On the dimension of products

by

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The object of this paper is to describe conditions under which it is true that

$$\text{(A)} \qquad \dim X \times Y = \dim X + \dim Y ,$$

where X and Y are compact Hausdorff spaces, $X \times Y$ denotes the direct product of X and Y, and dim denotes covering dimension. Our conditions involve the cohomological dimensions of X and Y with coefficients in the additive group R_p, p a prime, of those fractions which in lowest form contain no positive power of p in the denominator. Letting $D(X; G)$ denote the cohomological dimension of the space X with coefficients in the group or field G, we shall show that if (A) holds, then there is some prime p such that

$$(\mathrm{B}_p) \qquad D(X; R_p) = \dim X \quad \text{and} \quad D(Y; R_p) = \dim Y .$$

As a partial converse we also show that if X and Y are homologically locally connected in all dimensions and for some prime p the equations (B_p) hold, then (A) is true.

Since for any finite dimensional compact Hausdorff space X there is a prime p such that $D(X; R_p) = \dim X$, [3.a], and there is a compact metric space B such that $\dim B \times B = 3$, [4], the strengthening of the stated partial converse obtained by deleting the conditions that X and Y be homologically locally connected is false.

Our arguments rely on certain definitive theorems of M. Bockstein announced in [1], [2] and proved in [3]. We shall present alternate proofs of two of these theorems, namely the theorem on page 70 of [3.a] and the theorem on page 127 of [3.b]. These alternate proofs rely heavily on techniques due to Bockstein. Their merit lies in their comparative brevity and in possible independent interest of some of their algebraic lemmas.

We shall also use certain relations between the cohomology of the nerves of the terms of certain sequences of closed coverings of a space and the cohomology of the space. These relations may be found implicitly in papers of Solomon Lefschetz [12] and R. L. Wilder [14]; the first explicit statement of them is to be found in a paper of E. E. Floyd [10]. The particular statement we shall use is Theorem 1 of [9].

I. Algebraic preliminaries. In addition to the definitions and first consequences, the following properties of the functors $\otimes$ and Tor will be used:

If $0\to A\to B\to C\to 0$ is an exact sequence of groups (all groups in this paper being abelian) and G is a group, then the sequence

$$0\to \mathrm{Tor}(A, G)\to \mathrm{Tor}(B, G)\to \mathrm{Tor}(C, G)\to A\otimes G\to B\otimes G\to C\otimes G\to 0$$

is exact.

Tor commutes with direct limits; i. e., if $\{A_\lambda\}$, and $\{B_\lambda\}$, $\lambda\in\Lambda$, are direct systems of groups, then

$$\varinjlim_{\lambda\in\Lambda}\mathrm{Tor}(A_\lambda, B_\lambda)\cong \mathrm{Tor}(\varinjlim_{\lambda\in\Lambda} A_\lambda, \varinjlim_{\lambda\in\Lambda} B_\lambda)\,.$$

Proofs of these statements can be found in [8].

Throughout we use the symbol A_p, where A is a group and p is a prime number, to denote the p-primary part of A; i. e., the subgroup of those elements a of A such that $p^\alpha\cdot a=0$ for some integer α.

LEMMA 1.1. *If* $0\to A_p\xrightarrow{f} B\xrightarrow{g} C\to 0$ *is exact, then* $0\to A_p\xrightarrow{f} B_p\xrightarrow{g} C_p\to 0$ *is exact.*

It is necessary only to show that $g\colon B_p\to C_p$ is an epimorphism. Suppose $c\in C_p$. Then there is a $b\in B$ such that $g(b)=c$. For some α, $p^\alpha\cdot c=0$; and so, $p^\alpha\cdot b\in\operatorname{im} f$; i. e., there is an $a\in A$ such that $f(a)=p^\alpha\cdot b$. But for some γ, $p^\gamma\cdot a=0$ since $a\in A_p$; and so, $0=f(p^\gamma\cdot a)=p^{\alpha+\gamma}\cdot b$; i. e., $b\in B_p$.

LEMMA 1.2. *If B is a torsion group, $C_p=C$, $B_p=0$, and $g\colon B\to C$, then* $\operatorname{im} g=0$.

Suppose $0\neq c=g(b)$. Then for some α, $0=p^\alpha\cdot c=g(p^\alpha\cdot b)$. The element b is of order q, where $(p, q)=1$. There are integers n and m such that $1=nq+mp^\alpha$. Then $c=g(b)=g(nqb+mp^\alpha\cdot b)=g(mp^\alpha\cdot b)=0$.

LEMMA 1.3. *If p is a prime and G and H are groups, the sequence*

$$0\to G_p\otimes H_p\to(G\otimes H)_p\to([G/G_p]\otimes H_p)\otimes(G_p\otimes[H/H_p])\to 0$$

is exact.

The sequences $0\to G_p\to G\to G/G_p\to 0$ and $0\to H_p\to H\to H/H_p\to 0$ are exact. Hence, we have the commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & \mathrm{Tor}(G_p, H/H_p) \arrow[d] & \mathrm{Tor}(G, H/H_p) \arrow[d] & \mathrm{Tor}(G/G_p, H/H_p) \arrow[d] & \\
\ldots \arrow[r] & \mathrm{Tor}(G/G_p, H_p) \arrow[r] & G_p\otimes H_p \arrow[r] \arrow[d] & G\otimes H_p \arrow[r] \arrow[d] & G/G_p\otimes H_p \arrow[r] \arrow[d] & 0 \\
\ldots \arrow[r] & \mathrm{Tor}(G/G_p, H) \arrow[r] & G_p\otimes H \arrow[r] \arrow[d] & G\otimes H \arrow[r] \arrow[d] & G/G_p\otimes H \arrow[r] \arrow[d] & 0 \\
\ldots \arrow[r] & \mathrm{Tor}(G/G_p, H/H_p) \arrow[r] & G_p\otimes H/H_p \arrow[r] \arrow[d] & G\otimes H/H_p \arrow[r] \arrow[d] & G/G_p\otimes H/H_p \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

in which all horizontal and vertical rows are exact. Since the p-primary parts of G/G_p and H/H_p are zero, the p-primary parts of each of the Tor's appearing in the diagram is zero. Each of the Tor's is a torsion group. Furthermore, each of the groups $G_p\otimes H/H_p$, $G_p\otimes H$, $G\otimes H_p$, and $G/G_p\otimes H_p$ is its own p-primary part. Hence, by Lemma 2 we obtain the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G_p\otimes H_p \arrow[r] \arrow[d] & G\otimes H_p \arrow[r] \arrow[d] & G/G_p\otimes H_p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_p\otimes H \arrow[r] \arrow[d] & G\otimes H \arrow[r] \arrow[d] & G/G_p\otimes H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_p\otimes H/H_p \arrow[r] \arrow[d] & G\otimes H/H_p \arrow[r] \arrow[d] & G/G_p\otimes H/H_p \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since $(G/G_p\otimes H/H_p)_p=0$, by Lemma 1.1 we obtain the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G_p\otimes H_p \arrow[r, "i_1"] \arrow[d, "j_1"] & G\otimes H_p \arrow[r, "i_2"] \arrow[d, "j_3"] & G/G_p\otimes H_p \arrow[r] \arrow[d, "j_5"] & 0 \\
0 \arrow[r] & G_p\otimes H \arrow[r, "i_3"] \arrow[d, "j_2"] & (G\otimes H)_p \arrow[r, "i_4"] \arrow[d, "j_4"] & (G/G_p\otimes H)_p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_p\otimes H/H_p \arrow[r, "i_5"] \arrow[d] & (G\otimes H/H_p)_p \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Consider the sequence

$$0\to G_p\otimes H_p\xrightarrow{k_1}(G\otimes H)_p\xrightarrow{j_2}([G/G_p]\otimes H_p)\oplus(G_p\otimes[H/H_p])\to 0\,,$$

where $k_1=i_3j_1=j_3i_1$ and $k_2=(j_5^{-1}i_4, i_5^{-1}j_4)$. We shall show it is exact.

