whence, by Theorem 3.1, $B_{\cdot}(x^*)=0$. Thus $0=B_{\cdot}(x^*)=B_{\cdot}(x)-B_{\cdot}(u) = B_{\cdot}(x)-\chi_B(u) = B_{\cdot}(x)-\chi_A(\chi_A^{-1}(A_{\cdot}(x))) = B_{\cdot}(x)-A_{\cdot}(x)$.

We conclude with the following remarks. The formulation of the generalizations of Theorem 3.2 for the case of sequences not necessarily summable to zero is left to the reader. In Theorem 3.1 the methods A and B need not transform the sequences of X into sequences of the same space Y.

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Spaces of continuous functions (III)

(Spaces $C(\Omega)$ for Ω without perfect subsets)

by

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A topological space T is said to be *dispersed* (1) if it contains no perfect non-void subset. In this paper we present some investigations on spaces $C(\Omega)$ of real-valued continuous functions defined on dispersed compact Hausdorff spaces Ω.

In the main theorem we give a number of necessary and sufficient conditions for a compact Hausdorff space Ω to be dispersed (in terms of the space $C(\Omega)$, of its conjugate space $V(\Omega)$, or in terms of Borel measures on Ω). In particular, we prove that the *dispersedness of a compact Ω is an invariant of the linear dimension* (2) *of* $C(\Omega)$; conditions (5), (6), (8) and (9) give characterizations of dispersedness by invariants of linear dimension. The most essential condition is the following one:

In order that a compact Hausdorff space Ω be dispersed, it is necessary and sufficient that the condition $\dim_l X \leqslant \dim_l C(\Omega)$ *imply* $\dim_l X \geqslant \dim_l c_0$ *or X is of finite dimension.*

Other characterizations may easily be derived from the above.

By condition (10), the dispersedness of a compact Ω is necessary and sufficient for any Borel measure on Ω to be purely atomic, which enables us to derive some properties of the first and second conjugate to $C(\Omega)$. At the same time, these properties characterize the dispersedness of Ω, e. g. *a compact Ω is dispersed if and only if the space $V(\Omega)$ conjugate to $C(\Omega)$ contains no subspace isomorphic to the space L.*

In a further section of this paper we consider some singular properties of the spaces $C(\Omega)$ for dispersed Ω. The great importance of these spaces consists in the fact that many general questions concerning Banach spaces may be negatively solved by a suitable counter-example of a space $C(\Omega)$ with dispersed Ω.

(1) *clairsemé* in French (see [14], p. 46).

(2) The definition of linear dimension is given by Banach ([1], p. 193; see also [2]). The symbol $\dim_l X \leqslant \dim_l Y$ means that X is isomorphic to a subspace of Y, $\dim_l X = \dim_l Y$ means that $\dim_l X \leqslant \dim_l Y$ and $\dim_l X \geqslant \dim_l Y$, and so on.

1. Preliminaries. In the sequel, Ω will denote a compact Hausdorff space and $C(\Omega)$ will denote the Banach space of all real-valued continuous functions $x = x(t)$ defined on Ω with the norm $\|x\| = \max_{t\in\Omega}|x(t)|$ and with the natural ordering;

$V(\Omega)$ will denote the Banach lattice conjugate of $C(\Omega)$;

$\mathcal{I}$ will denote the closed interval $\langle 0, 1\rangle$;

$\mathcal{C}$ will denote the Cantor discontinuum;

ω and ω_τ will denote the least ordinal numbers of power $\aleph_0$ and $\aleph_\tau$ respectively;

α being an ordinal, $\Gamma(\alpha)$ or Γ_α will denote the set of all ordinals $\leqslant \alpha$ (with order topology); we shall also write C_α instead of $C(\Gamma_\alpha)$ or $C(\Gamma(\alpha))$;

$\gamma(\aleph_\tau)$ will denote the one-point compactification of an isolated set of power $\aleph_\tau$; thus, $C(\gamma(\aleph_\tau))$ is equivalent to the space $c(\aleph_\tau)$ of all transfinite sequences $x_\alpha = x(\alpha)$, $\alpha \leqslant \omega_\tau$, such that the set $\{\alpha: |x(\alpha) - x(\omega_\tau)| \geqslant \varepsilon\}$ is finite for all $\varepsilon > 0$; we shall write c in place of $c(\aleph_0)$;

$\beta(\aleph_\tau)$ will denote the Stone-Čech compactification of an isolated set of power $\aleph_\tau$; thus, the space $C(\beta(\aleph_\tau))$ is equivalent to the space $m(\aleph_\tau)$ of all bounded functions on a set of power $\aleph_\tau$; for $\tau = 0$ we shall write m instead of $m(\aleph_0)$;

$l(\aleph_\tau)$ will denote the space of all sequences $y_\alpha = y(\alpha)$, $\alpha < \omega_\tau$, such that the set $\{\alpha: y(\alpha) \neq 0\}$ is countable and $\sum_\alpha |y(\alpha)| < \infty$; we shall write l instead of $l(\aleph_0)$.

