j \geqslant n$, have only one successor, such a branch is unique.*

Now, fix the index n. Let $\mathfrak{G}_i$ be the sum of all branches emerging from W_n^i. We shall prove that

(2.1.4) *The sets $\mathfrak{G}_i$ are components of $\overline{U}_n - p$. Moreover $\bigcup\limits_{i=1}^{i_n} \mathfrak{G}_i = \overline{U}_n - p$, i. e. every component is of the form $\mathfrak{G}_i$.*

$\mathfrak{G}_i$ are connected subsets of U_n and since $W_n^i \subset \mathfrak{G}_i$, by (2.1.2)(b) $\mathfrak{G}_i$ are contained in different components of $\overline{U}_n - p$. Therefore in order to prove that $\mathfrak{G}_i$ are components of $\overline{U}_n - p$ it is sufficient to show that $\bigcup\limits_{i=1}^{i_n} \mathfrak{G}_i = \overline{U}_n - p$. This will also complete the proof of (2.1.4).

Let $q \in \overline{U}_n - p$. By (2.1.2) (a) and (b), $\overline{U}_n - p$ is the sum of elements W_i^j for $i \geqslant n$. Therefore $q \in W_i^j$ for some j and $i \geqslant n$. By (2.1.3) there exists a branch emerging from W_i^j. By (2.1.2) (c) every element has one predecessor, and therefore there exists a branch emerging from $W_{i-1}^{j_1}$ and containing q. If $n < i-1$, then we repeat this reasoning; after a finite number of steps we shall find a branch emerging from W_n^k, and therefore contained in $\mathfrak{G}_k$ and containing q. This completes the proof.

(2.1.5) *If the inclusion homomorphism $H_1(W_n^i) \to H_1(K)$ is trivial, then W_n^i may be imbedded in the plane.*

An elementary reasoning shows that the condition above implies that the inclusion homomorphism $H_1(\operatorname{Fr} W_n) \to H_1(W_n^i)$ is onto. Since W_n^i is a manifold with boundary F contained in $\operatorname{reg} K$, $F = \operatorname{Fr}(W_n^i)$ and (1.4.1) yields (2.1.5).

The criterion (2.1.5) is important because of the following lemma, which describes the situation when all branches emerging from a given element are composed of elements which may be imbedded in the plane. Roughly speaking, the lemma says that in that case the sum of these branches may be imbedded in a 2-element in such a manner that every point of the complement corresponds to a branch. Precisely:

(2.1.6) *Let $\mathfrak{G}$ be the sum of all branches emerging from W_N^i. If all elements of branches belonging to $\mathfrak{G}$ may be imbedded in the plane, then there exists a homeomorphism $h\colon \mathfrak{G} \to Q_2$ satisfying*

(a) *$h(\mathfrak{G})$ is a dense and open subset of Q_2;*

(b) *$Q_2 - h(\mathfrak{G})$ is a zero-dimensional subset of $I(Q_2)$;*

(c) *the set resulting from Q_2 by the identification of $Q_2 - h(\mathfrak{G})$ is homeomorphic with $\mathfrak{G} \cup p$;*

(d) *if $\{G_i\}$, $i = 1, 2, \ldots$, is a sequence of connected open subsets of $\mathfrak{G}$ satisfying $G_i \supset G_{i+1}$, $\delta(G_i) \to 0$ and $p \in \overline{G}_i$, then* $\operatorname{Lim} h(G_i)$ *exists and is just one point of $Q_2 - h(\mathfrak{G})$.*

First, we shall prove that

(i) Every element W_i^k, $i \geqslant 1$, has one and only one predecessor. Its intersection is a component of the boundary of W_i^k. Any other component of the boundary of W_i^k is also a component of the boundary of one and only one successor of W_i^k.

Suppose that W_{i-1}^j and W_{i-1}^m are predecessors of W_i^k. Then $W_{i-1}^j \cup \cup W_{i-1}^m \cup W_i^k$ is connected and contained in $\overline{U}_{i-1} - p$. Therefore by (2.1.2) (b) we have $j = m$. Now if W_i^k is a predecessor of W_{i+1}^m, then by (2.1.2), $W_i^k \cap W_{i+1}^m$ is a simple closed curve. Suppose that the simple closed curve C is a component of the boundary of W_i^k and of the boundary of W_{i+1}^m and W_{i+1}^j. Since $C \subset \operatorname{reg} K$, we have $j = m$, which completes the proof of (i).

Now let F_j, $j = 1, \ldots, j(i, N)$, be components of the boundary of W_N^i. We shall assume that F_1 is a component of the boundary of the predecessor of W_N^i. Let h_0 be such a homeomorphism of W_N^i into Q_2 that $h_0(F_1) = F(Q_2)$ and the diameter of every component of $Q_2 - h_0(W_N^i)$ is $< \frac{1}{2}$. Then $h_0(F_j)$, $j = 2, \ldots, j(i, N)$, is a simple closed curve in $I(Q_2)$ and a boundary of a component A_j of $Q_2 - h_0(W_N^i)$. By (i) there is one and only one successor of W_N^i having F_j as a component of the boundary. Let it be W_{N+1}^j.

By (1.4.2), $h_0|F_j$ may be extended to a homeomorphism $h_1^j\colon W_{N+1}^j \to \overline{A}_j$ in such a manner that the diameter of every component of $\overline{A}_j - h_1^j(W_{N+1}^j)$ is smaller that $\frac{1}{3}$. We define $h_1\colon \bigcup_j W_{N+1}^j \to \bigcup_j \overline{A}_j$ by $h_1(x) = h_1^j(x)$ for $x \in W_{N+1}^j$. Since W_{N+1}^j are disjoint and $\overline{A}_j$ are disjoint, the mapping h_1 is a homeomorphism. Moreover it is an extension of the homeomorphism h_0 and the extended mapping is also a homeomorphism.

This construction may be repeated, yielding a sequence of homeomorphisms $h_0, h_1, h_2, \ldots$ such that

(ii) The mapping $h\colon \mathfrak{G} \to Q_2$ defined by $h(x) = h_i(x)$ for $x \in W_{N+i}^j$ is a homeomorphism.

(iii) Let $\mathfrak{K}_i = \bigcup_{k<i} W_{N+k}^j$. Then $Q_2 - h(\mathfrak{K}_i)$ is a sum of a finite number of sets homeomorphic with $I(Q_2)$ and of diameter smaller than $1/i$.

(iv) Let $\mathfrak{G}_i^j$ be the sum of all branches emerging from W_{N+i}^j. Then $h(\mathfrak{G}_i^j)$ is contained in one component of $Q_2 - h(\mathfrak{K}_i)$.

These properties will serve to prove that h satisfies (a)-(d).

Since $Q_2 - h(\mathfrak{G}) = \bigcap_i \overline{Q_2 - h(\mathfrak{K}_i)}$, $Q_2 - h(\mathfrak{G})$ is closed; since $\overline{Q_2 - h(\mathfrak{K}_i)}$ form a monotone decreasing sequence of sets with components of diameter smaller than $1/i$, it is a zero-dimensional set. This proves (a) and (b).

Now let us observe that $\mathfrak{G} - \mathfrak{K}_i \subset U_{N+1} \cap \mathfrak{G}$; therefore the sequence $(\mathfrak{G} - \mathfrak{K}_i) \cup p$ is a sequence of neighbourhoods of p in $\mathfrak{G} \cup p$ with diameters converging to zero. Define $h'\colon Q_2 \to \mathfrak{G} \cup p$ by

$$h'(x) = \begin{cases} h^{-1}(x) & \text{for} \quad x \in h(\mathfrak{G}), \\ p & \text{for} \quad x \in Q_2 - h(\mathfrak{G}). \end{cases}$$

To prove (c) we have only to prove that h' is continuous. Let $x \in Q_2 - h(\mathfrak{G})$. For every i the point x has a neighbourhood contained in $Q_2 - h(\mathfrak{K}_i)$, and therefore h' maps this neighbourhood into $(\mathfrak{G} - \mathfrak{K}_i) \cup p$. Since by the foregoing remark the diameter of this set is arbitrarily small, it follows that h' is continuous, which proves (c).

Let G_i be a sequence of sets satisfying the conditions of (d). Then $h(G_i)$ is a sequence of connected sets satisfying $h(G_{i+1}) \subset h(G_i)$. Therefore $\operatorname{Lim} h(G_i) = \operatorname{Lim} \overline{h(G_i)} = \bigcap_i \overline{h(G_i)}$ exists and is connected. To prove that it is one point of $Q_2 - h(\mathfrak{G})$ it is sufficient to prove that it is contained in $Q_2 - h(\mathfrak{G})$, this last set being compact and zerodimensional.

Since $\delta(G_i) \to 0$ and $p \in \overline{G}_i$, for every i there is such an $n(i)$ that $G_i \subset U_{n(i)}$ and $n(i) \to \infty$ together with i. Let $k(i) = n(i) - N$. Therefore $k(i) \to \infty$ and $h(G_i) \subset h(U_{n(i)}) \subset Q_2 - h(\mathfrak{K}_{k(i)})$. Therefore

$$\bigcap_i \overline{h(G_i)} \subset \bigcap_i \overline{Q_2 - h(\mathfrak{K}_{k(i)})} \subset Q_2 - h(\mathfrak{G}),$$

which completes the proof of (d).