(1) Clearly, k_1 is a monomorphism.

(2) $\operatorname{im} k_1 \supset \ker k_2$. Suppose $k_2(g) = (j_5^{-1} i_4 g, i_5^{-1} j_4 g) = 0$. Then $i_4 g = j_4 g = 0$. There is an element $g_1 \in G_p \otimes H$ such that $i_3 g_1 = g$. Since i_5 is an isomorphism and $j_4 g = 0$, $j_2 g_1 = 0$. Thus, there is an element $g' \in G_p \otimes H_p$ such that $j_1 g' = g_1$ and $i_3 j_1 g' = g \in \operatorname{im} k_1$.

(3) $\ker k_2 \supset \operatorname{im} k_1$. Suppose $j_3 i_1 g' = g$. Then $0 = j_4 j_3 i_1 g' = j_4 g$ and $0 = i_2 i_1 g' = j_5^{-1} i_4 g$. Thus, $k_2 g = 0$.

(4) k_2 is an epimorphism. Suppose that (g_1, g_2) is an element of $([G/G_p] \otimes H_p) \oplus (G_p \otimes [H/H_p])$. There are elements $g_1' \in G \otimes H_p$ and $g_2' \in G_p \otimes H$ such that $i_2 g_1' = g_1$ and $j_2 g_2' = g_2$. Then $k_2(j_3 g_1' + i_3 g_2') = (j_5^{-1} i_4 j_3 g_1' + j_5^{-1} i_4 i_3 g_2', i_5^{-1} j_4 j_3 g_1' + i_5^{-1} j_4 i_3 g_2') = (i_2 g_1' + 0, 0 + j_2 g_2') = (g_1, g_2)$.

Lemma 1.4. $\operatorname{Tor}(H, G) \neq 0$ *if and only if for some prime p both H and G contain elements of order p.*

Proof. If $\operatorname{Tor}(H, G) \neq 0$ and H_t and G_t are the torsion subgroups of H and G, then $\operatorname{Tor}(H_t, G_t) \neq 0$ (since $\operatorname{Tor}(H, G) \cong \operatorname{Tor}(H_t, G_t)$). Since Tor commutes with direct limits, for some finitely generated subgroups H' and G' of H_t and G_t, $\operatorname{Tor}(H', G') \neq 0$. It follows easily that there are elements in H' and G' of the same prime order.

Let ${}_pH$ and ${}_pG$ denote the subgroups of H and G of all elements of order p. If both subgroups are non-trivial, then since they are vector spaces over Z_p, $\operatorname{Tor}({}_pH, {}_pG) \neq 0$. The two exact sequences

$$0 \to \operatorname{Tor}({}_pG, {}_pH) \to \operatorname{Tor}(G, {}_pH) \to \operatorname{Tor}(pG, {}_pH) \to {}_pG \otimes {}_pH \to \ldots,$$
$$0 \to \operatorname{Tor}(G, {}_pH) \to \operatorname{Tor}(G, H) \to \operatorname{Tor}(G, pH) \to G \otimes {}_pH \to \ldots$$

then imply in turn that $\operatorname{Tor}(G, {}_pH) \neq 0$ and that $\operatorname{Tor}(G, H) \neq 0$.

The group H is said to have property $P(p)$, p a prime, if there is some element of H/H_p which is not divisible by p, equivalently, if there is an element $h \in H$ such that $p^\gamma \cdot (ph' - h) \neq 0$ for any integer γ and element $h' \in H$.

Lemma 1.5. *If G contains an element of order p and H has property $P(p)$, then $G \otimes H \neq 0$. If Q_p denotes the additive group of p-adic rationals reduced modulo 1 and $Q_p \otimes H \neq 0$, then H has property $P(p)$.*

Proof. If H has property $P(p)$, then $K = (H/H_p)/p(H/H_p)$ is a non-zero vector space over Z_p. Since H/H_p contains no element of order p, $\operatorname{Tor}(H/H_p, G)$ contains no element of order p. It follows then from Lemma 1.2 that the sequence

$$0 \to \operatorname{Tor}(G, K) \to G \otimes H/H_p \to G \otimes H/H_p \to G \otimes K \to 0$$

is exact. Since G contains an element of order p, by the previous lemma $\operatorname{Tor}(G, K) \neq 0$. It follows that $G \otimes H/H_p \neq 0$. By tensoring the exact sequence $0 \to H_p \to H \to H/H_p \to 0$ by G it is seen that $G \otimes H \neq 0$.

If $H \otimes Q_p \neq 0$, then since $H_p \otimes Q_p = 0$, $Q_p \otimes H/H_p \neq 0$. If H did not have property $P(p)$, then $K = 0$ and the homomorphism $(1 \otimes p): Q_p \otimes H/H_p \to Q_p \otimes H/H_p$ is an isomorphism. Let a denote the least positive integer n such that there is an element of the form $1/p^n \otimes h' \in Q_p \otimes H/H_p$ which is not zero. Then

$$(1 \otimes p)(1/p^a \otimes h') = (1/p^a \otimes ph') = (1/p^{a-1} \otimes h') = 0,$$

which is a contradiction. Thus, H has property $P(p)$.

Lemma 1.6. *In order for $(G \otimes H)_p$ to be non-zero it is necessary and sufficient that either*

(1) *one of the groups G and H contains an element in its p-primary part which is not divisible by p and the other contains an element not divisible by p, or*

(2) *the p-primary part of one of the groups, say G, is isomorphic to the direct sum of copies of Q_p and the other, H, has property $P(p)$.*

Proof. By Lemma 1.3, $(G \otimes H)_p \neq 0$ if and only if either

(a) $G_p \otimes H_p \neq 0$, or

(b) $([G/G_p] \otimes H_p) \oplus (G_p \otimes [H/H_p]) \neq 0$.

Suppose $(G \otimes H)_p \neq 0$. Then if (a) is true, (1) is true. If (b) is true and (1) is false, then, supposing $G_p \otimes (H/H_p) \neq 0$, H has property $P(p)$ and every element of G_p is divisible by p, which implies that $G_p \cong \otimes Q_p$; and so, (2) is true.

Suppose that (1) is true; i. e., $pG_p \neq G_p$ and $pH \neq H$. Then G_p/pG_p and H/pH are non-trivial vector spaces over Z_p; and so, $(G_p/pG_p) \otimes (H/pH) \neq 0$. The exact sequences

$$\ldots \to \operatorname{Tor}(G_p/pG_p, H/pH) \to pG_p \otimes H/pH \to G_p \otimes H/pH \to G_p/pG_p \otimes H/pH \to 0$$

and

$$\ldots \to \operatorname{Tor}(G_p, H/pH) \to G_p \otimes pH \to G_p \otimes H \to G_p \otimes H/pH \to 0$$

show in turn that $G_p \otimes H/pH \neq 0$ and $G_p \otimes H \neq 0$. In the argument for Lemma 1.3 it was shown that

$$0 \to G_p \otimes H \to (G \otimes H)_p \to \big((G/G_p) \otimes H\big)_p \to 0$$

is exact. Hence, $(G \otimes H)_p \neq 0$.

If (2) is true, then $(G \otimes H)_p \neq 0$ by lemma 1.5.

Lemma 1.7. $A \otimes B \neq 0$ *if and only if either*

(1) *both A and B contain elements of infinite order, or*

(2) *for some prime p one of the conditions of Lemma 1.6 holds.*

Proof. $A \otimes B$ contains an element of infinite order if and only if both A and B contain elements of infinite order. Otherwise, $A \otimes B \neq 0$ if and only if for some prime p, $(A \otimes B)_p \neq 0$.

