f_x being the element of $C^\infty(a,b)$ defined by $y \to f(x,y)$. This is so because h_n is simply $\varphi^{(n)}(0)/n!$, where φ denotes the holomorphic vector-valued function $x \to f_x$; thus

$$h_n = (2\pi i)^{-1} \int_{|x|=r} \varphi(x)\, x^{-n-1} dx,$$

and (3.3) follows at once, just as for the Cauchy inequalities for a scalar-valued holomorphic function.

4. Other extensions. Theorems 1 and 2 admit extensions in which the function f takes its values in a separable Fréchet space F. $\mathcal{E}(D)$ would be replaced by the space $\mathcal{E}(D, F)$ of functions from $\overline{D}$ into F which are continuous on D and holomorphic on D; and $C^\infty(a,b)$ would be modified in like manner. $\mathcal{E}(D, F)$ will be a Fréchet space when equipped with the topology defined by the seminorms

$$\operatorname*{Sup}_{y \in D} p_n(g(y)),$$

where the p_n $(n = 1, 2, \ldots)$ are seminorms defining the topology of F. It is easily seen that $\mathcal{E}(D, F)$ will be separable whenever F has this property (cf. [4], p. 58, Proposition 5). (One might weaken continuity on $\overline{V}$ to weak continuity on $\overline{V}$, together with separability conditions on the function involved, but this would have little advantage from the point of view of applications.) Similar remarks apply to the space of vector-valued C^∞ functions.

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The two-norm spaces and their conjugate spaces

by

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In this paper we continue our investigations on the two-norm spaces, presented in the papers [2], [3], [5].

A two-norm space is a linear space X provided with two norms: $\|\ \|$ and a coarser (¹) one $\|\ \|^*$; these two norms lead to the following notion of limit: the sequence x_n is termed *γ-convergent* to x_0 (written $x_n \xrightarrow{\gamma} x_0$) if $\sup\limits_{n=1,2,\ldots} \|x_n\| < \infty$ and $\lim\limits_{n\to\infty} \|x_n - x_0\|^* = 0$. Thus, as regards the distributive functionals, three classes arise in a natural way: the spaces $\langle \Xi, \|\ \|\rangle$ and $\langle \Xi^*, \|\ \|^*\rangle$ conjugate to the normed spaces $\langle X, \|\ \|\rangle$ and $\langle X, \|\ \|^*\rangle$, respectively, and the space Ξ_γ of functionals sequentially continuous with respect to the convergence γ. Obviously $\Xi^* \subset \Xi_\gamma \subset \Xi$.

The triplet $\langle X, \|\ \|, \|\ \|^*\rangle$ is called the *two-norm space*. The space $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$ (²) seems to be the natural two-norm space conjugate to $\langle X, \|\ \|, \|\ \|^*\rangle$. We show that, analogously to the Banach space case, every two-norm space may be *canonically* embedded into its biconjugate two-norm space, with the preservation of both norms. The canonical mapping enables us to embed any two-norm space into a two-norm space sequentially complete with respect to the convergence γ; this process will be called the *γ-completion*.

The main purpose of this paper is the study of the interrelations of the two-norm spaces and of the concepts arising in connection with them. Some pages are devoted to the *γ-reflexive* spaces, i. e. such which are canonically embedded onto the biconjugate two-norm space; a characterization similar to the Banach space case is derived. We study also the *γ-compact* spaces, i. e. such that each γ-bounded sequence contains a subsequence which is γ-convergent (to an element); a detailed study is devoted

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(¹) The norm $\|\ \|^*$ is called *coarser* than $\|\ \|$ (or $\|\ \|$ is called *finer* than $\|\ \|^*$) if $\|x_n\| \to 0$ implies $\|x_n\|^* \to 0$.

(²) In the triplet-notation for a two-norm space the finer norm will always precede the coarser one; so in this case the norm $\|\ \|$ is coarser than $\|\ \|^*$.

to the spaces for which $\Xi_\gamma = \Xi$; these spaces are called *saturated* in the sequel.

1. Preliminaries. We recall first some notions and results to be found in [2], [3], and [5] and introduce some new ones, which are needed later.