2.2. We proceed to the proof of theorem 2.3. The following lemma is its special case.

(2.2.1) *Let K be a cyclic* ANR *and let $K - p \subset \operatorname{reg} K$ for a point $p \in K$. Then K may be obtained by the identification of a finite number of points in a compact manifold. If p does not locally disconnect K then K is a manifold.*

K is an infinite manifold; we shall consider a fixed decomposition of K with the notation as in (2.1.2). First, we shall prove that

(i) there exists such an N that for $n \geqslant N$ all elements W_n^i may be imbedded in the plane.

Since K is locally contractible in p, there exists such an N that the closure of the neighbourhood U_N is contractible in K to a point. Since all elements W_n^i for $n \geqslant N$ are contained in $\overline{U}_N$, the inclusion homomorphism $H_1(W_n^i) \to H_1(K)$ is trivial for $n \geqslant N$. Thus (2.1.5) yields (2.2.1).

(ii) There exists such an N that for $n \geqslant N$ every element W_n^i has only one successor.

Consider the triads $(\overline{U}_n; \overline{U}_{n+1}, \overline{U}_n - U_{n+1})$ and $(K; \overline{U}_{n+1}, K - U_{n+1})$. Let $A = \overline{U}_{n+1} \cap (\overline{U}_n - U_{n+1}) = \overline{U}_{n+1} \cap (K - U_{n+1})$. We have the following commutative diagram, where horizontal sequences are taken from the respective Mayer-Vietoris exact sequences and the vertical homomorphisms are induced by inclusions ([6], § 15):

$$\begin{array}{ccccc} H_0(\overline{U}_{n+1})+H_0(K-U_{n+1}) & \longleftarrow & H_0(A) & \overset{\psi}{\longleftarrow} & H_1(K) \\ \uparrow & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} \\ H_0(\overline{U}_{n+1})+H_0(\overline{U}_n-U_{n+1}) & \overset{\Delta}{\longleftarrow} & H_0(A) & \overset{\varphi}{\longleftarrow} & H_1(\overline{U}_n)\,. \end{array}$$

Suppose that the element W_n^i has two successors: W_{n+1}^j, W_{n+1}^k. Since $W_n^i\cap(W_{n+1}^j\cup W_{n+1}^k)$ is not connected and since components of this set are also components of A, we infer that there is an element $\alpha\in H_0(A)$ such that $\alpha\neq 0$ and the image of α in $H_0(\overline{U}_{n+1})$ is zero since $\overline{U}_{n+1}$ is connected, and the image of α in $H_0(\overline{U}_n-U_{n+1})$ is zero since W_n^i is connected. Thus $\Delta\alpha=0$ and there exists a $\beta\in H_1(\overline{U}_n)$ such that $\varphi\beta=\alpha$. Now, $0\neq i\alpha=i\varphi\beta=\psi j\beta$ implies that $j\beta\neq 0$, i. e. j: $H_1(\overline{U}_n)\to H_1(K)$ is not trivial. Thus we have proved that if W_n^i has more than one successor, then $\overline{U}_n$ is not contractible in K. Since K is locally contractible in p, this proves (ii).

By (i) and (ii) there is such an N that for $n\geqslant N$ all elements W_n^i are imbeddable in the plane and have only one successor. Let W_N^i, $i=1,\dots,i_N$, be all components of $\overline{U}_N-U_{N+1}$. Let $\mathfrak{G}_i$ be a branch emerging form W_N^i. By (2.1.3) there is one and only one such branch. Therefore, by (2.1.4) we have

(iii) $\bigcup\mathfrak{G}_i\cup p=\overline{U}_N$.

Let Q_i, $i=1,\dots,i_N$, be i_N different disjoint 2-elements. By (2.1.6) (c) the set $\mathfrak{G}_i\cup p$ is homeomorphic with a set obtained from Q_i by the identification of a compact zerodimensional subset of $I(Q_i)$; it follows easily from (2.1.6) (d) and from the fact that there is only one branch emerging from W_N^i that this subset consists of only one point. Therefore $\mathfrak{G}_i\cup p$ is homeomorphic with Q_i, the point p corresponding to a certain point $q_i\in I(Q_i)$. Let P be the space obtained from $\bigcup_i Q_i$ by the identification of all q_i. By (2.1.4), $(\mathfrak{G}_i\cup p)\cap(\mathfrak{G}_j\cup p)=p$ for $i\neq j$, and thus $\bigcup_i\mathfrak{G}_i\cup p$ is homeomorphic with P. By (iii), $\overline{U}_N$ is homeomorphic with P, which proves the first part of (2.2.1). The second part easily follows from the first: if p does not locally disconnect K, we have $i_N=1$. It follows that $\overline{U}_N$ is homeomorphic with a 2-element, i. e. $p\in\operatorname{reg}K$.

(2.2.2) *Let K be a connected* ANR. *Let B be a closed zero-dimensional subset of K such that $K-B$ is connected and contained in* $\operatorname{reg}K$. *Let $b\in B$. If $b\notin\overline{L}_K$ then $b\in\operatorname{reg}K$.* (L_K denotes as usual the set of local disconnecting points of K.)

First we shall prove that

(i) If U is a connected neighbourhood of a point $a\in B$ and if $U\cap L_K=0$, then $U-B$ is connected.

For if U satisfies the above condition then, U being open, we have $L_U=U\cap L_K$. Thus $L_U=0$ and (i) follows from (1.2.2).

Let K^* be a set obtained from K by the identification of B, let f: $K\to K^*$ be the identification mapping, and let $p=f(B)$. Since p does not disconnect K^*, K^* is an infinite manifold. We shall consider a fixed decomposition of K^* with the notation as in (2.1.2).

For every element W_n^i let $\mathfrak{G}_n^i$ denote the sum of all branches emerging from W_n^i.

(ii) For every $\varepsilon>0$ there is such an N that for $n\geqslant N$

$$\delta\big(\overline{f^{-1}(\mathfrak{G}_n^i)}\big)<\varepsilon\,.$$

For suppose that there exists such a sequence W_n^i that $\delta\big(\overline{f^{-1}(\mathfrak{G}_n^i)}\big)\geqslant\varepsilon>0$. Since $\overline{f^{-1}(\mathfrak{G}_n^i)}$ is a sequence of continua contained in K, it contains a convergent subsequence ([12], § 38, I, 1) whose limit is compact and connected and thus, being of positive diameter, a continuum. But this is impossible since this limit is a subset of a zerodimensional set B.

(iii) There exists such an N that W_n^i is imbeddable in the plane for $n\geqslant N$.

Let ε be such a positive number that every subset of K with diameter less than ε is contractible in K. By (ii) there exists such an N that for $n\geqslant N$, $\delta\big(f^{-1}(W_n^i)\big)<\varepsilon$, and it follows that for $n\geqslant N$, $H_1\big(f^{-1}(W_n^i)\big)\to H_1(K)$ is trivial. Since the mapping f maps homeomorphically $f^{-1}(W_n^i)$ onto W_n^i, it follows that $H_1(W_n^i)\to H_1(K^*)$ is also trivial. Thus (2.1.5) yields (iii).

Now, let ε be a positive number smaller than $\varrho(b,L_K)$. Let N be such a number that for $n\geqslant N$, W_n^i is imbeddable in the plane and $\delta\big(\overline{f^{-1}(\mathfrak{G}_n^i)}\big)<\varepsilon$. Let W_N^i, $i=1,\dots,i_N$, be components of $\overline{U}_N-U_{N+1}$ and let V be a neighbourhood of b in K so small that $V-B$ is connected (see (i)) and $f(V)\subset U_N$. By (2.1.4), $f(V-B)$ is contained in one of the sets $\mathfrak{G}_N^i$: we shall denote it by $\mathfrak{G}$. Let $W=\overline{f^{-1}(\mathfrak{G})}$. Then $f(W)=\mathfrak{G}\cup p$, $W\cap L_K=0$ and $\overline{V}\subset W$. To prove (2.2.2) it is sufficient to prove that there is a homeomorphism g: $W\to Q_2$ which maps W onto Q_2 and is such that $f(b)\in I(Q_2)$.

By (2.1.6) there is a homeomorphism h: $\mathfrak{G}\to Q_2$. Then hf: $W-B\to Q_2$ is a homeomorphism and since $hf(W-B)=h(\mathfrak{G})$, by (2.1.6) (b) $Q_2-hf(W-B)\subset I(Q_2)$. Therefore we have only to extend hf onto W in such a manner that the extended mapping will be a homeomorphism mapping W onto Q_2.

(iv) Let $a \in W \cap B$. There exists a sequence $\{V_i\}$, $i = 1, 2, \ldots$, of connected neighbourhoods of a such that $V_i \supset V_{i+1}$, $\delta(V_i) \to 0$, and $f(V_i - B)$ is a connected subset of $\mathfrak{G}$.