THEOREM 1.1. *$H^q(X; G) \neq 0$ if and only if either*

(1) both G and $H^q(X; Z)$ contain elements of infinite order, or for some prime p either

(2_p) $G_p \cong \oplus Q_p$ and $H^q(X; Z)$ has property $P(p)$,

(3_p) $\big(H^q(X; Z)\big)_p \cong \oplus Q_p$ and G has property $P(p)$,

(4_p) both G_p and $H^q(X; Z)$ contain elements not divisible by p,

(5_p) both $\big(H^q(X; Z)\big)_p$ and G contain elements not divisible by p, or

(6_p) both G and $H^{q+1}(X; Z)$ contain elements of order p.

If $\ldots \to H^i(X; Z) \xrightarrow{p} H^i(X; Z) \xrightarrow{j} H^i(X; Z_p) \to H^{i+1}(X; Z)$ is the Bockstein sequence induced by the sequence $0 \to Z \xrightarrow{p} Z \to Z_p \to 0$, then the condition in (6_p) that $H^{q+1}(X; Z)$ contains an element of order p is equivalent to the condition that j: $H^q(X; Z) \to H^q(X; Z_p)$ is not epimorphic.

Theorem 1.1 is an immediate consequence of Lemmas 1.6 and 1.7 and the Universal Coefficient Sequence [13]

$$0 \to H^q(X; Z) \otimes G \to H^q(X; G) \to \operatorname{Tor}\big(H^{q+1}(X; Z), G\big) \to 0 .$$

We are using Alexander-Spanier cohomology with compact supports; X is assumed to be locally compact Hausdorff.

II. Two theorems of Bockstein. For a group G either

(a) G contains elements of infinite order, or for some prime p either

(b_p) G has property $P(p)$,

(c_p) $G_p \cong \oplus Q_p$, or

(d_p) G contains an element of order p^a which is not divisible by p.

For the group G we define a colection of groups $\gamma(G)$ as follows:

(i) $Q \in \gamma(G)$ if and only if (a) is true,

(ii) $R_p \in \gamma(G)$ if and only if (b_p) is true,

(iii) $Q_p \in \gamma(G)$ if and only if (c_p) is true, and

(iv) $Z_p \in \gamma(G)$ if and only if (d_p) is true.

The cohomology dimension, $D(X; G)$, of a compact Hausdorff space X with coefficients in the group of field G is defined by

$$D(X; G) = \text{l.u.b.}\{i \mid H^i(X, A; G) \neq 0 \text{ for some closed } A \subset X\}.$$

THEOREM 2.1. *$D(X; G) = \text{l.u.b.}\{D(X; H) \mid H \in \gamma(G)\}$.*

Proof. We shall use the symbols (1), (2_p), ..., (6_p) to denote the statements in Theorem 1.1, with $H^q(X, A; Z)$ instead of $H^q(X; Z)$, (a) (b_p), ..., (d_p) to denote the statements in the introductory paragraph of this section, and (i), ..., (iv) to denote the statements in the definition of $\gamma(G)$.

A. $D(X; G) \geqslant \text{l.u.b.}\{D(X; H) \mid H \in \gamma(G)\}$.

If (i) holds and $D(X; Q) = q$, then (1) holds and $H^q(X, A; G) \neq 0$.

If (ii) holds and $D(X; R_p) = q$, then (b_p) holds for G and either (1), (3_p), or (5_p) holds for R_p. The corresponding one will also hold for G.

If (iii) holds and $D(X; Q_p) = q$, then (c_p) holds for G and either (2_p) or (6_p) holds for Q_p. Again the corresponding one holds for G.

If (iv) holds and $D(X; Z_p) = q$, then (d_p) holds for G and either (4_p) or (6_p) holds for Z_p. The corresponding one holds for G.

B. There is a group $H \in \gamma(G)$ such that $D(X; H) \geqslant D(X; G)$.

Let n be one of the properties (1), (2_p), ..., (6_p) which holds because $H^q(X, A; G) \neq 0$, where $q = D(X; G)$.

If $n = (1)$, then $Q \in \gamma(G)$ and $D(X; Q) \geqslant D(X; G)$.

If $n = (2_p)$, then $Q_p \in \gamma(G)$ and $D(X; Q_p) \geqslant D(X; G)$.

If $n = (3_p)$, then $R_p \in \gamma(G)$ and $D(X; R_p) \geqslant D(X; G)$.

If $n = (4_p)$, then $Z_p \in \gamma(G)$ and $D(X; Z_p) \geqslant D(X; G)$.

If $n = (5_p)$ and (4_p) does not hold, then every element of G_p is divisible by p. Let g denote an element of G which is not divisible by p. Suppose for some $g' \in G$ and integer γ, $p^\gamma \cdot (pg' - g) = 0$. Then $pg' - g \in G_p$ and so there is a $g'' \in G_p$ such that $pg'' = pg' - g$, or $g = p(g' - g'')$. Thus, G has property $P(p)$, $R_p \in \gamma(G)$ and $D(X; R_p) \geqslant D(X; G)$.

If $n = (6_p)$, either $Q_p \in \gamma(G)$ or $Z_p \in \gamma(G)$ and both $D(X; Q_p) \geqslant D(X; G)$ and $D(X; Z_p) \geqslant D(X; G)$.

COROLLARY 2.1.

(a) $D(X; Z) = \text{l.u.b.}\{D(X; R_p) \mid \text{for } p \text{ a prime}\}$.

(b) $D(X; Z_p) - 1 \leqslant D(X; Q_p) \leqslant D(X; Z_p)$.

(c) $D(X; Q_p) \leqslant \max\big(D(X; Q), D(X; R_p) - 1\big)$.

(d) $D(X; R_p) \leqslant \max\big(D(X; Q), D(X; Q_p) + 1\big)$.

(e) $D(X; Z_p) \leqslant D(X; R_p)$.

(f) $D(X; Q) \leqslant D(X; R_p)$.

These statements can be immediately verified by using Theorem 1.1. Several of them can be more directly verified from the Bockstein sequences induced by the exact sequences

$$0 \to Z_p \to Q_p \to Q_p \to 0 \quad \text{and} \quad 0 \to R_p \to Q \to Q_p \to 0 .$$

In order to prove the second theorem of Bockstein we need two more lemmas. The first of these is a restatement of a theorem of Aleksandrov.

LEMMA 2.1. *If X is a compact Hausdorff space and $D(X;G)=n$, there exist a point $x\in X$ and an open neighborhood U of x such that if V is an open neighborhood of x, $V\subset U$, then the homomorphism*

$$H^n(X, X-V; G)\to H^n(X, X-U; G)$$

induced by inclusion is non-trivial.

The dual statement for homology with coefficients in Z_p was proved in a recent paper of E. E. Floyd [11]. His proof does not use the fact that the coefficient group is Z_p and properly restated establishes the above lemma.

LEMMA 2.2. $D(X\times Y;G)=\text{l.u.b.}\{n|\ H^n(X\times Y, A\times Y\cup X\times B;G)\neq 0$ *for A and B closed subsets of X and Y, respectively*$\}$.

Proof. There exists a point $p\in X\times Y$ and an open neighbourhood W of p such that if S is an open neighbourhood of p, $S\subset W$, then, letting $n=D(X\times Y;G)$,

$$H^n(X\times Y, X\times Y-S; G)\to H^n(X\times Y, X\times Y-W; G)$$

is non-trivial. There are open sets U and V in X and Y, respectively, such that $p\in U\times V\subset W$.

$$(X-U)\times Y\cup X\times(Y-V)=X\times Y-U\times V.$$

Let $A=X-U$ and $B=Y-V$. Then

$$H^n(X\times Y, A\times Y\cup X\times B; G)\to H^n(X\times Y, X\times Y-W; G)$$

is non-trivial and $H^n(X\times Y, A\times Y\cup X\times B; G)\neq 0$.

THEOREM 2.2. *If X and Y are compact Hausdorff spaces and F is a field, then*

(a) $D(X\times Y;F)=D(X;F)+D(Y;F)$.