We shall often say *the space* $\langle X, \|\ \|, \|\ \|^*\rangle$ or simply $\langle X, \|\ \|, \|\ \|^*\rangle$ instead of *the two-norm space* $\langle X, \|\ \|, \|\ \|^*\rangle$. By a *subspace* of a two-norm space $\langle X, \|\ \|, \|\ \|^*\rangle$ we shall always mean a linear subset of X provided with the norms $\|\ \|$ and $\|\ \|^*$ restricted to that subset.

For any two-norm space $\langle X, \|\ \|, \|\ \|^*\rangle$ we shall suppose in the sequel that the norm $\|\ \|^*$ is coarser than $\|\ \|$. Hence there exists a constant K such that $\|x\|^* \leqslant K\|x\|$; so we may, and in fact shall, suppose in this paper that

(n_0) $\quad \|x\|^* \leqslant \|x\|$ *for any* $x \in X$.

Any two-norm space $\langle X, \|\ \|, \|\ \|^*\rangle$ such that

(n) $\quad \lim\limits_{n\to\infty} \|x_n - x_0\|^* = 0$ *implies* $\|x_0\| \leqslant \varliminf\limits_{n\to\infty} \|x_n\|$

will be called *normal* (let us notice that this condition was automatically assumed throughout the paper [5]); this is the case if and only if the ball $S = \{x: x \in X, \|x\| \leqslant 1\}$ is closed in the space $\langle X, \|\ \|^*\rangle$. In the sequel we shall deal mostly with normal two-norm spaces.

A two-norm space is called *γ-complete* if it is sequentially complete for the convergence γ, i. e. if it satisfies the condition

(n_1) *If* $(x_{p_n} - x_{q_n}) \xrightarrow{\gamma} 0$ *as* $p_n \to \infty$ *and* $q_n \to \infty$, *then* $x_0 \in X$ *exists such that* $x_n \xrightarrow{\gamma} x_0$.

The conjunction of the conditions (n) and (n_1) is equivalent to the following condition of Orlicz ([14], p. 240):

The ball $S = \{x: x \in X, \|x\| \leqslant 1\}$ *is a complete metric space with the distance* $\varrho(x, y) = \|x - y\|^*$.

According to the definition of Banach, a sequence x_n will be called *γ-bounded* if $t_n x_n \xrightarrow{\gamma} 0$ for any sequence t_n of reals tending to zero. Evidently the sequence x_n is γ-bounded if and only if $\sup\limits_{n=1,2,\ldots} \|x_n\| < \infty$.

A functional ξ defined on X will be called *γ-linear* if it is distributive and if $x_n \xrightarrow{\gamma} 0$ implies $\xi(x_n) \to 0$; the set of those functionals will be denoted by Ξ_γ. We shall deal also with the spaces $\langle \Xi, \|\ \|\rangle$ and $\langle \Xi^*, \|\ \|^*\rangle$ conjugate to $\langle X, \|\ \|\rangle$ and $\langle X, \|\ \|^*\rangle$ respectively. Thus

$$\|\xi\| = \sup\{\xi(x): x \in X, \|x\| \leqslant 1\},$$
$$\|\xi\|^* = \sup\{\xi(x): x \in X, \|x\|^* \leqslant 1\},$$
$$\|\xi\| \leqslant \|\xi\|^*, \qquad \Xi^* \subset \Xi_\gamma \subset \Xi, \qquad \Xi^* = \{\xi: \xi \in \Xi, \|\xi\|^* < \infty\}.$$

We shall use the following notation without further reference:

$$S = \{x: x \in X, \|x\| \leqslant 1\},$$
$$S^* = \{x: x \in X, \|x\|^* \leqslant 1\},$$
$$\Sigma = \{\xi: \xi \in \Xi, \|\xi\| \leqslant 1\},$$
$$\Sigma^* = \{\xi: \xi \in \Xi^*, \|\xi\|^* \leqslant 1\};$$

obviously $S \subset S^*, \Sigma^* \subset \Sigma$.