Since $\varrho(a, L_K) > 0$, then there exists a sequence of connected neighbourhoods V_i of a such that $V_1 \cap L_K = 0$, $V_i \supset V_{i+1}$ and $f(V_1) \subset U_N$. Thus $V_i - B$ is connected by (i) and it follows from (2.1.4) that $f(V_i - B)$ is contained in one of the sets $\mathfrak{G}_N^i$. Since $a \in \overline{f^{-1}(\mathfrak{G})}$, we have $(V_i - B) \cap f^{-1}(\mathfrak{G}) \neq 0$, i. e. $f(V_i - B) \cap \mathfrak{G} \neq 0$. Thus $f(V_i - B) \subset \mathfrak{G}$, which completes the proof of (iv).

Let $a \in W \cap B$ and let V_i be a sequence of neighbourhoods of a satisfying (iv). Then $f(V_i - B)$ satisfy the conditions of (2.1.6)(d) and there exists $\operatorname{Lim} hf(V_i - B)$ and is one point of $Q_2 - h(\mathfrak{G})$. Let $g'(a) = \operatorname{Lim} hf(V_i - B)$.

We assert that $g'(a)$ depends only on the point a. For if V_i' is another sequence of neighbourhoods of a satisfying (iv), then there exists a sequence V_i'' also satisfying (iv) and such that $V_i'' \subset V_i \cap V_i'$. But $\operatorname{Lim} hf(V_i' - B) = \operatorname{Lim} hf(V_i'' - B) = \operatorname{Lim} hf(V_i - B)$.

Therefore putting

$$g(x) = \begin{cases} hf(x) & \text{for} \quad x \in W - B, \\ g'(x) & \text{for} \quad x \in W \cap B \end{cases}$$

we obtain a continuous mapping of W into Q_2. But by (2.1.6)(a), $g(W)$ is dense in Q_2, and therefore, being compact, $g(W) = Q_2$. To complete the proof it remains to show that g is a homeomorphism. Since $g|W - B$ is a homeomorphism, $g(W - B) \cap g(W \cap B) = 0$, we have only to prove that if $a_1, a_2 \in W \cap B$ and $g(a_1) = g(a_2)$ then $a_1 = a_2$.

Let V_{i1} be a sequence of neighbourhoods for a_1 satisfying (iv), and let V_{i2} be an analogous sequence for a_2. Let $q = g'(a_1) = g'(a_2)$, i. e. $\operatorname{Lim} hf(V_{i1} - B) = \operatorname{Lim} hf(V_{i2} - B) = q$. Let $p_{i1} \in V_{i1} - B$, $p_{i2} \in V_{i2} - B$. Then $\lim p_{i1} = a_1$ and $\lim p_{i2} = a_2$. By (2.1.6)(b), $g(W \cap B)$ is zero-dimensional; therefore there exists a sequence of continua $\{L_i\}$ converging to q and such that $L_i \subset hf(W - B)$ and both $hf(p_{i1})$, $hf(p_{i2})$ belong to L_i, ([12], § 53, II, 1). By (2.1.6)(c), $h^{-1}(L_i)$ is a sequence of continua in $\mathfrak{G}$ converging to p, and thus $f^{-1}h^{-1}(L_i)$ is a sequence of continua in K. Taking if necessary a subsequence, we may assume that this last sequence is also convergent. Then $\operatorname{Lim} f^{-1}h^{-1}(L_i) \subset B$ and since B is zero-dimensional, $\operatorname{Lim} f^{-1}h^{-1}(L_i)$ is one point of B. Thus $\lim p_{i1} = \lim p_{i2}$, i. e. $a_1 = a_2$. The proof of (2.2.2) is then completed.

2.3. THEOREM. *Let K be an* ANR *of dimension* 2 *and satisfying* $\dim(K - \operatorname{reg} K) \leqslant 0$. *Then K is a polyhedron.*

Proof. By (1.3.1) we may assume that K is connected. We shall consider a decomposition of K into strongly cyclic elements and we will prove first that

(i) Every true strongly cyclic element is a polyhedron and is not contractible in itself.

Let E be a t.s.c.e. By (1.2.3), E is a retract of K, and thus E is an ANR. Moreover $\dim(E - \operatorname{reg} E) \leqslant 0$ and, by (1.2.4), L_E is a finite set not disconnecting E. Therefore by (2.2.2), $E - L_E \subset \operatorname{reg} E$. Now let E^* be the set obtained from E by the identification of L_E to a point p. Since L_E is finite, E^* is an ANR; since L_E does not disconnect E, $E^* - p$ is connected and contains only 2-regular points. Thus E^* satisfies all conditions of (2.2.1), therefore is a polyhedron and may be obtained from a manifold by the identification of a finite set of points. The same is then true for E, which proves (i).

By (i) and (1.2.5) there is only a finite number of t.s.c.e. $E_1, \ldots, E_n$. By (1.2.6), $E_i \cap E_j$ is finite for $i \neq j$ and therefore we infer from (i) and (1.3.1) that $E = \bigcup_i E_i$ is a polyhedron.

Now, since K is connected and $\dim(K - \operatorname{reg} K) \leqslant 0$, it follows that $\operatorname{reg} K$ is dense in K. Since $\operatorname{reg} K \subset E$, we have $E = K$. This completes the proof.

2.4. The following corollary gives a topological characterization of 2-dimensional closed pseudomanifolds among continua:

COROLLARY. *In order that a 2-dimensional continuum K be a closed pseudomanifold it is necessary and sufficient that K be an* ANR *such that* $\operatorname{reg}_2 K$ *is connected and* $\dim(K - \operatorname{reg}_2 K) \leqslant 0$.

Proof. Suppose that K satisfies the above conditions. By 2.3 it is then a polyhedron, and thus $K - \operatorname{reg} K$ is a finite set. Since K is connected, $K - \operatorname{reg} K$ is contained in the closure of $\operatorname{reg} K$. Therefore K is a homogenously 2-dimensional polyhedron such that $\operatorname{reg} K$ is connected. By [1], p. 403, K is a closed pseudomanifold. The condition is thus sufficient. It is known that it is necessary ([1], l. c.).

§ 3. The following theorem gives a topological characterization of 2-dimensional polyhedra.

3.1. THEOREM. *In order that a compact space K be a polyhedron of dimension* 2 *it is necessary and sufficient that $K = A \cup B$ where*

1° $K \in \mathrm{ANR}$, 2° $A \subset \operatorname{reg}_2 K$, 3° *$B$ is a graph,*

4° *almost all points $p \in B$ have arbitrarily small neighbourhoods U such that for every component S of $U - B$ the set $S \cup (U \cap B - p)$ is connected.*

The points which do not satisfy 4° will be called *singular*.

The conditions 1°-4° are obviously necessary in order that K be a polyhedron (as regards 4° it is easy to see that a singular point is a vertex in every triangulation). Thus we shall prove only that a space K satisfying 1°-4° is a polyhedron. By (1.3.1) we may assume that K is a Peano continuum; we may also assume that $A \neq 0$ and that $A = K-B$.

In 3.2 the proof of the theorem will be reduced to a proof of its special case. This will be done in 3.3-3.5. 3.6 contains some remarks concerning singular points and an example showing that condition 4° is essential. In 3.7 we prove that theorem 3.1 easily implies the well-known theorem about the equivalence for dimension 2 between locally triangulable spaces and triangulable spaces (polyhedra).

3.2. We shall prove now that theorem 3.1 follows from the

LEMMA. *Let K be a connected* ANR, $K = A \cup B$ *where*

(a) A *is homeomorphic with* $I(Q_2)$,

(b) $B \subset \bar{A}$,

(c) B *is a connected graph containing no more than one point of order* $\neq 2$ *and only this point may be singular.*

Then K is a polyhedron and B a subcomplex of some of its triangulations.

Suppose that the above lemma is true and let K satisfy the conditions of theorem 3.1. We have to prove that K is a polyhedron.

Let $i(K)$ be the space obtained from K by the identification of B, let i: $K \to i(K)$ be the identification mapping. By (1.1.1), $i(K)$ satisfies the conditions of theorem 2.3, and thus it is a polyhedron. Let $\mathrm{St}(p)$ be the closed star of the point $p = i(B)$ in a fixed triangulation of $i(K)$. Let $N_1, \ldots, N_k$ be closures of components of $\mathrm{St}(p)-p$ and let $K_j = i^{-1}(N_j)$, $j = 1, \ldots, k$. Since $K_i \cap K_j = B$ for $i \neq j$, we find by (1.3.1) that to prove that K is a polyhedron it is sufficient to prove that K_j are polyhedra and B is a subcomplex of every K_j. Let $A_j = K_j - B$. Then

(i) A_j is homeomorphic with Q_2 with an interior point removed.

Now we shall prove that

(ii) every K_j is an ANR.

To prove (ii) it is sufficient to prove that K_j is locally contractible at all points belonging to B.

Let $p \in B$, let ε be a positive number and let W be a connected neighbourhood of p in B such that $\delta(W) < \frac{1}{4}\varepsilon$. Since B is a graph, $\bar{W}$ is an AR provided ε is sufficiently small, which is what we suppose. We shall prove first that

(3.2.1) *There is a connected neighbourhood V_0 of p in K such that* (a) $\delta(V_0) < \frac{3}{4}\varepsilon$, (b) $V_0 \cap B = W$, $\bar{V}_0 \cap B = \bar{W}$.