$\mathcal{I}^{\aleph_\tau}$ will denote the Tychonoff cube with the product measure ν_τ (see [12], p. 157-158) and with the product topology; $L(\aleph_\tau)$ will denote the space of all functions $x(t)$ absolutely integrable on $\mathcal{I}^{\aleph_\tau}$ (or the equivalent space of all signed-measures on $\mathcal{I}^{\aleph_\tau}$ absolutely continuous with respect to ν_τ); we shall write L instead of $L(\aleph_0)$.

$c(\aleph_\tau)$, $m(\aleph_\tau)$, $l(\aleph_\tau)$ and $L(\aleph_\tau)$ are Banach spaces with the norms

$$\sup_\alpha |x(\alpha)|, \quad \sup_\alpha |x(\alpha)|, \quad \sum_\alpha |y(\alpha)| \quad \text{and} \quad \int_{\mathcal{I}^{\aleph_\tau}} |x(t)|\,d\nu_\tau,$$

respectively. Moreover, it is known that $l(\aleph_\tau)$ is equivalent to the space conjugate of $c(\aleph_\tau)$ and $m(\aleph_\tau)$ is equivalent to the space conjugate of $l(\aleph_\tau)$ for any $\tau \geqslant 0$.

α being an ordinal, $\Psi^{(\alpha)}$ will denote the α-th derivative of the set Ψ.

LEMMA 1. *Let σ be a continuous mapping of a compact set Ω onto another set Φ. Then the inclusion*

$$(*) \qquad \sigma(\Omega^{(\alpha)}) \subset \Phi^{(\alpha)}$$

is satisfied for every ordinal α.

Proof. Let $\alpha = 1$ and let $u \in \Phi^{(1)}$; then $\sigma^{-1}(u) \cap \Omega^{(1)} \neq 0$ (in the contrary case $\sigma^{-1}(u)$ as well as $\sigma[\sigma^{-1}(u)]$ must be closed, open and finite, whence $\sigma[\sigma^{-1}(u)] \cap \Phi^{(1)} = 0$). Next, let us assume (*) for an ordinal α; writing $\Psi = \sigma^{-1}(\Phi^{(\alpha)}) \cap \Omega^{(\alpha)}$, we obtain $\Psi^{(1)} \subset \Omega^{(\alpha+1)}$ and $\Phi^{(\alpha)} \subset \sigma(\Psi)$, whence

$$\sigma(\Omega^{(\alpha+1)}) \supset \sigma(\Psi^{(1)}) \supset [\sigma(\psi)]^{(1)} \supset \Phi^{(\alpha+1)}.$$

Finally, let us assume (*) for all $\alpha < \lambda$, λ being a limit number; we shall prove that $\sigma(\Omega^{(\lambda)}) \supset \Phi^{(\lambda)}$. Let $u \in \Phi^{(\lambda)}$; then $u \in \Phi^{(\alpha)}$ for $\alpha < \lambda$. Write $A = \sigma^{-1}(u)$ and $A_\alpha = A \cap \Omega^{(\alpha)}$. By the assumption of induction, all A_α are non-empty, whence, by the compactness of Ω and by $A_\alpha \supset A_{\alpha+1}$, the intersection $\bigcap_{\alpha<\lambda} A_\alpha = A \cap \bigcap_{\alpha<\lambda} \Omega^{(\alpha)} = A \cap \Omega^{(\lambda)}$ is also non-empty. t being a point of $A \cap \Omega^{(\lambda)}$, we have $\sigma(t) = u$.

LEMMA 2. *Let Ω be compact and let it be possible to choose, for every finite sequence $\varepsilon_1, \ldots, \varepsilon_n$ of numbers 0 and 1 ($n = 1, 2, \ldots$), a closed-and-open non-empty subset $\Omega_{\varepsilon_1,\ldots,\varepsilon_n}$ of Ω so that*

$$\Omega_0 \cup \Omega_1 = \Omega, \quad \Omega_0 \cap \Omega_1 = 0, \quad \Omega_{\varepsilon_1,\ldots,\varepsilon_{n-1},0} \cap \Omega_{\varepsilon_1,\ldots,\varepsilon_{n-1},1} = 0,$$

$$\Omega_{\varepsilon_1,\ldots,\varepsilon_{n-1},0} \cup \Omega_{\varepsilon_1,\ldots,\varepsilon_{n-1},1} = \Omega_{\varepsilon_1,\ldots,\varepsilon_{n-1}} \quad \textit{for } n = 1, 2, \ldots$$

Then the mapping

$$\sigma(t) = \{\varepsilon_1, \varepsilon_2, \ldots\} \quad \textit{for} \quad t \in \bigcap_{n=1}^{\infty} \Omega_{\varepsilon_1,\ldots,\varepsilon_n}$$

is continuous, and $\sigma(\Omega) = \mathcal{C}$.