Also

(b) $D(X\times Y;Q_p)=\max\{D(X;Q_p)+D(Y;Q_p),\ D(X\times Y;Z_p)-1\}$,

and

(c) $D(X\times Y;R_p)=\min\{\max[D(X\times Y;Q),\ D(X\times Y;Z_p),\ D(X;Q_p)+D(Y;Q_p)+1],\ \max[D(X\times Y;Q),\ D(X\times Y;Z_p),\ D(X;Q_p)+D(Y;R_p),\ D(X;R_p)+D(Y;Q_p)]\}$.

Proof. (a) Since for a field F (see Appendix),

$$\sum_{i+j=n} H^i(X,A;F)\otimes H^j(Y,B;F)\cong H^n(X\times Y, A\times Y\cup X\times B;F),$$

(a) follows immediately from Lemma 2.2.

(b) 1. $D(X\times Y;Q_p)\leqslant\max\{D(X;Q_p)+D(Y;Q_p),\ D(X\times Y;Z_p)-1\}$.

By Theorem 1.1. and Lemma 2.2, $D(X\times Y;Q_p)\geqslant d$ if and only if there are closed subsets A and B of X and Y, respectively, such that either

(α) $H^d(X\times Y, A\times Y\cup X\times B;Z)$ has property $P(p)$, or

(β) $\big(H^{d+1}(X\times Y, A\times Y\cup X\times B;Z)\big)_p\neq 0$.

If (α) holds, consider the relative Künneth sequence (see Appendix)

$$0\to\sum_{i+j=d} H^i(X,A)\otimes H^j(Y,B)\to H^d(X\times Y, A\times Y\cup X\times B)\to\sum_{i+j=d+1}\operatorname{Tor}\big(H^i(X,A), H^j(Y,B)\big)\to 0$$

(when the coefficient group is not written, it will be understood to be the integers). $H^d(X\times Y, A\times Y\cup X\times B)$ has property $P(p)$ if and only if its tensor product with Q_p is non-trivial. $\operatorname{Tor}(G,H)\otimes Q_p=0$ for any two groups G and H. Thus, $H^d(X\times Y, A\times Y\cup X\times B)$ has property $P(p)$ only if for some i and j, $i+j=d$, $H^i(X,A)\otimes H^j(Y,B)\otimes Q_p\neq 0$. But this is true if and only if both $H^i(X,A)$ and $H^j(Y,B)$ have property $P(p)$. Thus, if (α) holds, $D(X;Q_p)+D(Y;Q_p)\geqslant d$.

If (β) holds, then by the similar Künneth sequence with the dimension raised one, either

$$\sum_{i+j=d+1}\big(H^i(X,A)\otimes H^j(Y,B)\big)_p\neq 0$$

or

$$\sum_{i+j=d+2}\big(\operatorname{Tor}\big(H^i(X,A),\ H^j(Y,B)\big)\big)_p\neq 0.$$

Either of these is true if and only if for some integers i and j such that $i+j=d$, either

(i) $\big(H^{i+1}(X,A)\big)_p\cong\oplus Q_p$ and $H^j(Y,B)$ has property $P(p)$,

(ii) $H^{i+1}(X,A)$ has property $P(p)$ and $\big(H^j(Y,B)\big)_p\cong\oplus Q_p$,

(iii) $\big(H^{i+1}(X,A)\big)_p$ and $H^j(Y,B)$ have elements not divisible by p,

(iv) $H^{i+1}(X,A)$ and $\big(H^j(Y,B)\big)_p$ have elements not divisible by p, or

(v) $\big(H^{i+1}(X,A)\big)_p\neq 0\neq\big(H^{j+1}(Y,B)\big)_p$.

Each of the statements (i), (ii) and (v) implies that $D(X;Q_p)+D(Y;Q_p)\geqslant d$. Each of (iii) and (iv) implies that $D(X;Z_p)+D(Y;Z_p)-1\geqslant d$.

Thus, if $D(X\times Y;Q_p)\geqslant d$, either $D(X;Q_p)+D(Y;Q_p)\geqslant d$ or $D(X;Z_p)+D(Y;Z_p)-1\geqslant d$; i. e.,

$$D(X\times Y;Q_p)\leqslant\max[D(X;Q_p)+D(Y;Q_p),\ D(X\times Y;Z_p)-1].$$

(b) 2. We next show that the oppositely directed inequality holds.

If $D(X; Q_p) = i$ and $D(Y; Q_p) = j$ and $i+j = d$, then for some closed subsets A and B of X and Y, respectively, either

(i) $H^i(X, A)$ and $H^j(Y, B)$ have property $P(p)$,

(ii) $H^i(X, A)$ has property $P(p)$ and $\big(H^{j+1}(Y, B)\big)_p \neq 0$,

(iii) $\big(H^{i+1}(X, A)\big)_p \neq 0$ and $H^j(Y, B)$ has property $P(p)$, or

(iv) $\big(H^{i+1}(X, A)\big)_p \neq 0 \neq \big(H^{j+1}(Y, B)\big)_p$.

If (ii) or (iii) is true, then $\big(H^{d+1}(X\times Y, A\times Y \cup X\times B)\big)_p \neq 0$; and so, $D(X\times Y; Q_p) \geqslant d$. If (i) is true but $\big(H^{i+k}(X, A)\big)_p = 0 = \big(H^{j+k}(Y, B)\big)_p$ for $k \geqslant 1$; i. e., (ii) and (iii) are both false, then since $H^i(X, A)\otimes H^j(Y,B)$ has property $P(p)$ and

$$\Big(\sum_{i+j=d+1} \operatorname{Tor}\big(H^i(X, A),\ H^j(Y, B)\big)\Big)_p = 0\,,$$

we find upon tensoring the relative Künneth sequence of $(X, A)\times(Y, B)$ by Q_p that

$$\sum_{i+j=d} H^i(X, A)\otimes H^j(Y, B)\otimes Q_p \cong H^d(X\times Y,\ A\times Y\cup X\times B)\otimes Q_p\,.$$

Since the term of the left is non-zero, $H^d(X\times Y, A\times Y\cup X\times B)$ has property $P(p)$ and $D(X\times Y; Q_p) \geqslant d$.

If (iv) is true, then $\Big(\operatorname{Tor}\big(H^{i+1}(X, A),\ H^{j+1}(Y, B)\big)\Big) \neq 0$. Consider the relative Künneth sequence

$$0 \to \sum_{i+j=d+1} H^i(X, A)\otimes H^j(Y, B) \xrightarrow{n^*} H^{d+1}(X\times Y,\ A\times Y\cup X\times B)$$
$$\xrightarrow{m^*} \sum_{i+j=d+2} \operatorname{Tor}\big(H^i(X, A),\ H^j(Y, B)\big) \to 0\,.$$

Either $\big(H^{d+1}(X\times Y, A\times Y\cup X\times B)\big)_p \neq 0$, in which case $D(X\times Y; Q_p) \geqslant d$, or $\big(H^{d+1}(X\times Y,\ A\times Y\cup X\times B)\big)_p = 0$ and there is an element $\gamma \in H^{d+1}(X\times Y, A\times Y\cup X\times B)$ of infinite order which is not in $\operatorname{im} n^*$ and such that for some a, $p^a\cdot\gamma \in \operatorname{im} n^*$. If for each i and j, $i+j = d+1$, either every element of $H^i(X, A)$ or every element of $H^j(Y, B)$ were divisible by p, then $\operatorname{im} n^*$ would be divisible by p. Thus, since $p^a\cdot\gamma \in \operatorname{im} n^*$, there would be an element $\beta \in \operatorname{im} n^*$ such that $p^a\cdot\beta = p^a\cdot\gamma$. Since $\gamma \notin \operatorname{im} n^*$, $\beta-\gamma \neq 0$ and $\big(H^{d+1}(X\times Y,\ A\times Y\cup X\times B)\big)_p \neq 0$, but this is a contradiction. Thus, for some i and j, $i+j = d+1$, there are elements of $H^i(X, A)$ and of $H^j(Y, B)$ not divisible by p. This implies that

$$D(X; Z_p) + D(Y; Z_p) - 1 \geqslant d\,.$$

If this last inequality holds, then $D(X\times Y; Z_p)-1 \geqslant d$. By (b) of Corollary 2.1, $D(X\times Y; Q_p) \geqslant D(X\times Y; Z_p)-1$.