Let

$$G(W) = \mathop{E}_{x \in K}[\varrho(x, W) < \varrho(x, B-W)], \qquad H(W) = \mathop{E}_{x \in K}[\varrho(x, W) < \tfrac{1}{4}\varepsilon],$$
$$V_1 = G(W) \cap H(W)$$

and let V_0 be a connected component of V_1 containing p. Since $G(W)$ and $H(W)$ are open and K is locally connected, it follows that V_0 is a connected neighbourhood of p in K. From $V_0 \subset H(W)$ it follows that V_0 satisfies (a). It satisfies also (b). It is known ([12], § 15, XIII) that $G(W) \cap B = W$. It is easy to see that $\overline{G(W)} \cap B = \bar{W}$. Therefore

$$V_0 \cap B \subset V_1 \cap B = G(W) \cap H(W) \cap B = H(W) \cap W = W$$

and

$$\bar{V}_0 \cap B \subset \bar{V}_1 \cap B = \overline{G(W) \cap H(W)} \cap B \subset \overline{G(W)} \cap \overline{H(W)} \cap B = \overline{H(W)} \cap \bar{W} = \bar{W}.$$

On the other hand $W \subset G(W)$, $W \subset H(W)$, $W \subset V_0$ and $\bar{W} \subset B$, thus

$$W \subset V_0 \cap B$$

and

$$\bar{W} \subset \bar{V}_0 \cap B.$$

All four relations together give (b).

Let V_0 be a neighbourhood of p in K satisfying (3.2.1) with ε so small that V_0 is contained in $\bigcup K_j$. Then every component of $V_0 - B$ is contained in a set K_j. We fix the index j; let S_j be the sum of those components of $V_0 - B$ which are in K_j. Let r' be a retraction of $\bigcup_{i \neq j} S_i \cup W$ to W; such a retraction exists since by (3.2.1) (b), W is closed in V_0 and thus also in $\bigcup_{i \neq j} S_i \cup W$. Since K is locally connected, the function

$$r(x) = \begin{cases} r'(x) & \text{for} \quad x \in \bigcup_{i \neq j} S_i \cup W, \\ x & \text{for} \quad x \in S_j \cup W \end{cases}$$

is continuous and is a retraction of $\bigcup_i S_i \cup W$ to $S_j \cup W$.

Now let G be a neighbourhood of p in K contained in V_0 and contractible in V_0. Then $G_j = G \cap K_j$ is a neighbourhood of p in K_j and $G_j \subset S_j \cup W \subset V_0 \cap K_j$. Since $S_j \cup W$ is a retract of V_0, G_j is contractible in $S_j \cup W$, which completes the proof of (ii).

We shall now consider the set $\bar{A}_j$. Since B is a graph, $\dim\left(\mathrm{Fr}_{K_j}(\bar{A}_j)\right) \leqslant 0$ and it follows that $\bar{A}_j$ is a locally connected subcontinuum of K_j. Therefore by (ii) and (1.2.1) there is only a finite number of components of $K_j - \bar{A}_j$ whose boundaries contain more than one point. But there is also a finite number of components whose boundaries contain only one point. For only a finite number among them contains points of order $\neq 2$ and the closures of those which contain only points of order 2 are simple closed curves intersecting $\bar{A}_j$ at one point only. By (ii) they are finite in number.

It follows that $\overline{K_j - \bar{A}_j}$ is a sum of a finite number of closed and connected subsets of B, whence it is a polyhedron; moreover this polyhedron intersects $\bar{A}_j$ at a finite set of points. Thus also $B_j = B \cap \bar{A}_j$ is a graph and to prove that K_j is a polyhedron with B as a subcomplex it is sufficient to show that $\bar{A}_j$ is a polyhedron with B_j as a subcomplex.

Since $K_j = \overline{K_j - \bar{A}_j} \cup \bar{A}_j$ and the intersection of these sets is a finite set of points, we infer from (ii) and [12], § 48, III, 1 that

(iii) $\bar{A}_j$ is an ANR.

We have shown before that

(iv) B_j is a graph contained in $\bar{A}_j$.

Now, by 3.1, 4°, B_j contains only a finite number of singular points. We identify them, this will not change (i), (iii) and (iv). We identify also to a point the simple closed curve which is the boundary of $\bar{A}_j$; the space obtained after these identifications satisfies all conditions of lemma 3.2; therefore is a polyhedron and the set obtained by this identification from B_j is a subcomplex. Now, there exists by (i) a simple closed curve J in A_j which separates $\bar{A}_j$ into just two components F_1 and F_2 such that F_1 contains the simple closed curve which is the boundary of $\bar{A}_j$ and F_2 contains B_j. Then $\bar{F}_2$ is a polyhedron with B_j as a subcomplex, $\bar{F}_1$ is a 2-dimensional annulus, thus a polyhedron, and since $\bar{F}_1 \cap \bar{F}_2 = J$, it follows from (1.3.1) that $\bar{A}_j = \bar{F}_1 \cup \bar{F}_2$ is a polyhedron and B_j a subcomplex of $\bar{A}_j$. This completes the proof that theorem 3.1 follows from lemma 3.2.

We proceed now to the proof of lemma 3.2. K will be a space satisfying all conditions from 3.2 and we shall assume that $A = K - B$. The sole singular point of B will be denoted by b. By i we shall denote the identification mapping which maps B into one point and is a homeomorphism in A.

3.3. Construction of normed neighbourhoods. We shall prove first that

(3.3.1) *For every $p \in B$ there are arbitrarily small neighbourhoods V such that $V \cap B$ is connected, $\bar{V} \cap B$ is an* AR *and $\bar{V} \cap A$ is a manifold with boundary (non-compact).*

Let ε be a positive number and let W be a connected neighbourhood of p in B with $\delta(W) < \frac{1}{4}\varepsilon$. We shall assume that $\bar{W}$ is an AR and let V_0 be a neighbourhood of p satisfying (3.2.1).

Now let T be a triangulation of $K - B$ with all simplexes of diameter less than $\frac{1}{8}\varepsilon$ and such that the diameter of simplexes converges to zero together with the distance from B. Let T' be a barycentric subdivision of T. We add to $\bar{V}_0$ all simplexes of T which intersect $\bar{V}_0$ and if in this sum a vertex of a simplex is a local disconnecting point we add the star of that vertex in T'. Let the interior of this sum be V. Then V is a connected neighbourhood of p. $\delta(V) < \varepsilon$, $\bar{V} \cap B = \bar{V}_0 \cap B = \bar{W}$, $V \cap B = W$, and $\bar{V} \cap A$ is a manifold with boundary, since $\bar{V} \cap A$ is an infinite polyhedron lying in A and without local disconnecting points. This proves (3.3.1).

Let again $p \in B$. We shall prove that

(3.3.2) *There are arbitrarily small connected neighbourhoods V of p such that $V \cap B$ is connected and $i(\bar{V})$ is homeomorphic to the sum of 2-dimensional elements every two of which have only one common point lying on the boundary of both, this point being common to all elements.*

A neighbourhood U of a point $p \in B$ is said to be *normed* if it satisfies (3.3.2) and if $U \cap B - p$ does not contain the singular point b.

We proceed to the proof of (3.3.2). Let $p \in B$ and let U be a neighbourhood of p in K with diameter less than ε. Let U_1 be a neighbourhood of p in K contractible in U and let V be a neighbourhood of p in K satisfying (3.3.1) and such that $\bar{V} \subset U_1$. Then $\bar{V}$ is contractible in U and the inclusion homomorphism $H_1(\bar{V}) \to H_1(\bar{U})$ is trivial. We shall prove that also the inclusion homomorphism $H_1(i(\bar{V})) \to H_1(i(\bar{U}))$ is trivial. To this end observe first that in the mapping $\bar{V} \to i(\bar{V})$ the inverse-image of a point is either a point or the set $\bar{V} \cap B$. It follows that all inverse-images are acyclic and therefore by the Vietoris Mapping Theorem [2] the induced homomorphism $H_1(\bar{V}) \to H_1(i(\bar{V}))$ is an isomorphism onto. Thus the commutativity of the diagram

$$\begin{array}{ccc} H_1(\bar{V}) & \to & H_1(\bar{U}) \\ \downarrow & & \downarrow \\ H_1(i(\bar{V})) & \to & H_1(i(\bar{U})) \end{array}$$

implies that the homomorphism $H_1(i(\bar{V})) \to H_1(i(\bar{V}))$ is trivial.

Now, $i(K)$ is a 2-sphere; it follows from elementary duality theorems that all components of $i(K) - i(\bar{V})$ except one are contained in $i(\bar{U})$. Let W be the sum of $i(\bar{V})$ and of all those components of $i(K) - i(\bar{V})$ which are contained in $i(\bar{U})$. Thus W does not disconnect $i(K)$ and is locally connected at all points with the exception — perhaps — of the point $i(B) = q$. Therefore it is locally connected. Let S be the closure of a component of $W - q$. Since $\mathrm{Fr}_W S = q$, S is locally connected. Moreover S does not disconnect the sphere $i(K)$ and no point disconnects S

(since only q can disconnect W). But this implies ([12], § 54, IV, 11) that S is homeomorphic with Q_2. Therefore W is homeomorphic with the sum of 2-dimensional elements every two of which have only one common point lying on the boundary of both, this point being common to all elements.