X being a Banach space, X^* will denote its conjugate space. A series $\sum x_n$ of elements of X will be termed w. u. c. (*weakly unconditionally convergent*) if $\sum_{n=1}^{\infty} |\xi(x_n)| < \infty$ for every $\xi \in X^*$ (we do not assume $\sum x_n$ to be weakly convergent to an element of X). The further terminology and notation used in this paper follows Banach's monograph [1] and that of Day [8].

LEMMA 3. *In order that a Banach space X contain isomorphically the space c, it is necessary and sufficient that there exist in X a sequence x_n such that $\overline{\lim}_{n\to\infty} \|x_n\| > 0$ and that $\sum x_n$ be w. u. c.*

Lemma 3 is given in [18], p. 797 and p. 160. The condition $\overline{\lim}\|x_n\| > 0$ may be replaced by $\|x_n\| = 1$ for $n = 1, 2, \ldots$

LEMMA 4. *Let x_n be a sequence of elements of a space $C(\Omega)$. In order that the series $\sum x_n$ be w. u. c., it is necessary and sufficient that there exist a number K such that $\sum_{n=1}^{\infty} |x_n(t)| \leqslant K$ for all $t \in \Omega$.*

This is a well-known consequence of a theorem of Banach (see [1], p. 224).

By the Riesz representation theorem on the general form of linear functionals over $C(\Omega)$ (see [21], p. 326, and [13], p. 1012), the space $V(\Omega)$ is equivalent and lattice-isomorphic to the space of all regular Borel signed-measures on Ω with usual addition and multiplication by scalars, with the norm $\|\mu\| = |\mu|(\Omega)$ and with the natural ordering (see [1], p. 122). By the Radon-Nikodym theorem it follows that, for any fixed measure μ_0, the set Y of all signed-measures absolutely continuous with respect to μ_0 is a closed linear subspace and a lattice-complete l-ideal in $V(\Omega)$ (in the sense of G. Birkhoff, see [5], p. 222 and 232), and a projection of norm 1 transforming $V(\Omega)$ on Y exists.

2. Main theorem. *Ω being a compact Hausdorff space, the following statements* ([3]) *are equivalent:*

(0) *Ω is dispersed,*

(1) *every continuous image of Ω is dispersed,*

(2) *Ω is 0-dimensional and the Cantor discontinuum $\mathcal{C}$ is not a continuous image of Ω,*

(3) *the unit interval J is not a continuous image of Ω,*

(4) *every separable subspace of $C(\Omega)$ is contained in a subspace equivalent to a space $C(\Gamma_\alpha)$, where $\alpha < \omega_1$,*

(5) *every infinitely dimensional subspace of $C(\Omega)$ contains isomorphically the space c,*

(6) *no subspace of $C(\Omega)$ is isomorphic to l,*

(7) *no subspace of $C(\Omega)$ is isometric to $C(J)$,*

(8) *for every separable subspace X of $C(\Omega)$ the space X^* is also separable.*

(9) *every bounded subset of $C(\Omega)$ is conditionally weakly (sequentially) compact, i. e. for every bounded sequence x_n of elements of $C(\Omega)$ a subsequence x_{n_k} exists such that for every $\xi \in V(\Omega)$ the sequence $\xi(x_{n_k})$ is convergent,*

(10) *every regular Borel atomless measure on Ω is identically zero,*

(11) *every linear functional over $C(\Omega)$ is of the form*

$$\xi(x) = \sum_{n=1}^{\infty} a_n x(t_n),$$

where $t_1, t_2, \ldots$ is a fixed sequence of points of Ω and $\sum_{n=1}^{\infty} |a_n| < \infty$,

([3]) Some implications between these statements are known. Excluding some that are trivial, the implications (0) → (1) and (0) → (10) → (11) have been proved by W. Rudin [20], whose proofs are different and somewhat more concise.