Thus, if either $D(X; Q_p)+D(Y; Q_p) \geqslant d$ or $D(X\times Y; Z_p)-1 \geqslant d$, then $D(X\times Y; Q_p) \geqslant d$, and (b) is proved.

(c) 1. $D(X\times Y; R_p) \leqslant \min[\max\{\ \},\ \max\{\ \}]$.

By (d) of Corollary 2.1

$$D(X\times Y; R_p) \leqslant \max[D(X\times Y; Q),\ D(X\times Y; Q_p)+1]\,.$$

Thus by (b) of this theorem

$$D(X\times Y; R_p) \leqslant \max[D(X\times Y; Q),\ D(X\times Y; Z_p);\ D(X; Q_p)+D(Y; Q_p)+1]\,.$$

$D(X\times Y; R_p) \geqslant d$ if there exist closed subsets A and B of X and Y, respectively, such that either

(α) $H^d(X\times Y,\ A\times Y\cup X\times B)$ contains an element of infinite order, or

(β) $H^d(X\times Y,\ A\times Y\cup X\times B)_p \neq 0$.

The statement (α) holds if and only if $D(X\times Y; Q) \geqslant d$. If ($\beta$) holds, then either

$$\sum_{i+j=d} \big(H^i(X, A)\otimes H^j(Y, B)\big)_p \neq 0\,,$$

or

$$\sum_{i+j=d+1} \Big(\operatorname{Tor}\big(H^i(X, A),\ H^j(Y, B)\big)_p\Big) \neq 0\,.$$

Hence, if (β) holds, then for some integers i and j, $i+j = d$, either

(i) $\big(H^i(X, A)\big)_p \cong \oplus Q_p$ and $H^j(Y, B)$ has property $P(p)$,

(ii) $H^i(X, A)$ has property $P(p)$ and $\big(H^j(Y, B)\big)_p \cong \oplus Q_p$,

(iii) $\big(H^i(X, A)\big)_p$ and $H^j(Y, B)$ both contain elements not divisible by p,

(iv) $H^i(X, A)$ and $\big(H^j(Y, B)\big)_p$ both contain elements not divisible by p, or

(v) $\big(H^{i+1}(X, A)\big)_p \neq 0 \neq \big(H^j(Y, B)\big)_p$.

If (i) is true, then $D(X; R_p)+D(Y; Q_p) \geqslant d$ and if (ii) is true, then $D(X; Q_p)+D(Y; R_p) \geqslant d$. If either (iii) or (iv) is true, then $D(X\times Y; Z_p)$

$\geqslant d$. If (v) holds, then $D(X;Q_p)+D(Y;R_p)\geqslant d$ and $D(X;R_p)+D(Y;Q_p)\geqslant d$. We have shown that if $D(X\times Y;R_p)\geqslant d$, then either

$$D(X;Q_p)+D(Y;R_p)\geqslant d,\quad D(X;R_p)+D(Y;Q_p)\geqslant d,$$
$$D(X\times Y;Q)\geqslant d,\quad \text{or}\quad D(X\times Y;\ Z_p)\geqslant d.$$

Thus,

$$D(X\times Y;R_p)\leqslant \max[D(X\times Y;Q),\ D(X\times Y;Z_p),\ D(X;Q_p)+D(Y;R_p),\ D(X;R_p)+D(Y;Q_p)].$$

Hence, the inequality (c)1 is proved.

(c)2. Suppose that strict inequality holds in (c)1. We shall show that this leads to a contradiction. By (e) and (f) of Corollary 2.1 it is seen that

$$D(X\times Y;R_p)\geqslant \max[D(X\times Y;Q),\ D(X\times Y;Z_p)].$$

Under our supposition

(α) $D(X\times Y;R_p)<D(X;Q_p)+D(Y;Q_p)+1$,

(β) $D(X\times Y;R_p)<\max[D(X;Q_p)+D(Y;R_p),D(X;R_p)+D(Y;Q_p)]$.

Let $i=D(X;Q_p)$, $i'=D(X;R_p)$, $j=D(Y;Q_p)$, $j'=D(Y;R_p)$, and $d=D(X\times Y;R_p)$.

(α) then states that $d<i+j+1$. According to (α), there are closed subsets A and B of X and Y, respectively, such that either

(i) $\big(H^{i+1}(X,A)\big)_p\neq 0\neq \big(H^{j+1}(Y,B)\big)_p$,

(ii) $\big(H^{i+1}(X,A)\big)_p\neq 0$ and $H^j(Y,B)$ has property $P(p)$,

(iii) $H^i(X,A)$ has property $P(p)$ and $\big(H^{j+1}(Y,B)\big)_p\neq 0$, or

(iv) both $H^i(X,A)$ and $H^j(Y,B)$ have property $P(p)$.

If (i) were true, then $\big(\mathrm{Tor}\,(H^{i+1}(X,A),\ H^{j+1}(Y,B))\big)_p\neq 0$ and the relative Künneth sequence implies that $H^{i+j+1}(X\times Y,\ A\times Y\cup X\times B)$ either contains an element of infinite order or an element of order p. Either would imply that $d=D(X\times Y;R_p)\geqslant i+j+1>d$. If (ii) or (iii) were true, $\big(H^{i+j+1}(X\times Y,\ A\times Y\cup X\times B)\big)_p\neq 0$, which implies, as above, that $d>d$. Since (ii) and (iii) are false, statement (iv) implies that $H^{i+j}(X\times Y,\ A\times Y\cup X\times B)$ has property $P(p)$; and so, $d=D(X\times Y;R_p)\geqslant i+j>d-1$. Thus, (α) implies that $d=i+j$ and that (iv) is true.

The statement (β) implies that either

(1) $i+j'>i+j$ or

(2) $i'+j>i+j$.

If (1) holds, then $j'>j$ and for some closed subset B' of Y either $\big(H^{j'}(Y,B')\big)_p\neq 0$ or $H^{j'}(Y,B')$ contains an element of infinite order. This, together with the fact that $H^i(X,A)$ has property $P(p)$, implies that $H^{i+j'}(X\times Y,\ A\times Y\cup X\times B')$ either contains an element of infinite order or an element of order p. In either case, $i+j=d=D(X\times Y;R_p)\geqslant i+j'>i+j$. The statement (2) similarly leads to a contradiction. Thus, equality holds in (c).

III. Relative cohomology in locally connected spaces. A compact Hausdorff space X is *cohomologically locally connected* in all dimensions through n, $\mathrm{cl}c^n$, with respect to a coefficient group G, if for each point $x\in X$ and closed neighborhood U of x, there is a closed neighborhood V of x, $V\subset U$, such that the inclusion homomorphism

$$\widetilde{H}^i(U;G)\to\widetilde{H}^i(V;G)$$

is trivial for all $i\leqslant n$, where $\widetilde{H}^*$ denotes reduced cohomology.

If $U=\{u_\alpha\}_{\alpha\in A}$ and $V=\{v_\beta\}_{\beta\in B}$ are finite indexed collections of compact sets and $j\colon B\to A$ is an inclusion mapping (i. e., $v_\beta\subset u_{j(\beta)}$ for all $\beta\in B$) such that for every $\beta\in B$ the inclusion homomorphism

$$\widetilde{H}^i(u_{j(\beta)};G)\to\widetilde{H}^i(v_\beta;G)$$

is trivial for all $i\leqslant n$, then V is said to be an *n-refinement of* U with respect to G. If for every subset $B'\subset B$, the inclusion homomorphism

$$\widetilde{H}^i\Big(\bigcap_{\beta\in B'}u_{j(\beta)};G\Big)\to\widetilde{H}^i\Big(\bigcap_{\beta\in B'}v_\beta;G\Big)$$

is trivial for all $i\leqslant n$, then V is said to be a *strong n-refinement* of U with respect to G. This concept has been used explicitly by several authors. The following is a restatement of a theorem of Floyd [10] (see Theorem 2.3 of [9]).