Now let $V_1 = V \cup i^{-1}(\mathrm{Int}(W))$. Then V_1 is a neighbourhood of p, $V_1 \subset U$, and thus $\delta(V_1) < \varepsilon$; obviously V_1 is connected and $i(\overline{V}_1) = i(\overline{V}) \cup i\left(\overline{i^{-1}(\mathrm{Int}(W))}\right) = i(\overline{V}) \cup \overline{\mathrm{Int}(W)} = W$. This completes the proof of (3.3.2).

3.4. Properties of normed neighbourhoods. We adopt in 3.4 the following notation: p will be a fixed point of B, U will be a normed neighbourhood of p, S will be a component of $U-B$ and $L = \overline{S} \cap B$. q will denote a fixed point of $\mathrm{Fr}_A(S)$, F will denote $\overline{\mathrm{Fr}_A(S)}$. By (3.3.2), $\mathrm{Fr}_A(S)$ is an arc without extremities; such is therefore also every component of $\mathrm{Fr}_A(S)-q$. The closures in K of those two components will be denoted F_1, F_2.

(3.4.1) *F, F_1, F_2 are continua, $F_j \cap B$ is connected and non-void.*

The first part is obvious. Were $F_j \cap B$ void, we should have $F_j \subset A$ and therefore $i(F_j) \cap i(B) = 0$. This is impossible, since $i(B) \subset i(F_j)$. Now, since $i(F_j \cap B)$ is one of the extremities of the arc $i(F_j)$, it follows from (1.1.2) that $F_j \cap B$ is connected.

(3.4.2) *L is connected.*

$i(L)$ does not locally disconnect $i(\overline{S})$, the last set being a 2-element. Thus (3.4.2) follows from (1.1.2).

(3.4.3) *Let T be an arbitrary subset of A. Then $\mathrm{Fr}_B(\overline{T} \cap B) \subset b \cup \left(\overline{\mathrm{Fr}_A(T)} \cap B\right)$. In particular, if p is not the singular point b, then $\mathrm{Fr}_B(L) \subset F \cap B$.*

First, we shall prove that if $p \in \mathrm{Fr}_B(L)$ then p is singular. For suppose that $p \in \mathrm{Fr}_B(L)$ and let V be an arbitrary neighbourhood of p contained in U. Let S' be a component of $V-B$ contained in S. Such a component does exist since $p \in \overline{S}$. Now, since no point of B is of order 1, we may write $V \cap B - p = L_1 \cup L_2$ where $(L_1 \cap \overline{L}_2) \cup (\overline{L}_1 \cap L_2) = 0$. Since L is a connected subset of B and $p \in \mathrm{Fr}_B(L)$, one of L_i, say L_2, is disjoint with L. Therefore

$$\overline{L}_2 \cap (S' \cup L_1) = (\overline{L}_2 \cap S') \cup (\overline{L}_2 \cap L_1) = 0\,.$$

But $S' \subset S$ implies $\overline{S}' \cap L_2 \subset \overline{S} \cap L_2 \subset \overline{S} \cap B = L$ and because of $L \cap L_2 = 0$ we have

$$L_2 \cap \overline{(S' \cup L_1)} = (L_2 \cap \overline{S}') \cup (L_2 \cap \overline{L}_1) = 0\,.$$

Therefore $S' \cup L_1 \cup L_2 = S' \cup (V \cap B - p)$ is not connected for some component S' of every sufficiently small neighbourhood of p, which means that p is singular.

Now let T be an arbitrary subset of A. Let $t \in \mathrm{Fr}_B(\overline{T} \subset B)$ and suppose that $t \notin \overline{\mathrm{Fr}_A(T)}$. We shall prove that t is singular, i. e. $t = b$.

Let V be a normed neighbourhood of t so small that $V \cap \overline{\mathrm{Fr}_A(T)} = 0$. Let S' be such a component of $V-B$ that $S' \cap T \neq 0$. Such a component does exist since $t \in \overline{T}$. Since S' is connected, $S' \subset A$, $S' \cap T \neq 0$ and $S' \cap \mathrm{Fr}_A(T) = 0$ thus $S' \subset T$. It follows that $\overline{S}' \cap B \subset \overline{T} \cap B$, and therefore $t \in \mathrm{Fr}_B(\overline{S}' \cap B)$. But we have proved that this implies that t is singular; therefore the proof is completed.

(3.4.4) *L contains at least one component of $U \cap B - b$. In particular*

(a) *L is a continuum containing p;*

(b) *if p is not singular then L is an arc containing $U \cap B$.*

Since $L = \overline{S} \cap B \supset F \cap B = (F_1 \cup F_2) \cap B \neq 0$ and $F_i \cap U = 0$, we infer that there is a point $t' \in L - U$. Since U is connected, there is a point $t'' \in L \cap U$. By (3.4.2), L contains the arc $t't''$ and thus also a point different from p and belonging to U. That point is then certainly different from b; let T be the component of that point in $U \cap B - b$. Then $T \cap L \neq 0$. Were T not contained in L, there would exist a point t satisfying $t \in T \cap \mathrm{Fr}_B(L)$. In particular t would belong to U, and thus U would be a normed neighbourhood for t. Since t is not singular, the inclusion $t \in \mathrm{Fr}_B(L)$ contradicts (3.4.3). This completes the proof.

(3.4.5) *p is accessible from S.*

Let V be a normed neighbourhood of p contained in U. Let T be the sum of those components of $V-B$ which are contained in S. Then $T \cup p$ is a neighbourhood of p in $S \cup p$ and moreover it is a connected set, since by (3.4.4), p belongs to the closure of every component forming T. Therefore $S \cup p$ is locally connected, which proves (3.4.5) by [12], § 45, III, 7.

By (3.4.5) there is an arc $J = pq$ contained with the exception of the extremities in S. Since S is homeomorphic with $I(Q_2)$, $S-J$ has just two components S_1, S_2, each of them homeomorphic with S. Now, F_1 is contained either in $\overline{S}_1$ or in $\overline{S}_2$. We suppose that $F_1 \subset \overline{S}_1$. Then $F_2 \subset \overline{S}_2$.

(3.4.6) *$\overline{S}_i \cap B$ is a continuum, $i = 1, 2$. If p is not singular then $\overline{S}_i \cap B$ is an arc and $\overline{S}_1 \cap \overline{S}_2 \cap B = p$.*

First, $\overline{S}_i \cap B$ contains $F_i \cap B$ and the last set is not empty by (3.4.1). On the other hand $\overline{S}_i \cap B$ contains p, and $p \notin F_i$. Therefore $\overline{S}_i \cap B$ contains two different points and, being connected, is a continuum.

Now, suppose that p is not singular. By (3.4.4), L is an arc. Therefore $\overline{S}_i \cap B$, being subcontinua of L, are arcs, and their sum is L. To complete the proof we have to show that if $p \in \bar{S}_i \cap B$ for $i = 1, 2$, then p is an extremity of $\bar{S}_i \cap B$, $i = 1, 2$.

Suppose that p is an interior point of $\bar{S}_1 \cap B$. Then $\mathrm{Fr}_B(\bar{S}_1 \cap B)$ intersects both components of $L-p$. On the other hand $\overline{\mathrm{Fr}_A(S_i)} = J \cup F_i$ and we infer from (3.4.3) that $\mathrm{Fr}_B(\bar{S}_i \cap B) \subset b \cup p \cup (F_i \cap B)$. Since neither p nor b belong to $\mathrm{Fr}_B(\bar{S}_1 \cap B)$, we have $\mathrm{Fr}_B(\bar{S}_1 \cap B) \subset F_1 \cap B$. It follows that $F_1 \cap B$ intersects both components of $L-p$ and since $F_1 \cap B$ does not contain p we infer that it is not connected, which contradicts (3.4.1) and completes the proof.

(3.4.7) *$U-B$ has a finite number of components.*

Suppose first that p is not singular and suppose that (3.4.7) is false. Let $S_1, S_2, \ldots$ be an infinite sequence of components of $U-B$, let $F_i = \overline{\mathrm{Fr}_A(S_i)}$. The sequence $\{F_i\}$ contains a convergent subsequence: we may suppose simply that F_i converge to a set F_0. Since F_i are continua, F_0 is also a continuum, which is contained in B, since K is locally connected. Let p_n, q_n be extremities of the segment $L_n = \bar{S}_n \cap B$. By (3.4.3), $p_n, q_n \in F_n$, by (3.4.4), $L_n \supset U \cap B$. It follows that $F_0 \supset U \cap B$, and thus $p \in F_0$, which is impossible, since F_n are disjoint with U.