In the paper [5] we adopted weaker hypotheses: the space $\langle X, \|\ \|^*\rangle$ was supposed to be of B_0-type only. One may, however (as shown in the proof of Theorem 2 of [5]), always introduce a (homogeneous) norm $\|\ \|_1^*$ in X, finer than $\|\ \|^*$, leading in $\langle X, \|\ \|, \|\ \|_1^*\rangle$ to the same γ-convergence as that in $\langle X, \|\ \|, \|\ \|^*\rangle$.

The following theorems are of principal importance for this paper:

Theorem A. *Let the space* $\langle X, \|\ \|, \|\ \|^*\rangle$ *be normal; then the set* Ξ_γ *is equal to the closure of* Ξ^* *in the space* $\langle \Xi, \|\ \|\rangle$.

Theorem B. *Let the space* $\langle X, \|\ \|, \|\ \|^*\rangle$ *be normal. Then the set* Ξ_γ *is strictly norming for* $\langle X, \|\ \|\rangle$ ([3]) *and*

$$\|x\| = \sup\{\xi(x): \xi \in \Xi_\gamma, \|\xi\| \leqslant 1\}$$

for each $x \in X$.

Theorem A was proved in [5] (p. 130). Theorem B results by a theorem of [4] (p. 109), since as shown in [5] (proposition 1.5) the set Ξ^* as well as the set Ξ_γ are norming and the set Ξ_γ is closed in $\langle \Xi, \|\ \|\rangle$ ([15], p. 57).

An operation U from $\langle X, \|\ \|, \|\ \|^*\rangle$ to $\langle Y, \|\ \|, \|\ \|^*\rangle$ will be called *γ-γ-linear* if it is distributive and if $x_n \xrightarrow{\gamma} 0$ implies $U(x_n) \xrightarrow{\gamma} 0$. Since γ-γ-linear operations transform bounded sequences into bounded ones, we get

1.1. Proposition. *Any γ-γ-linear operation is linear as an operation from* $\langle X, \|\ \|\rangle$ *to* $\langle Y, \|\ \|\rangle$.

Given a subset A of the two-norm space $\langle X, \|\ \|, \|\ \|^*\rangle$, we shall denote by $\gamma(A)$ the set of all the limits of γ-convergent sequences of ele-

([3]) A subset Γ of Ξ is called *strictly norming for* $\langle X, \|\ \|\rangle$ if every sequence x_n such that $\sup\limits_{n=1,2,\ldots} |\xi(x_n)| < \infty$ for every $\xi \in \Gamma$ is necessarily bounded.

A subset Γ of Ξ is called *norming for* $\langle X, \|\ \|\rangle$ if there exists an $r > 0$ such that the functional

$$\|x\|_0 = \sup\{|\xi(x)|: \xi \in \Gamma \cap r\Sigma\}$$

is a norm equivalent to the norm $\|\ \|$. Every strictly norming set is norming, the converse not being true in general.

ments of A. In general, $\gamma(A) \neq \gamma(\gamma(A))$ ([5], p. 133). A subset A of X will be termed *γ-closed* if $\gamma(A) = A$. Since the intersection of any family of γ-closed sets is γ-closed, there exists for any set A a smallest γ-closed set, $\bar{\gamma}(A)$, containing A. Let us write $\gamma_0(A) = A$, $\gamma_\alpha(A) = \gamma\big(\bigcup_{\beta<\alpha}\gamma_\beta(A)\big)$ for any ordinal $\alpha \geqslant 1$. We obviously have the

1.2. PROPOSITION. *The set $\bar{\gamma}(A)$ is identical with $\gamma_{\omega_1}(A)$ where ω_1 is the smallest uncountable ordinal.*

2. The conjugate two-norm spaces. Given a two-norm space $\langle X, \|\ \|, \|\ \|^*\rangle$, the space $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$ is obviously a two-norm space satisfying the condition (n_0); it will be called the *γ-conjugate space* to $\langle X, \|\ \|, \|\ \|^*\rangle$.

2.1. PROPOSITION. *The space $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$ is a normal, γ-complete two-norm space.*

Proof. Let $\|\xi_n\|^* \leqslant K$ and $\|\xi_n - \xi_m\| \to 0$ as $n, m \to \infty$. Then $\xi_n(x) - \xi_m(x) \to 0$ for every $x \in X$ whence $\xi_0(x) = \lim_{n\to\infty} \xi_n(x)$ is a linear functional and $\|\xi_0\|^* \leqslant K$.