THEOREM F. *If X is a compact Hausdorff space, $U_0, U_1, \ldots, U_{n+1}$ is a sequence of finite collections of closed subsets of X such that for each i, $0\leqslant i\leqslant n$, U_{i+1} strongly n-refines U with respect to G, $\tau\colon U_{n+1}\to U_0$ is the composition of the inclusion mappings of U_{i+1} into U_i, and U_i^* denotes the union of the elements of U_i, then there is a natural commutative diagram*

$$\begin{array}{ccc} H^i(\mathcal{U}_0;G) & \xrightarrow{\pi_0^*} & H^i(U_0^*;G)\\ \downarrow{\scriptstyle\tau^*} & & \downarrow{\scriptstyle r^*}\\ H^i(\mathcal{U}_{n+1};G) & \xrightarrow{\pi_{n+1}^*} & H^i(U_{n+1}^*;G)\end{array}$$

in which $\operatorname{im}\pi_{n+1}^*\supset\operatorname{im}r^*$ *for* $i\leqslant n$ *and* $\ker\tau^*\supset\ker\pi_0^*$ *for* $i\leqslant n+1$.

In the conclusion of the above theorem the symbol $\mathcal{U}_i$ denotes the geometric nerve of the collection U_i.

In a $\mathrm{cl}c^n$ space strong n-refinements can be found in the following way: Let U be a finite collection of closed subsets of X, U' be a star refinement of U, and V be an n-refinement of U'. Then V is a strong n-refinement of U.

We now make an observation which will enable us to obtain a relative form of Theorem F.

LEMMA 3.1. *If*

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[d, "i_1"] & C \arrow[r, "j_2"] \arrow[d, "i_2"] & E \arrow[d, "i_3"] \\ B \arrow[r, "k_1"'] & D \arrow[r, "k_2"'] & F \end{tikzcd}$$

is a commutative diagram of groups and homomorphisms such that $\operatorname{im} i_2 \supset \operatorname{im} k_1$ *and* $\ker j_2 \supset \ker i_2$, *then there is a natural homomorphism* j: $B \to E$ *such that*

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "j_2 j_1"'] & B \arrow[d, "k_2 k_1"] \arrow[dl, "j"'] \\ E \arrow[r, "i_3"'] & F \end{tikzcd}$$

is commutative.

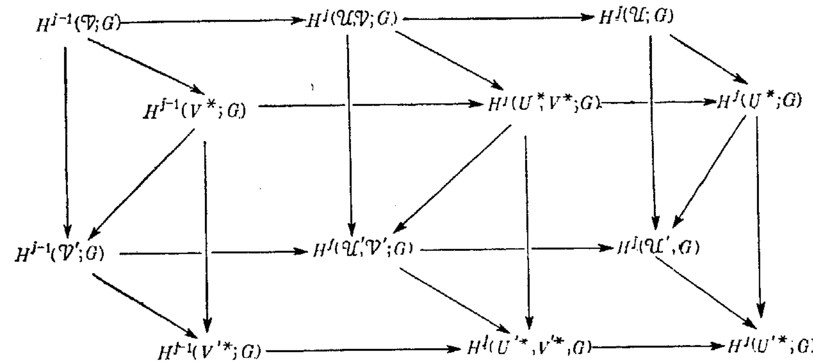
The homomorphism j is defined as follows: for $b \in B$, there is an element $c \in C$ such that $i_2(c) = k_1(b)$. Let $j(b) = j_2(c)$. The proof that j is a well-defined homomorphism and is natural is a routine verification.

LEMMA 3.2. *If X is a compact Hausdorff space, $U_0, U_1, \ldots, U_{2n+2}$ is a sequence of finite collections of closed subsets of X such that for each i, $0 \leqslant i \leqslant 2n+1$, U_{i+1} strongly n-refines U_i, then for $j \leqslant n$ there is a natural commutative diagram*

$$\begin{tikzcd} H^j(\mathcal{U}_0; G) \arrow[r, "\pi_0^*"] \arrow[d, "r^*"'] & H(U_0^*; G) \arrow[d, "r^*"] \arrow[dl, "c^*"'] \\ H^j(\mathcal{U}_{2n+2}; G) \arrow[r, "\pi_{2n+2}^*"'] & H^j(U_{2n+2}^*; G)\,. \end{tikzcd}$$

THEOREM 3.1. *If X is a compact Hausdorff space, $U_0, U_1, \ldots, U_{8n+8}$ is a sequence of finite collections of closed subsets of X such that for each i, $0 \leqslant i \leqslant 8n+7$, U_{i+1} strongly n-refines U_i, and V_i is a subcollection of U_i such that V_{i+1} strongly n-refines V_i, then, letting $U = U_0$, $V = V_0$,*

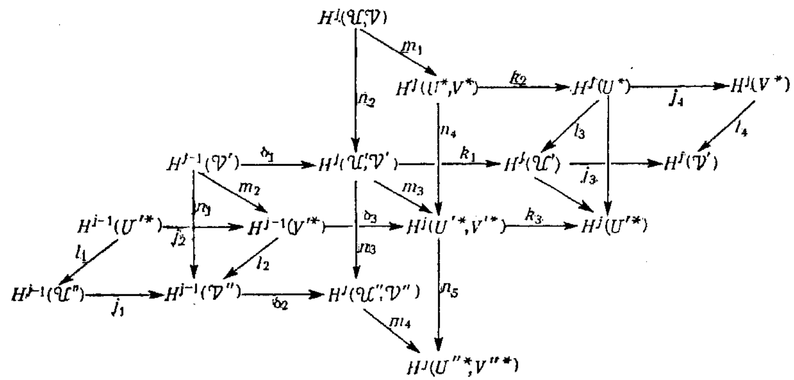
$U' = U_{8n+8}$, and $V' = V_{8n+8}$, for $j \leqslant n$ the following natural diagram is commutative:



Proof. We shall show that in the commutative diagram

$$\begin{tikzcd} H^j(\mathcal{U}_0, \mathcal{V}_0; G) \arrow[r, "m^*"] \arrow[d, "n^*"] & H^j(U_0^*, V_0^*; G) \arrow[d, "k^*"] \\ H^j(\mathcal{U}_{4n+4}, \mathcal{V}_{4n+4}; G) \arrow[r, "l^*"] & H^j(U_{4n+4}^*, V_{4n+4}^*; G)\,, \end{tikzcd}$$

$\operatorname{im} l^* \supset \operatorname{im} k^*$ and $\ker n^* \supset \ker m^*$ for $j \leqslant n$. This combined with Lemma 3.1 implies the theorem. By Lemmas 3.1 and 3.2 we have the commutative diagram (omitting the coefficient group)



in which $U_0 = U$, $U_{2n+2} = U'$, $U_{4n+4} = U''$, $V_0 = V$, $V_{2n+2} = V'$, and $V_{4n+4} = V''$.

$$\operatorname{im} m_4 \supset \operatorname{im} n_5 n_4 .$$

Suppose $\beta = n_5 n_4 \alpha$, where $\alpha \in H^j(U^*, V^*)$.