Now let U be a normed neighbourhood of the singular point b. Since $U \cap B$ is a connected neighbourhood in the graph B, there is only a finite number of components of $U \cap B - b$, denote it by $L_1, \ldots, L_n$. Let T_i, $i = 1, 2, \ldots, n$, be the set of all such components S of $U-B$ that $\bar{S} \supset L_i$. Let V be a normed neighbourhood of a point from L_i, so small that $V \subset U$. Then every component forming T_i contains one component of $V-B$, and no two contain the same component of $V-B$. Therefore T_i has no more components than $V-B$, i. e. a finite number. But by (3.4.4) every component of $U-B$ belongs to one of T_i. Thus there is only a finite number of components of $U-B$.

(3.4.8) *If U and V are normed neighbourhoods of two points of the same component of $B-b$, then $U-B$ and $V-B$ have the same number of components.*

Since a normed neighbourhood for a non singular point is a normed neighbourhood for all points which are near, it will be sufficient to prove (3.4.8) in the case where U and V are neighbourhoods of the same point. Since two normed neighbourhoods of the same point contain in their common part another normed neighbourhood, we may assume that $U \supset V$. Since by (3.4.4) the closure of every component of $U-B$ contains the point p, it contains also a component of $V-B$. To complete the proof we have to show that it contains only one such component.

Suppose that S is a component of $U-B$ and that S_1, S_2 are two different components of $V-B$ contained in S. Let $a_i \in S_i$, let L_i be an arc $a_i p$ contained in $S_i \cup p$ and let L_3 be an arc $a_1 a_2$ in S. We may assume that $L_3 \cap (L_1 \cup L_2) = a_1 \cup a_2$. Then $L = L_1 \cup L_2 \cup L_3$ is a simple closed curve in $S \cup p$. Since S is homeomorphic with $I(Q_2)$, $L \cap S$ disconnects S and one of the components of the complement of $L \cap S$ in S is closure-disjoint with $\mathrm{Fr}_A S$. Denote it by D and let $E = \bar{D} \cap B$. Since $\overline{\mathrm{Fr}_A(D)} = L$, we have by (3.4.3), $\mathrm{Fr}_B(E) \subset L \cap B = p$. But this is impossible: Since $p \notin \overline{\mathrm{Fr}_A(S_1)}$ and $E \cap \overline{\mathrm{Fr}_A(S_1)} \neq 0$, it follows that E contains more than one point. Thus so does its boundary and this contradiction completes the proof of (3.4.8).

Using again the notation from (3.4.6) we shall prove the following.

(3.4.9) *Let p_n, q_n be two sequences of points such that $p_n, q_n \in S$ and $\lim p_n = p = \lim q_n$. There is a sequence of arcs $p_n q_n \subset S$ such that $\delta(p_n q_n) \to 0$.*

If all points p_n, q_n belong to the same one of the sets S_i, then the arc $p_n q_n$ may be found also in the same set S_i.

Let ε be a positive number and let $V \subset U$ be a normed neighbourhood of p with diameter less than ε. By (3.4.8) there is one and only one component S' of $V-B$ contained in S. Therefore there exists such an N that for $n > N$ both p_n and q_n belong to S'. But S' is arcwise connected, hence there is an arc $p_n q_n \subset S'$. It follows that $\delta(p_n q_n) < \varepsilon$, which completes the proof of the first part of (3.4.9).

Suppose now that $p_n, q_n \in S_1$. We have proved that there is a sequence of arcs $p_n q_n$ in S such that

(i) $$\delta(p_n q_n) \to 0\,.$$

Let p'_n be the first and q'_n the last — starting from p_n — point of the set $p_n q_n \cap J$. By (i), $p'_n, q'_n \to p$. Let $p'_n q'_n$ be the arc in J with extremities p'_n, q'_n. We then have

(ii) $$\delta(p'_n q'_n) \to 0\,.$$

Let $p_n p'_n, q_n q'_n$ be subarcs of the arc $p_n q_n$ determined by the corresponding extremities. Let $L_n = p_n p'_n \cup p'_n q'_n \cup q'_n q_n$. We have by (i) and (ii)

(iii) $$\delta(L_n) \to 0\,.$$

Now, L_n is an arc in $\bar{S}_1$ with extremities p_n, q_n in S_1. An elementary reasoning (e. g. by using a triangulation of S in which J is a subcomplex) shows that there is an arc L'_n in S_1 with the same extremities as L_n and arbitrarily close to L_n. By (iii) we shall have again $\delta(L'_n) \to 0$. The proof is then completed.

3.5. The triangulation of K. We shall use the following notation: b will be as usual the singular point, components of $B-b$ will be denoted by B_a, $a = 1, \ldots, n$. Every set B_a is an open arc, $\bar{B}_a = B_a \cup b$ is a simple closed curve.

If $p \in B$ and U is a normed neighbourhood of p, then the number of components of $U-B$ will be denoted by $k(p)$. By (3.4.7) and (3.4.8), $k(p)$ is a natural number independent of U and if $p, q \in B_a$ then $k(p) = k(q)$.

Chose in each component B_a a point b_a, $a = 1, \ldots, n$, and let $b_0 = b$. Let U_a be a normed neighbourhood of b_a, $i = 1, \ldots, n$. We shall assume that U_a are so small that $\overline{U}_a \cap \overline{U}_\beta = 0$ for $a \neq \beta$. Choose a point a in $A - \bigcup_a U_a$. Let $S_a^1, \ldots, S_a^{k(b_a)}$ be all components of $U_a - B$. Let a_a^β be a fixed point $\mathrm{Fr}_A(S_a^\beta)$.

The sum of n arcs with extremities x, y and with disjoint interiors will be called the curve $\theta_n(x, y)$.

Let $L_a'^\beta$ be an arc with extremities b_a and a_a^β contained in $S_a^\beta \cup a_a^\beta \cup b_a$. Such an arc exists by (3.4.5). Let L''^β_a be the arc with extremities $a_a^\beta a$ contained in $(A - \bigcup_a U_a) \cup a_a^\beta$; we suppose moreover that $L_a''^\beta \cap L_\gamma''^\delta = a$ if $a \neq \gamma$ or $\beta \neq \delta$. Put $L_a^\beta = L_a'^\beta \cup L_a''^\beta$ and $K_1 = B \cup \bigcup_{a,\beta} L_a^\beta$.

K_1 is a graph and B a subcomplex of K_1. Let us observe that $\bigcup_\beta L_a^\beta$ is the curve $\theta_n(a, b_a)$ with $n = k(b_a)$, and that two such different curves have only the point a in common. Since the identification mapping i identifies all points b_a, $i(K_1)$ is homeomorphic with a θ_n-curve with $n > 1$. Since $i(K)$ is homeomorphic with a 2-sphere, $i(K_1)$ disconnects $i(K)$ and every component of the complement is homeomorphic with $I(Q_2)$. The inverse images in K of those components we shall denote by $T_1', \ldots, T_m'$ and the closure in K of T_j' will be denoted by T_j. Since the mapping i is a homeomorphism in A, it follows that T_j' are components of $A - K_1$ and are homeomorphic with $I(Q_2)$. Since $i(T_j)$ is homeomorphic with Q_2, it follows from (1.1.2) that $T_j \cap B$ is connected for all j.

Let T' be one of the sets T' and let $T = \overline{T}'$. Let S be the component S_a^β of $U_a - B$.

(3.5.1) $K_1 \cap S$ *disconnects* S *into just two components* S_1, S_2 *which satisfy*

(a) $\overline{S}_i \cap B$ *is a continuum; if* $a \neq 0$ *then* $\overline{S}_1 \cap \overline{S}_2 \cap B = b_a$;

(b) *if* $T' \cap S \neq 0$ *then* $T' \cap S$ *is one of the sets* S_1, S_2 *and* $T \supset L_a^\beta$.

Since $S \cap K_1 = L_a'^\beta - (a_a^\beta \cup b_a)$, the situation is as in (3.4.6), and this proves (a). We shall prove (b). It follows from $S \cap T' \neq 0$ that one of the sets S_1, S_2, for instance S_1, satisfies $S_1 \cap T' \neq 0$. But S_1 is a connected subset of $K - K_1$, T' is a component of $K - K_1$, and thus $S_1 \subset T'$. Now, since K_1 disconnects S, it follows that the θ_n-curve $i(K_1)$ disconnects $i(S)$. But $i(S)$ is an open subset of a 2-sphere $i(K)$, thus $i(S_1)$ and $i(S_2)$ are contained in different components of $i(K) - i(K_1) = i(A) - i(K_1)$, i. e. S_1 and S_2 are contained in different components of $A - K_1$. Hence $S_1 \subset T'$ implies $S_2 \cap T' = 0$.

Finally, since $\overline{S}_1 \supset L_a'^\beta$, also $T \supset L_a'^\beta$, and therefore $T \supset L_a^\beta$, which proves (b).

Now, $\overline{\mathrm{Fr}_A(T)}$ is a sum of two arcs from the system L_a^β. Suppose that there are arcs L_a^β and L_γ^δ. We shall prove that

(3.5.2) *One of these arcs contains* b_0, *the second does not.*

First, we prove that

(i) If $b_\varkappa \in T \cap B$ then $b_\varkappa \in L_a^\beta \cup L_\gamma^\delta$ (i. e. either $\varkappa = a$ or $\varkappa = \gamma$).