(12) *$V(\Omega)$ is equivalent to a space $l(\aleph_\tau)$, or $\dim_l V(\Omega) < \infty$,*

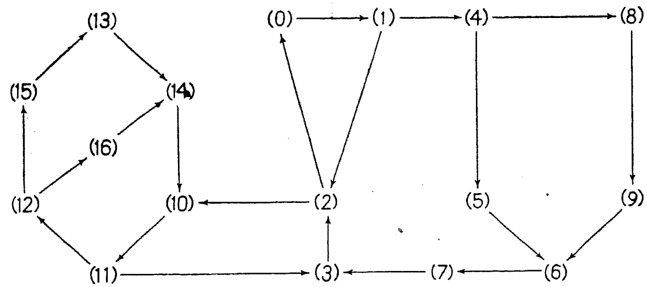
(13) *$V(\Omega)$ contains no subspace isomorphic to L,*

(14) *$V(\Omega)$ contains no lattice-complete l-ideal equivalent (in the linear-metric-lattice sense) to a space $L(\aleph_\tau)$,*

(15) *every separable subspace of $V(\Omega)$ is contained in a subspace of $V(\Omega)$ isometric to l,*

(16) *the second conjugate to $C(\Omega)$ is equivalent to a space $m(\aleph_\tau)$.*

Proof. We shall prove the following implications:



(0) → (1) follows immediately from Lemma 1.

(1) → (4). Let X_0 be a separable subspace of $C(\Omega)$ and let X_1 be the smallest subring with unit spanned on X_0. X_1 being separable, by a theorem of Eidelheit [9], X_1 is equivalent to a space $C(\Omega_0)$, where Ω_0 is compact and metrisable. By a theorem of M. H. Stone (see [23], p. 475), Ω_0 is a continuous image of Ω, whence, if we assume (1), Ω_0 is countable, and, by a theorem of Mazurkiewicz and Sierpiński [16], Ω_0 is homeomorphic to a space Γ_α with $\alpha < \omega_1$, whence $X_0 \subset X_1 \equiv C_\alpha$.

(4) → (8). Since every Borel measure on a countable set must be purely atomic, the space conjugate to C_α is (for $\omega \leqslant \alpha < \omega_1$) equivalent with the space l. Thus, under assumption (4), condition (8) follows by the known fact that if $X \subset Y$ and if Y^* is separable, then X^* is also separable (if a sequence $\eta_1, \eta_2, \ldots$ is strongly dense in Y^*, then the restricted functionals $\eta_1|X, \eta_2|X, \ldots$ are dense in X^*).

(8) → (9) is a consequence of the following well-known theorem: if X^* is separable, every bounded set in X is conditionally weakly compact (an analogous theorem is in [1], p. 123).

(9) → (6), since the weak and strong convergences in l are equivalent (see [1], p. 137).

(4) → (5). Assuming (4) we may restrict ourselves to the case $C(\Omega) = C_\alpha$, α being countable; we have to prove that any infinite

dimensional subspace X of C_α contains a subspace X_1 isomorphic to c (for $\alpha=\omega$ this theorem has been proved by Banach, see [1], p. 194). Let us suppose, a contrario, that there exists an ordinal $\lambda<\omega_1$ such that the space C_λ contains a subspace X whose linear dimension is uncomparable with the linear dimension of c; moreover, let us suppose that λ is the least number with this property (it is easily seen that λ must be a limit number). Write

$$X_0=\{x\in X\colon\ x(\lambda)=0\};$$

since the set of all x such that $x(\lambda)=0$ has the deficiency 1 in C_λ, X_0 must be infinitely dimensional. Let α be any ordinal smaller than λ and let $\varepsilon>0$. We shall prove that there exists a function $x\in X_0$ such that

$$(**)\qquad \|x\|=1 \quad\text{and}\quad |x(t)|<\varepsilon \quad\text{for all}\quad t\leqslant\alpha.$$

Let us denote by Z_α the set $X_0|\Gamma_\alpha$, i. e. the set of all functions $z(t)$ belonging to C_α for which there exist functions $x(t)$ belonging to X_0 such that $x(t)=z(t)$ for $t\leqslant\alpha$, and let us distinguish two cases.

A. Z_α is finitely dimensional. Then, since X_0 is infinitely dimensional, there exist functions x_1,x_2 of X_0 such that $x_1\neq x_2$ and $x_1(t)=x_2(t)$ for $t\leqslant\alpha$, whence the function $x=(x_1-x_2)\|x_1-x_2\|^{-1}$ satisfies $(**)$.

B. Z_α is infinitely dimensional. Then, by assumption and by $Z_\alpha\subset C_\alpha$, we have $\dim_l Z_\alpha\geqslant\dim_l c$, whence, by Lemmas 3 and 4, there exists a sequence z_n of elements of Z_α such that

$$\|z_n\|_{C_\alpha}=\sup_{t\leqslant\alpha}|z_n(t)|=1 \quad\text{and}\quad 1\leqslant\sup_{t\leqslant\alpha}\sum_{n=1}^{\infty}|z_n(t)|=K<\infty.$$

Let x_n be functions belonging to X_0 such that $x_n(t)=z_n(t)$ for $t\leqslant\alpha$ and for $n=1,2,\ldots$ Obviously, $\|x_n\|=\sup\limits_{t\leqslant\lambda}|x_n(t)|\geqslant\|z_n\|_{C_\alpha}=1$. By the assumption about λ and by Lemma 3, the series Σx_n is not w. u. c. (in C_λ). Hence, by Lemma 4, there exist a point $t\in\Gamma_\lambda/\Gamma_\alpha$ and integers $m_1<m_2<\ldots<m_i$ such that $|x_{m_1}(t)+\ldots+x_{m_i}(t)|>K/\varepsilon$, whence the function $x=(x_{m_1}+\ldots+x_{m_i})\|x_{m_1}+\ldots+x_{m_i}\|^{-1}$ satisfies $(**)$.