Now let us denote by $\langle \mathfrak{X}, \|\ \|\rangle$ and $\langle \mathfrak{X}^*, \|\ \|^*\rangle$ the spaces conjugate to $\langle \Xi, \|\ \|\rangle$ and $\langle \Xi^*, \|\ \|^*\rangle$, respectively. Thus

$$\|\mathfrak{x}\| = \sup\{\mathfrak{x}(\xi): \xi \in \Xi \cap \Sigma\} \quad \text{for} \quad \mathfrak{x} \in \mathfrak{X},$$

$$\|\mathfrak{x}\|^* = \sup\{\mathfrak{x}(\xi): \xi \in \Xi^* \cap \Sigma^*\} \quad \text{for} \quad \mathfrak{x} \in \mathfrak{X}^*.$$

$\langle \mathfrak{X}, \|\ \|\rangle$ and $\langle \mathfrak{X}^*, \|\ \|^*\rangle$ are the second conjugate spaces to $\langle X, \|\ \|\rangle$ and $\langle X, \|\ \|^*\rangle$ respectively. Next, let us denote by $\langle \mathfrak{X}^{(\gamma)}, \|\ \|\rangle$ the space conjugate to $\langle \Xi^*, \|\ \|\rangle$; the norm is equal in this case to [4]

$$\|\mathfrak{x}\| = \sup\{\mathfrak{x}(\xi): \mathfrak{x} \in \Xi^* \cap \Sigma\}.$$

Thus $\langle \mathfrak{X}^{(\gamma)}, \|\ \|, \|\ \|^*\rangle$ is γ-conjugate to $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$, whence it is the second γ-conjugate to $\langle X, \|\ \|, \|\ \|^*\rangle$.

Finally, in accordance with the notation adopted, let us denote by $\mathfrak{X}_\gamma$ the set of the γ-linear functionals on $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$. The space $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$ being normal, the set $\mathfrak{X}_\gamma$ is identical with the closure of $\mathfrak{X}^{(\gamma)}$ in $\langle \mathfrak{X}^*, \|\ \|^*\rangle$, and $\mathfrak{X}^{(\gamma)} \subset \mathfrak{X}_\gamma \subset \mathfrak{X}^*$.

2.2. PROPOSITION. *Let the space $\langle X, \|\ \|, \|\ \|^*\rangle$ be normal. Then the space $\langle \mathfrak{X}^{(\gamma)}, \|\ \|\rangle$ may be identified with the space conjugate to $\langle \Xi_\gamma, \|\ \|\rangle$.*

[4] We denote the norm of linear functionals on $\langle \Xi, \|\ \|\rangle$ and on $\langle \Xi^*, \|\ \|\rangle$ by the same symbol $\|\ \|$; this will be useful and will not cause confusion. In particular, as follows by 2.4, the norm of the functional $\mathfrak{x}(\xi) = \xi(x)$ is the same in both cases.

Proof. The space $\langle \Xi_\gamma, \|\ \|\rangle$ is identical with the completion of the space $\langle \Xi^*, \|\ \|\rangle$ (by Theorem A), whence their conjugate spaces may be considered as identical, and

$$\|\mathfrak{x}\| = \sup\{\mathfrak{x}(\xi): \xi \in \Xi^* \cap \Sigma\} = \sup\{\mathfrak{x}(\xi): \xi \in \Xi_\gamma \cap \Sigma\}.$$

However, the weak topologies $\sigma(\mathfrak{X}^{(\gamma)}, \Xi^*)$ and $\sigma(\mathfrak{X}^{(\gamma)}, \Xi_\gamma)$ are different [5], and $\sigma(\mathfrak{X}^{(\gamma)}, \Xi_\gamma)$ is not coarser than the topology of the norm $\|\ \|^*$.