For if $b_\varkappa \in T \cap B - (L_a^\beta \cup L_\gamma^\delta)$, then there is a component S of $U_\varkappa - B$ such that $S \cap (L_a^\beta \cup L_\gamma^\delta) = 0$ and $S \cap T' \neq 0$. Thus $S \subset T'$. But this is impossible, since $T' \cap K_1 = 0$ and $S \cap K_1 \neq 0$ by (3.5.1).

Next, we shall prove that

(ii) $T \cap B$ is a continuum.

Since $L_a^\beta \subset T$, we have $b_a \in T$, and thus $T' \cap U_a \neq 0$. Hence there exists a component S of $U_a - B$ satisfying $T' \cap S \neq 0$. By (3.5.1) (b) we have $T' \supset S_1$ where S_1 is one of components of $S - K_1$; by (3.5.1) (a), $\overline{S}_1 \cap B$ is a continuum. Since $T \cap B$ is connected and contains $\overline{S}_1 \cap B$, it is a continuum.

Now, since $\overline{\mathrm{Fr}_A(T)} = L_a^\beta \cup L_\gamma^\delta$, we have by (3.4.3)

(iii) $\mathrm{Fr}_B(T \cap B) \subset b_0 \cup b_a \cup b_\gamma$.

Now we can prove (3.5.2). First suppose that $a = 0$ and $\gamma = 0$. Then by (iii), $\mathrm{Fr}_B(T \cap B) \subset b_0$, thus by (ii) we infer that $T \cap B$ contains one of the sets $B_\varkappa$. Therefore it contains also one of the points $b_\varkappa$ with $\varkappa \neq 0$. This contradicts (i).

Suppose now that $a \neq 0$ and $\gamma \neq 0$. Then b_a and b_γ are in different components of $B - b_0$, and since $T \cap B$ is a continuum containing b_a and b_γ, it contains also b_0, which again contradicts (i). This completes the proof of (3.5.2).

(3.5.3) *If* $T' \cap U_a \neq 0$ *then* T' *intersects only one component* S *of* $U_a - B$. *If* S_1 *and* S_2 *are components of* $S - K_1$, *then* $T' \cap U_a = S_1$ *or* $T' \cap U_a = S_2$.

If $T' \cap U_a \neq 0$, then $T' \cap S_a^\beta \neq 0$ for an $\beta, 1 \leqslant \beta \leqslant k(b_a)$. Suppose that also $T' \cap S_a^\gamma \neq 0$, $1 \leqslant \gamma \leqslant k(b_a)$. Then by (3.5.1) (b), $L_a^\beta \subset T$ and $L_a^\gamma \subset T$. Thus by (3.5.2) it follows that $\beta = \gamma$, i. e. $T' \cap S \neq 0$ for one and only one component S of $U - B$. Applying to this component (3.5.1) (b) we have $T' \cap S = S_1$, and since $T' \cap U = T' \cap S$, we have $T' \cap U = S_1$.

By (3.5.2) we shall assume henceforth that $\overline{\mathrm{Fr}_A(T)} = L_0^\beta \cup L_\gamma^\delta$, with $\gamma \neq 0$. Then

(3.5.4) $T \cap B$ *is an arc with extremities* b_0, b_γ.

We shall prove first that $T\cap B\subset \bar{B}_\gamma$. For if $T\cap B_a\neq 0$ with $a\neq\gamma$ then $\mathrm{Fr}_B(T\cap B)\subset b_\gamma\cup b_0$ gives $B_a\subset T\cap B$, and thus $b_a\in T\cap B$, $a\neq 0,\gamma$. This contradicts (3.5.2) (i).

Now we shall prove that $T\cap B$ is an arc. Since $\bar{B}_\gamma$ is a simple closed curve and $T\cap B$ is a continuum contained in $\bar{B}_\gamma$, it is sufficient to show that $\bar{B}_\gamma - T\cap B\neq 0$.

Consider the neighbourhood U_γ. Since $b_\gamma\in T$, we have $T'\cap U_\gamma\neq 0$. We infer from (3.5.3) that $T'\cap U_\gamma = S_1$.

Now, $U_\gamma\cap B-b$ has two components G_1, G_2 and by (3.5.1) (a) one of them, say G_2, is disjoint with S_1. Let $g\in G_2$ and let $V\subset U_\gamma$ be a neighbourhood of g such that $V\cap S_1 = 0$. Then $V\cap T' = V\cap U_\gamma\cap T' = V\cap S_1 = 0$ and this implies that $g\notin T$.

Thus $T\cap B$ is an arc. Since $\mathrm{Fr}_B(T\cap B)\subset b_0\cup b_\gamma$, hence $T\cap B$ is an arc with extremities b_0, b_γ.

(3.5.5) $\mathrm{Fr}(T') = T-T'$ *is a simple closed curve. T is a Peano continuum.*

Since T' is open, $\mathrm{Fr}(T') = T-T'$. Now, $\mathrm{Fr}(T') = L_0^\beta\cup L_\gamma^\delta\cup(T\cap B)$. Since $L_0^\beta\cup L_\gamma^\delta$ is an arc with extremities b_0, b_γ and interesecting $T\cap B$ only at these points, we infer from (3.5.4) that $\mathrm{Fr}(T')$ is a simple closed curve. Hence T is a Peano continuum by [12], § 44, III, 4.

(3.5.6) *Let p be an interior point of the arc $L = T\cap B$ and let U be a normed neighbourhood of p. There is one and only one component S of $U-B$ contained in T' (i. e. $T'\cap U = S$).*

Let $T' = T_1', T_2', \ldots, T_m'$ be all such components of $K-K_1$ that p belongs to the closure of T_j'. By (3.5.4) we have $\bar{T}_j'\cap B = L$, $j = 1, \ldots, m$. Each of $k(p)$ components of $U-B$ is contained in one of the sets T_j'. Each of T_j' contains at least one of the components of $U-B$. If there were more than one component of $U-B$ contained in T', we should have $m<k(p)$. We will show that this is impossible.

Let $S_\gamma^1, \ldots, S_\gamma^{k(b_\gamma)}$ be components of $U_\gamma - B$. By (3.5.1) (a) there are two components $S_{1\gamma}^\beta, S_{2\gamma}^\beta$ of $S_\gamma^\beta - K_1$ and one of them, say $S_{1\gamma}^\beta$, satisfies $\bar{S}_{1\gamma}^\beta\cap B\subset L$. By (3.5.3) it follows that for every T_j' there is such an index β that $T_j'\cap U_\gamma$ is one of $S_{1\gamma}^\beta, S_{2\gamma}^\beta$. Since $\bar{S}_{2\gamma}^\beta\cap B - L\neq 0$, by (3.5.1) (a), and $\bar{T}_j'\cap B = L$, whence $T_j'\cap U_\gamma = S_{1\gamma}^\beta$. Hence the number of the sets T_j' is not smaller than the number of components $S_{1\gamma}^\beta$ with a fixed γ, i. e. $m\geqslant k(b_\gamma)$. But $p\in B_k$, and thus $k(p) = k(b_\gamma)$, i. e. $m\geqslant k(p)$. The proof is thus completed.

(3.5.7) *T' is uniformly arcwise connected.*

Let p_n, q_n be two sequences of points in T' such that $\varrho(p_n, q_n)\to 0$ and suppose that

(i) every arc in T' with extremities p_n, q_n is of diameter greater than a fixed positive number ε.

Obviously we may assume that the sequences p_n, q_n are convergent; let

(ii) $\lim p_n = \lim q_n = p$.

Now, since $T-T\cap B$ is homeomorphic with the half-plane, we have $p\in T\cap B$. First, suppose that p is an interior point of $T\cap B$ and let U be the normed neighbourhood of p. It follows from (ii) that, starting from a certain index, all pairs p_n, q_n belong to $U\cap T'$. Thus by (3.5.6) we infer that there is a component S of $U-B$ such that, starting from a certain index, all pairs p_n, q_n belong to S. Then (i) contradicts (3.4.9).

Now suppose that p is one of the extremities of the arc $T\cap B$, i. e. let $p = b_a$. By (ii) it follows that almost all pairs p_n, q_n belong to $U_a\cap T'$; therefore the same pairs belong, by (3.5.3), to S_1. Hence (i) again contradicts (3.4.9) and this completes the proof of (3.5.7).

We shall say that an arc J with extremities a_1, a_2 *spans* $\mathrm{Fr}(T')$ if $J\subset T$ and $J\cap\mathrm{Fr}(T') = a_1\cup a_2$. We will show that

(3.5.8) *Every arc J spanning $\mathrm{Fr}(T')$ disconnects T. No true closed subset of an arc spanning $\mathrm{Fr}(T')$ disconnects T.*

Since T' is homeomorphic with $I(Q_2)$, $T'-J$ has two components C_1, C_2. Were $T-J$ connected, it would mean that there exists a point $p\in\bar{C}_1\cap\bar{C}_2 - J$. Suppose that such a point does exist. Let $p_n^i\to p$, $p_n^i\in C_i$, $i = 1, 2$. We then have $\lim\varrho(p_n^1, p_n^2) = 0$ and $\lim\varrho(p_n^1, J) = \varepsilon > 0$. Since every arc $p_n^1p_n^2$ joining p_n^1 and p_n^2 in T' intersects J, we have $\delta(p_n^1p_n^2) > \varrho(p_n^1, J)$. Therefore $\lim\delta(p_n^1p_n^2)\geqslant\varepsilon > 0$ for every sequence of arcs joining p_n^1 and p_n^2 in T'. But this contradicts (3.5.7) and thus proves that J disconnects T.