Let $\gamma = k_2 \alpha$. Then $j_3(l_3\gamma) = l_4 j_4 k_2 \alpha = 0$, and there is a element $\delta \in H^j(U', V')$ such that $k_1\delta = l_3\gamma$. Since $k_3(m_3\delta - n_4\alpha) = 0$, there is an element $\lambda \in H^{j-1}(V'^*)$ such that $\delta_3\lambda = -m_3\delta + n_4\alpha$. Then

$$m_4\delta_2 l_2\lambda = -n_5 m_3\delta + n_5 n_4\alpha = n_5 n_4\alpha - m_4 n_3\delta ;$$

and so,

$$n_5 n_4 \alpha = m_4(\delta_2 l_2\lambda + n_3\delta) . \qquad \ker m_1 \subset \ker n_3 n_2 .$$

Suppose $\alpha \in H^j(U, V)$ and $m_1\alpha = 0$. Then $k_1 n_2\alpha = l_3 k_2 m_1\alpha = 0$, and there is an element $\beta \in H^{j-1}(V')$ such that $\delta_1\beta = n_2\alpha$. Then $\delta_3 m_2\beta = m_3\delta_1\beta = m_3 n_2\alpha = n_4 m_1\alpha = 0$. Thus, there is an element $\gamma \in H^{j-1}(U'^*)$ such that $j_2\gamma = m_2\beta$. Then $j_1 l_1\gamma = l_2 m_2\beta = n_1\beta$ and $n_3 n_2\alpha = n_3\delta_1\beta = \delta_2 n_1\beta = \delta_2 j_1 l_1\gamma = 0$.

IV. Conditions under which $\dim X \times Y = \dim X + \dim Y$.

THEOREM 4.1. *If X and Y are finite dimensional compact Hausdorff spaces and $\dim X \times Y = \dim X + \dim Y$, then there is a prime p such that*

$$D(X; R_p) = \dim X \quad \textit{and} \quad D(Y; R_p) = \dim Y .$$

Proof. Suppose that there is no such prime p. Let $m = \dim X$ and $n = \dim Y$, and let A and B be closed subsets of X and Y, respectively, such that $H^{n+m}(X \times Y, A \times Y \cup X \times B; Z) \neq 0$. If $H^m(X, A; Z)$ is not a torsion group, then $H^m(X, A; R_p) \neq 0$ for every prime p and since $D(Y; R_p) = n$ for some prime p, we have a contradiction. Thus, both $H^m(X, A; Z)$ and $H^n(Y, B; Z)$ are torsion groups. If $\big(H^m(X, A; Z)\big)_p \neq 0$, then $H^m(X, A; R_p) \neq 0$ and $D(X; R_p) = m$. Hence, by our supposition for every prime p either

$$\big(H^m(X, A; Z)\big)_p = 0 \quad \text{or} \quad \big(H^n(Y, B; Z)\big)_p = 0 .$$

Since torsion groups are direct sums of their p-primary parts,

$$H^m(X, A; Z) \oplus H^n(Y, B; Z) \cong \oplus_{p,q}\Big(\big(H^m(X, A; Z)\big)_p \otimes \big(H^n(Y; B; Z)\big)_q\Big),$$

p and q running through all primes. Then

$$H^m(X, A; Z) \otimes H^n(Y, B; Z) \cong \oplus_p \Big(\big(H^m(X, A; Z)\big)_p \otimes \big(H^n(Y, B; Z)\big)_p\Big) = 0 .$$

Since $\operatorname{Tor}\big(H^i(X, A; Z), H^j(Y, B; Z)\big) = 0$ for $i+j > m+n$, by the relative Künneth sequence $H^{n+m}(X \times Y,\ A \times Y \cup X \times B; Z) = 0$, which is a contradiction.

THEOREM 4.2. *Let $F(p)$ denote the class of all compact, clc^∞ (over Z) Hausdorff spaces X such that $D(X; R_p) = \dim X$. If $X, Y \in F(p)$, then $X \times Y \in F(p)$.*

Proof. We shall show that if $X \in F(p)$, then $D(X; Z_p) = \dim X$. This, together with Theorem 2.2 (a), will imply our theorem.

Suppose $X \in F(p)$. Since $D(X; R_p) = \dim X = n$, and the sum theorem of classical dimension theory holds for cohomological dimension, there is a point $x \in X$ such that for every closed neighborhood N of x, $D(N; R_p) = n$. Since X is clc^n, there is a closed neighborhood N of x such that $H^n(X; Z) \to H^n(N; Z)$ is trivial. If this homomorphism were not an epimorphism, $D(X; Z) \geqslant n+1$. Thus, $H^n(N; Z) = 0$. There is a closed subset A of N such that $H^n(N, A; R_p) \neq 0$. This is possible only if $H^n(N, A; Z)$ contains an element γ which is either of infinite order or in $\big(H^n(N, A; Z)\big)_p$. By the sequence

$$H^{n-1}(N; Z) \to H^{n-1}(A; Z) \xrightarrow{\delta^*} H^n(N, A; Z) \to 0 ,$$

there is an element $\eta \in H^{n-1}(A; Z)$ such that $\gamma = \delta^*(\eta)$. There is a closed neighborhood B of A such that $H^{n-1}(B; Z)$ contains an element η' mapping onto η under the inclusion homomorphism. Then $H^n(X, B; Z)$ contains an element γ' mapping onto γ under the inclusion homomorphism.

Let U_0 denote a finite set of closed subsets of X whose interiors cover X and such that no element of U_0 intersects both A and $X - B$, and let $U_1, U_2, \ldots, U_{8n+8}$ be a sequence of closed coverings of X such that for each i, $0 \leqslant i \leqslant 8n+7$, U_{i+1} strongly n-refines U_i. Let $U = U_0$ and $U' = U_{8n+8}$, and let $V = \{u \in U \mid u \cap A \neq \emptyset\}$ and $V' = \{u \in U' \mid u \cap A \neq \emptyset\}$. Then by Theorem 3.1 we have the commutative diagram

$$\begin{tikzcd}
& H^n(\mathcal{U}, \mathcal{V}; Z) \arrow[r] \arrow[d] & H^n(\mathcal{U}', \mathcal{V}'; Z) \arrow[d] & \\
H^n(X, B; Z) \arrow[r] & H^n(X, V^*; Z) \arrow[r] \arrow[ur, dashed] & H^n(X, V'^*; Z) \arrow[r] & H^n(N, A; Z) .
\end{tikzcd}$$

It is clear from this diagram that $\operatorname{im}\big(H^n(X, B; Z) \to H^n(N, A; Z)\big)$ is finitely generated. If γ is of infinite order, there is an element $\tilde{\gamma}$ in that image of infinite order which is not divisible within the image. If γ is of order a power of p, there is an element γ'' in that image of order a power of p which is not divisible by p within that image. Let μ be an element of $H^n(X, B; Z)$ which maps onto $\tilde{\gamma}$ if γ is of infinite order or maps onto γ'' if γ is of order a power of p. Then in either case μ is not divisible by p. It follows from Theorem 1.1 (4_p) that $H^n(X, B; Z_p) \neq 0$; and so, $D(X; Z_p) = n$.

By Corollary 2.1 (e) if $D(X; Z_p) = \dim X$, then $D(X; R_p) = \dim X$. Thus, for compact Hausdorff spaces X which are clc^∞ (over Z),

$$X \in F(p) \quad \text{if and only if} \quad D(X; Z_p) = \dim X .$$

By Theorem 8 of [9] if X and Y are compact Hausdorff spaces both of which are clc^∞ (over Z), then $X\times Y$ is clc^∞ (over Z). If X and Y are in $F(p)$, then $D(X; Z_p) = \dim X$ and $D(Y; Z_p) = \dim Y$; by Theorem 2.2 (a), $D(X\times Y; Z_p) = \dim X + \dim Y$. Since $D(X\times Y; Z_p) \leqslant \dim X\times Y \leqslant \dim X + \dim Y$, $\dim X\times Y = D(X\times Y; Z_p)$, and as noted above $\dim X\times Y = D(X\times Y; R_p)$. Since $X\times Y$ is clc^∞ (over Z), $X\times Y \in F(p)$.