Now, let z_1 be any element of X_0 of norm 1. Since $\lim\limits_{t\to\lambda}z_1(t)=z_1(\lambda)=0$, there exists an ordinal $\delta_1<\lambda$ such that $|z_1(t)|<\frac{1}{2}$ for $t>\delta_1$. Next, by $(**)$, there exists a function $z_2\in X_0$ such that $\|z_2\|=1$ and that $|z_2(t)|<\frac{1}{2}$ for $t\leqslant\delta_1$. Let us assume $z_1,z_2,\ldots,z_k$ (of norm 1) to have been defined, and let δ_k be an ordinal such that $|z_i(t)|<1/2^k$ for $i=1,2,\ldots,k$ and for $t\geqslant\delta_k$. By $(**)$, there exists a function $z_{k+1}\in X_0$ such that $\|z_{k+1}\|=1$ and $|z_{k+1}(t)|<1/2^k$ for $t\leqslant\delta_k$. It is easily seen that $\sum\limits_{n=1}^{\infty}|z_n(t)|\leqslant 2$ for all $t\in\Gamma_\lambda$, whence the series $\sum z_n$ is w. u. c. Thus, by

Lemma 3, the space C_λ contains isomorphically c, which contradicts our assumption.

$(5)\to(6)$, since the space l (being weakly complete) does not contain c isomorphically.

non $(3)\to$ *non* (7) by the quoted theorem of Stone ([23], p. 475).

non $(2)\to$ *non* (3). If Ω is not 0-dimensional, it contains an infinite closed connected subset Ω_0 (see [5], p. 174). Ω_0 is homeomorphic to a subset Ω_1 of a Tychonoff cube $\mathcal{I}^{\aleph_\tau}$; evidently, the projection of Ω_1 onto a suitable axis is a continuous mapping of Ω_1 onto a compact connected infinite subset of that axis, which means that a continuous mapping of Ω_0 onto $\mathcal{I}$ exists. Ω being normal, by the Tietze extension theorem any continuous mapping of Ω_0 onto $\mathcal{I}$ may be extended to the whole Ω.

On the other hand, if Ω is 0-dimensional, we apply the well-known theorem stating that $\mathcal{I}$ is a continuous image of $\mathcal{C}$.

non $(0)\to$ *non* (2). Let Ω be non-dispersed and 0-dimensional, let Ψ be a perfect non-void subset of Ω, and let $\Psi=\Psi_0\cup\Psi_1$ be a decomposition of Ψ into two disjoint, closed, infinite and relatively (in Ψ) open subsets. Ω being 0-dimensional and compact, there exist closed-and-open subsets Ω_0 and Ω_1 of Ω such that $\Psi_0\subset\Omega_0$, $\Psi_1\subset\Omega_1$, $\Omega_0\cap\Omega_1=0$, $\Omega_0\cup\Omega_1=\Omega$. Since Ψ_1 and Ψ_2 are open in Ψ, they are perfect (see [14], p. 46). Repeating the above reasoning we can choose closed non-dispersed sets Ω_{00}, Ω_{01}, Ω_{10} and Ω_{11} such that

$$\Omega_{00}\cup\Omega_{01}=\Omega_0,\quad \Omega_{00}\cap\Omega_{01}=0,\quad \Omega_{10}\cup\Omega_{11}=\Omega_1,\quad \Omega_{10}\cap\Omega_{11}=0,$$

and so forth; by Lemma 2 a continuous mapping of Ω onto $\mathcal{C}$ exists.