Let J span $\mathrm{Fr}(T')$ and let N be a closed true subset of J. Then $T'-N$ is connected. But $T'-N\subset\bar{T}'-N = \bar{T}'-\bar{N}\subset\overline{T'-N}$, and since $T-N = \bar{T}'-N$, $T-N$ is connected.

By theorem II from [7] we infer from (3.5.5) and (3.5.8) that

(3.5.9) *T is homeomorphic with the 2-simplex.*

Now let C be the complex of the triangulation of $\mathrm{Fr}(T')$ defined by the points a, b_0, b_γ as unique vertices. Then the 1-dimensional simplexes of this triangulation are arcs $T\cap B = b_0b_\gamma$, $L_0^\beta = ab_0$, $L_\gamma^\delta = ab_\gamma$. If T_1' is another components of $K-K_1$ and C_1 an analogous triangulation of $\mathrm{Fr}(T_1')$, then $T\cap T_1$ is a subcomplex of both C and C_1. By (1.3.1) the sum of all sets T is a polyhedron and the sum of all sets $T\cap B$ is its subcomplex. But the first sum is the set K and the second is the set B, which proves lemma 3.2. Since theorem 3.1 follows from lemma 3.2, the proof of theorem 3.1 is then completed.

3.6. Remarks. Conditions 1°-3° from 3.1 need no explanation. Condition 4° may be replaced by any of the following three:

(3.6.1) *Almost all points $p \in K$ have the following property: for every $\varepsilon > 0$ there is $\delta > 0$ such that for every pair of points $x \in A$, $y \in B$ satisfying $\varrho(p, x) + \varrho(p, y) < \delta$ exists in $A \cup y$ a continuum L containing x and y and of diameter less than ε.*

(3.6.2) *Almost all points $p \in K$ have arbitrarily small neighbourhoods U such that for efery $q \in U \cap B$, $\mathrm{Fr}(U)$ is a deformation retract of $\overline{U} - q$.*

(3.6.3) *There is a finite subset $B_0 \subset B$ such that for every subset T of A we have* $\mathrm{Fr}_B(\overline{T} \cap B) \subset B_0 \cup \overline{\mathrm{Fr}_A(T)}$.

We omit the proofs, which are rather easy. (3.6.3) is identical with (3.4.3) and it is easy to verify that in the proof of theorem 3.1 we use only (3.4.3).

It is worthwhile to observe that Condition 4° is essential, i. e. that there exist spaces which satisfy conditions 1°-3° from 3.1 but which are not polyhedra. A simple example is as follows.

Let Q be the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. We shall consider a semi-continuous decomposition of Q with the following elements:

(a) pairs of points $(1/n, y), (y+1/n, 0)$ where $0 \leqslant y \leqslant 1/2^n$, $n = 2, 3, \ldots$;
(b) all other individual points of Q.

Let Q^* be the hyperspace of this decomposition. By (1.1.1), Q^* is an ANR. It is easy to see that Q^* may be decomposed into the sum of two subsets, one of them being a simple closed curve and the second being homeomorphic with an open 2-element. Thus Conditions 1°-3° are satisfied.

Now, were Q^* a polyhedron, the images in Q^* of points $(1/n, 1/2^n)$ would be vertices of the triangulation. But this is imposisble since Q^* is compact and there are infinitely many images of points $(1/n, 1/2^n)$.

3.7. We shall say that the space K is *locally polyhedral* if every point $p \in K$ has in K neighbourhoods homeomorphic with an open subset of a polyhedron.

THEOREM. *A compact locally polyhedral 2-dimensional space is a polyhedron.*

Proof. Suppose that K is locally polyhedral and that $\dim K = 2$. Obviously K is an ANR. Let $A = \mathrm{reg} K$, $B = K - A$. B is then locally a graph, and thus a graph. Conditions 1°-3° from 3.1 are then fulfilled. So is also Condition 4°: since K is compact, it may be covered by a finite number of polyhedral neighbourhoods. Since every such neighbourhood contains only a finite number of singular points, there is only a finite number of singular points in K. Thus theorem 3.1 yields 3.7.

§ 4. As we have already remarked, the notions of graph and of the set $\mathrm{reg} K$, which we use in theorem 3.1, only indirectly belong to the set-theoretical topology. The aim of this section is to obtain a characterization of 2-dimensional polyhedra in which only purely set-theoretical notions appear. Such a characterization will be given in theorem 4.2. In 4.1 we shall introduce some auxiliary notions.

The whole content of this section is strongly linked with [10].

4.1. Relative r-points. Let K be a space and A a subset of K. A point $p \in A$ will be called an *r-point of* K rel. A if p has arbitrarily small neighbourhoods U in K such that, for every $q \in U \cap A$, $\mathrm{Fr}_K(U)$ is a deformation retract of $\overline{U} - q$. An r-point of K rel. K will be called an *absolute r-point* of K or, simply, an *r-point* of K.

Now, for every n-dimensional compact space K we introduce a sequence of subspaces as follows:

K_1 will be the set of those r-points of K at which $\dim_p K = n$;
K_i, $i \geqslant 2$, will be the set of r-points of K rel. $K - \bigcup_{j<i} K_j$.

Finally, we shall introduce the following inductive definition: A zero-dimensional space will be called an r-polyhedron if it is finite. An n-dimensional compact space K, $n>0$, will be called an n-dimensional r-polyhedron if it is an ANR and if the sets K_i defined above satisfy the conditions:

(a) K_1 is an open set;
(b) $K = \bigcup_i K_i$;
(c) $K - \bigcup_{j \leqslant i} K_j$ is an r-polyhedron of dimension $\leqslant n - i$.

The definition of r-polyhedra thus uses only the simplest notions from set-theoretical topology.

4.2. THEOREM. *R-polyhedra of dimension $\leqslant 2$ are polyhedra.*

Proof. Suppose that K is an r-polyhedron of dimension 1. By Theorem 3 in [10] we have $K_1 \in \mathrm{reg}_1 K$. By definition $K - K_1$ is finite. Therefore K is a polyhedron.

Now suppose that K is 2-dimensional r-polyhedron. Let $A = K_1$, $B = K - K_1$. Again by theorem 3 in [10] we have $A \subset \mathrm{reg}_2 K$. By definition B is an r-polyhedron of dimension $\leqslant 1$, and thus a graph as we have just proved. Thus K satisfies Conditions 1°-3° from theorem 3.1. Now, by definition, $K - K_1 - K_2 = B - K_2$ is finite. But this is equivalent to (3.6.2) and (3.6.2) implies Condition 4° from theorem 3.1. Therefore this last theorem yields 4.2. This completes the proof.

It is easy to see that polyhedra are r-polyhedra. It is not known whether r-polyhedra are always polyhedra. Probably the answer is negative.

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General continuum hypothesis and ramifications *

by

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1. Introduction and summary. Let $\mathcal{W}$ be a well-ordered set; for any set S, let

$$S(\mathcal{W}) \quad \text{or} \quad S^{\mathcal{W}} \tag{1}$$

denote the system of all functions on $\mathcal{W}$ to S; in particular, if α, β are ordinal numbers, let $\alpha(\beta)$ be the set of all the β-sequences of ordinals $<\alpha$, i. e.

$$\alpha(\beta) = I\alpha(I\beta)\,. \tag{2}$$

For example, $2(\omega_1)$ is the set of all the ω_1-sequences of digits 0, 1.

Let us put

$$TS(\mathcal{W}) = \bigcup_X S(X)\,, \tag{3}$$

X running over all initial segments of $\mathcal{W}$. Consequently, $T2(\omega_1)$ is the set of all the dyadic sequences whose length is $\leqslant \omega_1$. The set (3) is regarded as ordered by the relation

$$=\!\!| \quad \text{meaning:} \quad \textit{to be an initial portion of.}$$

In particular $-\!\!|$ means $=\!\!|$ and $\neq$.

One easily proves that the set (3) is a *tree*, i. e. that for every point x of (3) the set of all the elements each of which is $-\!\!|\, x$ is well-ordered.

The investigation of sets $T2(\omega_\sigma)$ and, in general, of sets of the form (3) is very important and involves enormous difficulties. In particular, we showed that the *problem whether every non countable subset of $T2(\omega_1)$ contains an uncountable chain or an uncountable antichain is equivalent to the Suslin problem* (cf. Kurepa, [1], p. 106, 124, 132, $P_4 \leftrightarrow P_5$).

In particular, the following two propositions are mutually equivalent:

(A) *Every subset S of $T2(\omega_1)$ of cardinality $\aleph_1$ such that every antichain of S is $\leqslant \aleph_0$ contains a chain of cardinality $\aleph_1$*;

(*) The second part of the results was presented 23. 12. 1953 in Beograd at the Mathematics Institute of the Serbian Academy of Sciences. For the first part see Kurepa [2].