The first γ-conjugate space depends on the norm $\|\ \|^*$ essentially, that is, when two starred norms $\|\ \|_1^*$ and $\|\ \|_2^*$ give rise (together with $\|\ \|$) to the same γ-convergence, the spaces Ξ_1^* and Ξ_2^* (conjugate to $\langle X, \|\ \|_1^*\rangle$ and $\langle X, \|\ \|_2^*\rangle$ respectively) need not be identical. The second γ-conjugate space $\mathfrak{X}^{(\gamma)}$, however, depends only on the space $\langle \Xi_\gamma, \|\ \|\rangle$. More precisely:

2.3. PROPOSITION. *Let $\|\ \|_1^*$ and $\|\ \|_2^*$ be two coarser norms in a normed space $\langle X, \|\ \|\rangle$, satisfying* (n) *and leading to the same class Ξ_γ of γ-linear functionals. Then the spaces $\mathfrak{X}^{(\gamma)}$ are equal in both cases.*

This follows immediately by 2.2. Let us remark that two coarser norms $\|\ \|_1^*$ and $\|\ \|_2^*$ leading to the same class Ξ_γ may determine different γ-convergencies. On the other hand, the norm $\|\ \|^*$ in $\mathfrak{X}^{(\gamma)}$ determines $\|\ \|^*$ in X uniquely.

Given an element $x \in X$, the formula $\mathfrak{x}_x(\xi) = \xi(x)$ determines a functional which is linear on $\langle \Xi, \|\ \|\rangle$ and on $\langle \Xi^*, \|\ \|^*\rangle$. It is well known that the mapping $x \to \mathfrak{x}_x$, called *canonical*, embeds isometrically the space $\langle X, \|\ \|\rangle$ into $\langle \mathfrak{X}, \|\ \|\rangle$, and also embeds isometrically $\langle X, \|\ \|^*\rangle$ into $\langle \mathfrak{X}^*, \|\ \|^*\rangle$; the canonical mapping, since it defines linear functionals on $\langle \Xi^*, \|\ \|\rangle$, embeds the space $\langle X, \|\ \|, \|\ \|^*\rangle$ into $\langle \mathfrak{X}^{(\gamma)}, \|\ \|, \|\ \|^*\rangle$. The canonical mapping restricted as above will be called *γ-canonical*.

2.4. PROPOSITION. *Let the space $\langle X, \|\ \|, \|\ \|^*\rangle$ be normal. Then the γ-canonical mapping embeds $\langle X, \|\ \|, \|\ \|^*\rangle$ into $\langle \mathfrak{X}^{(\gamma)}, \|\ \|, \|\ \|^*\rangle$ with the preservation of both norms $\|\ \|$ and $\|\ \|^*$, i. e. $\|\mathfrak{x}_x\| = \|x\|$ and $\|\mathfrak{x}_x\|^* = \|x\|^*$ for $x \in X$, $\mathfrak{x}_x \in \mathfrak{X}^{(\gamma)}$. Conversely, the preservation of norms by the γ-canonical mapping implies condition* (n) *in $\langle X, \|\ \|, \|\ \|^*\rangle$.*

Proof. By Theorem B

$$\|\mathfrak{x}_x\| = \sup\{\xi(x): \xi \in \Xi^*, \|\xi\| \leqslant 1\} = \|x\|,$$

and by the definition of the norm $\|\ \|^*$ in $\mathfrak{X}^*$

$$\|\mathfrak{x}_x\|^* = \sup\{\xi(x): \xi \in \Xi^*, \|\xi\|^* \leqslant 1\} = \|x\|^*.$$

[5] Given a linear subset Ω of the algebraic dual of X, $\sigma(X, \Omega)$ denotes the weakest topology on X for which the functionals $\omega(x)$ are continuous for $\omega \in \Omega$; concerning the details see [7].

On the other hand, the space $\langle\mathfrak{X}^{(\gamma)}, \| \|, \| \|^*\rangle$ is normal (by 2.1) and any subspace of a normal two-norm space is also normal.