non $(10)\to$ *non* (2). Let μ be a non-trivial regular atomless Borel measure on Ω. Then μ, considered on the Boolean algebra of open-and-closed subsets of Ω, is finitely additive, and the values of $\mu(A)$ for open-and-closed A are dense in the interval $\langle 0,\mu(\Omega)\rangle$ (see [12], p. 169 and [13], p. 1011). Thus there exist open-and-closed subsets Ω_0 and Ω_1 of Ω such that

$$\Omega_0\cup\Omega_1=\Omega,\qquad \Omega_0\cap\Omega_1=0,\qquad \tfrac{1}{3}\leqslant\mu(\Omega_0)\leqslant\tfrac{2}{3},\qquad \tfrac{1}{3}\leqslant\mu(\Omega_1)\leqslant\tfrac{2}{3},$$

and similarly, there exist open-and-closed sets $\Omega_{00},\Omega_{01},\Omega_{10},\Omega_{11}$ such that $\Omega_{00}\cup\Omega_{01}=\Omega_0$, $\Omega_{00}\cap\Omega_{01}=0$, $\Omega_{10}\cup\Omega_{11}=\Omega_1$, $\Omega_{10}\cap\Omega_{11}=0$ and $\frac{1}{9}\leqslant\mu(\Omega_{ik})\leqslant\frac{4}{9}$ for $i,k=1,2$, and so forth; by Lemma 2 a continuous mapping of Ω onto $\mathcal{C}$ exists.

$(10)\to(11)$ follows by the quoted Riesz representation theorem on the general form of linear functionals on the space $C(\Omega)$ for a compact Ω (see [13], p. 1009 and 1012).

non (3) $\to$ *non* (11). Let σ be a continuous mapping of Ω onto $\mathcal{T}$. It is known that the set X_0 of all functions of the form $x(t) = z(\sigma(t))$ with $t \in \Omega$ and $z \in C(\mathcal{T})$ is a subspace of $C(\Omega)$ isometric to $C(\mathcal{T})$. The functional

$$\eta(z) = \int_0^1 z(v)\,dv$$

is linear over $C(\mathcal{T})$; let ξ be the corresponding functional over X_0 and let $\bar{\xi}$ be the Hahn-Banach extension of ξ onto $C(\Omega)$. Assuming a contrario condition (11), we have $\bar{\xi}(x) = \sum_{n=1}^{\infty} a_n x(t_n)$ for all $x \in C(\Omega)$, with fixed $t_1, t_2, \ldots$ and $a_1, a_2, \ldots$ In particular, we have

$$\bar{\xi}(x) = \xi(x) = \sum_{n=1}^{\infty} a_n x(t_n) = \sum_{n=1}^{\infty} a_n z(\sigma(t_n)) = \sum_{n=1}^{\infty} a_n z(v_n) = \int_0^1 z(v)\,dv$$

for all $x \in X_0$, $z \in C(\mathcal{T})$ and $v_n = \sigma(t_n) \in \mathcal{T}$, which is impossible (4).

(15) $\to$ (13) follows from the fact that $\dim_l l < \dim_l L$ (see [2], p. 108).

(13) $\to$ (14). It suffices to prove that $L(\aleph_\tau)$ contains isomorphically the space L. Since $\mathcal{T}^{\aleph_\tau} = \mathcal{T}^{\aleph_\tau} \times \mathcal{T}^{\aleph_0}$ (in the set-theoretical and measure sense), we have (5) $L(\aleph_\tau) = Y_1 \times L(\aleph_0)$.

non (10) $\to$ *non* (14). Let μ be a non-trivial atomless Borel measure on Ω. By a theorem of D. Maharam [15], there exist a sequence $\tau_1, \tau_2, \ldots$ of ordinals ($\tau_i \geqslant 0$) and a sequence $A_1, A_2, \ldots$ of disjoint Borel subsets of Ω such that $\Omega = \bigcup_{n=1}^{\infty} A_n$ and such that each space A_n considered (with respect to μ) as a measure space (modulo sets of μ-measure zero) is isomorphic (modulo a multiplicative constant) to the measure space $\mathcal{T}^{\aleph_{\tau_n}}$. Let Y_0 be the set of all signed-measures of $V(\Omega)$ absolutely continuous with respect to μ and vanishing outside A_1. Evidently Y_0 is a lattice-complete l-ideal in $V(\Omega)$ and Y_0 is equivalent and lattice-isomorphic with $L(\aleph_{\tau_1})$.

non (14) $\to$ *non* (16). Let Y_0 be a lattice-complete l-ideal in $V(\Omega)$ equivalent and lattice-isomorphic to $L(\aleph_\tau)$, and let Y_1 be the orthogonal

(4) We are indebted to Mr. S. Mrówka for this argument in the proof.