V. Remarks. V. G. Boltyanskii has constructed [4] a sequence $\{B_p\}$ of two dimensional compact metric spaces, one for each prime p, such that $1 = D(B_p; Q) = D(B_p; Z_q) = D(B_p; Q_q)$ for every prime q and $D(B_p; R_q) = 1 + \delta_{pq}$. ($H^2(B_p, A; Z) \cong \oplus Q_p$ if it is non-zero.) In Theorem 11 of [9] the author showed that if X is compact Hausdorff, clc^∞ (over Z), and $D(X; Q) = \dim X$, then for every compact Hausdorff space Y, $\dim X\times Y = \dim X + \dim Y$ (X is dimensionally full-valued). If $D(X; Q) < \dim X$, then $D(X; Q_p) < \dim X$ for every prime p, and it follows from Theorem 2.2 (c) that $\dim X\times B_p = \dim X + 1$ for every prime p. Thus, if X is compact Hausdorff and clc^∞ (over Z), it is dimensionally full-valued if and only if $D(X; Q) = \dim X$. This is a slight strengthening of Corollary 2 of [9].

It would be interesting to know if every compact Hausdorff, clc^∞ (over Z) space X is dimensionally full-valued. If there is such a space X which is not dimensionally full-valued, then $D(X; Q) < \dim X$. (It is not difficult to see that $2 < \dim X$.) If $n = \dim X$ and k is a positive integer, then by Theorem 2.2 (a) and Theorem 4.2

$$\dim X^k = kn \quad \text{and} \quad D(X^k; Q) \leqslant kn - k .$$

If Y is any closed subspace of X^k whose dimension exceeds $kn-k$, then $D(Y; Q) < \dim Y = m$. By the sequence

$$\ldots \to H^m(Y, B; R_p) \to H^m(Y, B; Q) \to H^m(Y, B; Q_p) \to 0$$

induced by the sequence $0\to R_p \to Q \to Q_p \to 0$, we see that for every prime p, $D(Y; Q_p) < \dim Y$. By Theorem 2.2 (c) it is seen that $\dim(Y\times B_p) = \dim Y + 1$ for every p. Thus, in a sense Y is maximally dimensionally deficient.

It might be supposed that no space could have such pathological dimension properties as this. That is not so, however. Pontrjagin has constructed a sequence $\{P_p\}$ of two-dimensional compact metric spaces, one for each prime p, such that $D(P_p; Z_p) = 2$ and $D(P_p; Q) = 1$. By Theorem 2.2 (a), $D(P_p^k; Z_p) = 2k = \dim P_p^k$ and $D(P_p^k; Q) = k$. The same argument as above then applies to show this space has the properties described above. The space P_p is clc^0 but it is not clc^1 (over Z).

Appendix. As the only published proof of the Künneth theorem in its exact sequence formulation known to the author is for the algebraic case, in this appendix an argument is sketched for Čech cohomology of locally compact Hausdorff spaces (equivalently, of compact pairs of Hausdorff spaces).

THE KÜNNETH SEQUENCE. *If (X, A) and (Y, B) are compact pairs of Hausdorff spaces and F is a field, then the sequence*

$$0 \to \sum_{i+j=n} H^i(X, A)\otimes H^j(Y, B) \to H^n(X\times Y,\ A\times Y \cup X\times B) \to \sum_{i+j=n+1} \operatorname{Tor}\big(H^i(X, A),\ H^j(Y, B)\big) \to 0$$

is exact and

$$\sum_{i+k=n} H^i(X, A; F)\otimes H^j(Y, B; F) \cong H^n(X\times Y,\ A\times Y\cup X\times B; F).$$

Furthermore, both the sequence and isomorphism are natural.

Proof. Let T and T' denote locally compact Hausdorff spaces and let $[C]$ and $[C']$ denote fine couvertures (Z or F couvertures) [7] on T and T'. Let π and π' denote the projection maps of $T\times T'$ onto T and T', respectively. Then $\pi^{-1}([C])\circ\pi'^{-1}([C'])$ is a fine couverture on $T\times T'$; and so, $H_c^*(T\times T') \cong H^*\big(\pi^{-1}([C])\circ\pi'^{-1}([C'])\big)$. But $\pi^{-1}([C])\circ\pi'^{-1}([C']) \cong C\otimes C'$, where C and C' denote the underlying algebras of the couvertures $[C]$ and $[C']$. Since C and C' are free group complexes, the algebraic Künneth theorems hold [8]; i. e.,

$$0\to \sum_{i+j=n} H^i(C)\otimes H^j(C') \to H^n(C\otimes C') \to \sum_{i+j=n+1} \operatorname{Tor}\big(H^i(C),\ H^j(C')\big) \to 0 ,$$

where $\otimes$ and Tor are over the group Z or field F as the case may be. It should be noted that Tor over a field is zero. Since $H_c^i(T) \cong H^i(C)$, $H_c^j(T') \cong H^j(C')$, and $H_c^n(T\times T') \cong H^n(C\otimes C')$,

$$0\to \sum_{i+j=n} H_c^i(T)\otimes H_c^j(T') \to H_c^n(T\times T') \to \sum_{i+j=n+1} \operatorname{Tor}\big(H_c^i(T),\ H_c^j(T')\big) \to 0 ,$$

where $H_c^*(\ )$ denotes cohomology with compact supports.

For the pairs (X, A) and (Y, B), let $T = X - A$ and $T' = Y - B$. Then $H^i(X, A) \cong H_c^i(T)$ and $H^j(Y, B) \cong H_c^j(T')$. Since $T\times T' = (X-A)\times(Y-B) = X\times Y - (A\times Y)\cup(X\times B)$, $H^n(X\times Y,\ A\times Y\cup X\times B) \cong H_c^n(T\times T')$. Making the appropriate substitutions in the above exact sequence, we obtain the Künneth sequence. All of the above isomorphism as well as the algebraic Künneth sequence are natural. Thus, naturality holds in the topological case.

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On cyclically ordered groups

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A relation $[x, y, z]$ which is defined on all ordered triplets of different elements x, y, z of a group G is called a *cyclic order* if it has the following properties:

I. *Either* $[x, y, z]$ *or* $[z, y, x]$,

II. $[x, y, z]$ *implies* $[y, z, x]$,

III. $[x, y, z]$ *and* $[y, u, z]$ *implies* $[x, u, z]$,

IV. $[x, y, z]$ *implies* $[uxv, uyv, uzv]$ *for* $u, v \in G$.

A group on which a cyclic order is defined will be called a *cyclically ordered group* (for references see [1]).

The natural order of points on a directed circle defines a cyclic order on the group of multiplication of complex numbers of absolute value one. We shall denote this group by K and the cyclic order on K by (x, y, z) [1].

If Γ is a (linearly) ordered group, then a cyclic order $[x, y, z]$ is defined on Γ by

$$[x, y, z] \equiv x < y < z \quad \text{or} \quad y < z < x \quad \text{or} \quad z < x < y .$$

We shall say that this cyclic order is *generated by the order* on Γ.

Cyclically ordered groups can be obtained by the following construction. Let Γ be an ordered group and let $[x, y, z]$ be the cyclic order generated by the order on Γ. We consider the direct product $\Gamma \times K$ (its elements are pairs $\langle x, a\rangle$, $x \in \Gamma$, $a \in K$) and we define a cyclic order on this group by

$$[\langle x, a\rangle, \langle y, b\rangle, \langle z, c\rangle] \equiv \begin{cases} (a, b, c) & \text{in } K \quad \text{if} \quad a \neq b \neq c \neq a, \\ x < y & \text{in } \Gamma \quad \text{if} \quad a = b \neq c, \\ y < z & \text{in } \Gamma \quad \text{if} \quad b = c \neq a, \\ z < x & \text{in } \Gamma \quad \text{if} \quad c = a \neq b, \\ [x, y, z] & \text{in } \Gamma \quad \text{if} \quad a = b = c . \end{cases}$$

This cyclic order on $\Gamma \times K$ will be called the *natural cyclic order*. Evidently every subgroup of $\Gamma \times K$ is also a cyclically ordered group. The aim of this paper is to prove that there exist no other cyclically ordered groups, i. e.

[1] A more precise definition is given in the remark to Lemma 1.