(5) In general, $\langle A_1, \mathfrak{A}_1, \mu_1\rangle$ and $\langle A_2, \mathfrak{A}_2, \mu_2\rangle$ being measure spaces, the correspondence $Tx(a, b) = x(a)$, $x(\cdot) \to Tx(\cdot, \cdot)$ (where $x \in Y_1 = L^1(A_1, \mathfrak{A}_1, \mu_1)$ and $Tx \in Z = L^1(A_1 \times A_2, \mathfrak{A}_1 \times \mathfrak{A}_2, \mu_1 \times \mu_2)$) establishes an equivalence between Y_1 and a subspace Z_1 of Z, and the correspondence

$$Uz(u, w) = \int_{A_2} z(u, v)\,dv, \qquad z(\cdot, \cdot) \to Uz(\cdot, \cdot)$$

establishes, by Fubini's theorem, a projection of Z onto Z_1.

complement of Y_0, i. e. the set of all $y_1 \in V(\Omega)$ such that $|y_0| \wedge |y_1| = 0$ for all $y_0 \in Y_0$. Since $V(\Omega)$ is a complete vector lattice (see [20], p. 179) by a theorem of F. Riesz (see [5], p. 233, and [20], p. 185-186), $V(\Omega)$ may be represented as the direct sum $Y_0 \times Y_1$, and (since $|y_0| \wedge |y_1| = 0$ for $y_0 \in Y_0$, $y_1 \in Y_1$) we have $\|y_0 + y_1\| = \|y_0\| + \|y_1\|$ for all $y_0 \in Y_0$, $y_1 \in Y$. Hence the space Ξ conjugate to $V(\Omega)$ is equivalent and lattice-isomorphic to the Cartesian product of the spaces Ξ_0 (conjugate to $L(\aleph_\tau)$) and Ξ_1 (conjugate to Y_1); moreover, the norm in Ξ is given by the formula

$$\|(\xi, \eta)\| = \max(\|\xi\|, \|\eta\|)$$

(see [1], p. 192). By a theorem of S. Kakutani (see [13], p. 1023) there exist compact Hausdorff spaces Ω, Ω_0, Ω_1 such that the spaces Ξ, Ξ_0, Ξ_1 are equivalent to $C(\Omega)$, $C(\Omega_0)$, $C(\Omega_1)$, respectively, and, by a theorem of S. Eilenberg ([10], p. 577), Ω is homeomorphic to the discrete union $\Omega_0 \cup \Omega_1$ (Ω_0 and Ω_1 being considered as disjoint). Since Ξ_0 is equivalent to the space $L^\infty(\mathcal{T}^{\aleph_\tau})$ of essentially bounded measurable functions on $\mathcal{T}^{\aleph_\tau}$, and since Ω_0 is the Stone space corresponding to the atomless Boolean algebra of all measurable subsets of $\mathcal{T}^{\aleph_\tau}$ (up to sets of ν_τ-measure zero), Ω_0 is dense-in-itself. Since every dense-in-itself subset of $\beta(\aleph_\alpha)$ is nowhere dense in $\beta(\aleph_\alpha)$, the space Ω is not homeomorphic to any space $\beta(\aleph_\alpha)$, whence Ξ is not equivalent with any space $m(\aleph_\alpha)$.

The implications (1) $\to$ (2), (6) $\to$ (7), (11) $\to$ (12), (12) $\to$ (15) and (12) $\to$ (16) are trivial.

3. Singular examples. It is known that numerous examples of Banach spaces possessing special properties (such as the extension property, the property of isometrically containing all separable Banach spaces, etc.) are to be found among the spaces $C(\Omega)$ or their conjugate spaces. In a sense, the M-spaces and L-spaces occupy two extreme, mutually opposite places among the Banach spaces, while the reflexive spaces take a central place.

In particular, the spaces $C(\Omega)$ with dispersed Ω have some singular properties. Firstly, by condition (12), the space conjugate to $C(\Omega)$ with dispersed infinite Ω is equivalent to the space $l(\aleph_\tau)$, where $\aleph_\tau$ denotes the power of Ω. Thus, the space $V(\Omega)$ depends only on the power of Ω; however, the isomorphic properties of $C(\Omega)$ depend also on the topological properties of Ω, and there are many examples of *non-isomorphic Banach spaces with equivalent conjugate spaces*, even for metrisable Ω (i. e. for separable $C(\Omega)$ and $V(\Omega)$). E. g. the spaces $C(\Gamma(\omega))$ and $C(\Gamma(\omega^\omega))$ are not isomorphic (6); they give a negative answer to a problem of Banach ([1], p. 243); the first example of this kind is given in [4], p. 250.

(6) The proof of this theorem will be published in the next paper.

Further singular examples are the following:

the space $c(\aleph_1)$ is not isomorphic to a strictly convex space (Day [7], p. 521).

in the space $c(\aleph_1)$ the functional $p(x) = \|x\|$ is not a pointwise limit of polynomial functionals (see [17], p. 180-181).

Alexandroff and Urysohn constructed a compact, non-metrisable Hausdorff space Q_M of power $2^{\aleph_0}$, containing a countable dense set $T = \{t_1, t_2, \ldots\}$ of isolated points (see [24], p. 936, and [22]). We are indebted to S. Mrówka for this example.

Let us write $Z = C(\mathcal{J}) \times C(Q_M)$; then

$$\dim_l C(\mathcal{J}) < \dim_l Z < \dim_l m.$$

Indeed, Z is equivalent to a subspace of m, since there exists a countable set $\zeta_1, \zeta_2, \ldots$ of linear functionals over Z such that $\|z\| = \sup\limits_{n=1,2,\ldots} |\zeta_n(z)|$ for all $z \in Z$. On the other hand, $\dim_l Z = \dim_l C(\mathcal{J})$ is impossible (by the non-separability of Z), and $\dim_l Z = \dim_l m$ is also impossible (since the space conjugate to Z is of power $\mathfrak{c} = 2^{\aleph_0}$ and the space conjugate to m is of power $2^{\mathfrak{c}}$; see [11], p. 81-83).

It may be proved that the space $C(Q_M)$ is not equivalent to any subspace of a Banach space with an unconditional Schauder basis (7) of arbitrary power; however, for any separable subspace X_0 of $C(Q_M)$ there exists a subspace X_1 isomorphic to c (in particular, it follows that X_1 possesses an unconditional basis) and such that $X_0 \subset X_1 \subset C(Q_M)$. On the other hand, the space $l(\mathfrak{c})$ conjugate to $C(Q)$ possesses an absolute basis.

The linear dimensions of $C(Q_M)$ and of $c(\mathfrak{c})$ are uncomparable ($\dim_l C(Q_M) \geqslant \dim_l c(\mathfrak{c})$ is impossible, since $C(Q_M)$ is isomorphic to a strictly convex space, and $\dim_l C(Q_M) \leqslant \dim_l c(\mathfrak{c})$ is impossible, since $c(\mathfrak{c})$ possesses an unconditional basis), although their conjugate spaces are equivalent.

W. Bogdanowicz has proved that any polynomial functional on the space c is weakly continuous (see [6] and [17], p. 179). More generally, all spaces $C(\Omega)$ with dispersed Ω have this property (the proof of this fact is analogous to that of Bogdanowicz, but it is more complicated). It may be proved that if X is a Banach space, if X^* possesses an unconditional basis and if X^* has the property of Schur, i. e. if the weak (with respect to X^{**}) convergence is equivalent to the strong convergence, then any polynomial functional on X is weakly continuous.

(7) The definition of an unconditional basis can be found in [8], p. 73; the definition of an unconditional (absolute) basis of power $\aleph_\tau$ in [3].

4. Problems. 1° Let Ω be compact and let every polynomial functional on $C(\Omega)$ be weakly continuous. Is Ω dispersed?

2° Let X be a separable Banach space such that every infinitely dimensional subspace of X contains isomorphically the space c. Does there exist a countable ordinal a such that X is isomorphic to a subspace of C_a?

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A central limit theorem for stochastic processes with independent increments

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1. The central limit theorem for sequences $\{Y_k\}$ of random variables (vectors) states, roughly speaking, that whatever are the probability distributions of the particular random variables (vectors), provided that some assumptions are satisfied, the sequence of probability distributions of suitably normed sums ξ_n of Y_k $(k=1,\ldots,n)$ converges as $n\to\infty$ to the corresponding normal probability distribution. The central limit theorem has been generalized in [10] and [5] to random elements in Banach spaces. However, we often have to deal with stochastic processes whose realizations form — by the choice of a convenient distance — a metric non linear function-space. In this case the central limit theorem can be formulated in the following way: Consider a sequence of real stochastic processes $Y_k(t)$ with realizations belonging to some metric, complete and separable function - space $\mathfrak{A}$. Denote by $\xi_n(t)$ a suitably normed sum of $Y_k(t)$ $(k=1,\ldots,n)$ and by $\xi_0(t)$ a Gaussian stochastic process with realizations in $\mathfrak{A}$. Let P^{ξ_n} and P^{ξ_0} denote the probability measures in $\mathfrak{A}$ induced by the finite dimensional distributions ([8], § III,4) of $\xi_n(t)$ and $\xi_0(t)$ respectively. We shall say that the central limit theorem holds if,

$$P^{\xi_n}\Rightarrow P^{\xi_0}, \quad \text{as} \quad n\to\infty. \tag{1}$$

As we know, relation (1) means by definition that for any bounded and continuous function $f(x)$, where $x\in\mathfrak{A}$, the relation

$$\lim_{n\to\infty}\int_{\mathfrak{A}} f(x)\,dP^{\xi_n}=\int_{\mathfrak{A}} f(x)\,dP^{\xi_0}$$

holds.

If we limit ourselves to perfect measures ([6] chap. 1 § 3) then, as Prohorov ([11] Theorem 1.8) has shown, relation (1) holds if and only if for any real function $f(x)$, $x\in\mathfrak{A}$, continuous almost everywhere (P^{ξ_0}) in $\mathfrak{A}$ the sequence of probability distributions of $f[\xi_n(t)]$ converges as