Now let $\langle \boldsymbol{X}^*, \| \|^*, \| \|\rangle$ be the γ-conjugate space to $\langle\mathfrak{X}^{(\gamma)}, \| \|, \| \|^*\rangle$; the canonical mapping of Ξ^* into $\boldsymbol{X}^*$ is given by

$$\zeta \to \mathfrak{z}_\zeta(\mathfrak{z}) = \mathfrak{z}(\zeta).$$

Let us denote by $\boldsymbol{X}_\gamma$ the space of the γ-linear functionals on $\langle\mathfrak{X}^{(\gamma)} \| \|, \| \|^*\rangle$. Since the γ-canonical mapping of $\langle \Xi^*, \| \|^*, \| \|\rangle$ embeds Ξ^* into the space $\langle \boldsymbol{X}^*, \| \|^*, \| \|\rangle$ γ-conjugate to $\langle\mathfrak{X}^{(\gamma)}, \| \|, \| \|^*\rangle$ with the preservation of both norms, and since the canonical mapping embeds $\langle \Xi, \| \|^*\rangle$ into the space $\langle \boldsymbol{X}, \| \|\rangle$ conjugate to $\langle\mathfrak{X}^{(\gamma)}, \| \|^*\rangle$ with the preservation of the norm $\| \|^*$, we infer by Theorem A

2.5. Proposition. *Let $\langle X, \| \|, \| \|^*\rangle$ be normal; then the canonical mapping of $\langle \Xi, \| \|^*\rangle$ into $\langle \boldsymbol{X}, \| \|^*\rangle$ embeds Ξ_γ into $\boldsymbol{X}_\gamma$.*

3. γ-reflexive two-norm spaces. A two-norm space $\langle X, \| \|, \| \|^*\rangle$ will be called *γ-reflexive* if it is normal and if the γ-canonical mapping embeds $\langle X, \| \|, \| \|^*\rangle$ onto $\langle\mathfrak{X}^{(\gamma)}, \| \|, \| \|^*\rangle$, or equivalently, if each linear functional on $\langle \Xi_\gamma, \| \|\rangle$ is of the form $\mathfrak{x}(x) = \xi(x)$ with $x \in X$. Each γ-reflexive space is obviously γ-complete.

Let $\langle X, \| \|, \| \|^*\rangle$ be normal. The space conjugate to X equipped with the topology $\sigma(X, \Xi_\gamma)$ is equal to Ξ_γ. Let us consider the strong topology $\beta(\Xi_\gamma, X)$ of Ξ_γ: the basis of neighbourhoods of zero is composed, for this topology, of all polar sets of bounded (for the topology $\sigma(X, \Xi_\gamma)$) subsets of X. Let $A \subset X$ be bounded for $\sigma(X, \Xi_\gamma)$; then

$$\sup\{|\xi(x)| : x \in A\} < \infty$$

for every $\xi \in \Xi_\gamma$, whence $A \subset nS$ for some n, since by Theorem B the set Ξ_γ is strictly norming. Conversely, each ball nS is bounded for the topology $\sigma(X, \Xi_\gamma)$. Indeed, let V be a basic neighbourhood of zero for $\sigma(X, \Xi_\gamma)$, i. e. let

$$V = \bigcap_{i=1}^{n} \{x : |\xi_i(x)| \leqslant 1\}$$

with $\xi_i \in \Xi_\gamma$; $\Xi_\gamma \subset \Xi$ implies $|\xi_i(x)| \leqslant M_i\|x\|$, whence $nS \subset n(M_1 + \ldots + M_n)V$. Since the polar set of nS is equal to $n^{-1}\Sigma$, we obtain

3.1. Lemma. *Let the space $\langle X, \| \|, \| \|^*\rangle$ be normal, then the strong topology $\beta(\Xi_\gamma, X)$ of the space Ξ_γ, since it is conjugate to X provided with the topology $\sigma(X, \Xi_\gamma)$, is identical with the topology of the norm $\| \|$.*

3.2. Theorem. *The space $\langle X, \| \|, \| \|^*\rangle$ is γ-reflexive if and only if the ball S is compact for the weak topology $\sigma(X, \Xi_\gamma)$.*

Proof. By Lemma 3.1 and Proposition 2.2 the space conjugate to $\langle \Xi_\gamma, \beta(\Xi_\gamma, X)\rangle$ is equal to $\mathfrak{X}^{(\gamma)}$, whence our theorem follows from a theorem of Koethe-Dieudonné-Schwartz ([8], p. 79).

3.3. Remark. *In Theorem 3.2 the topology $\sigma(X, \Xi_\gamma)$ may be replaced by $\sigma(X, \Xi^*)$.*

Proof. The proof follows from a theorem of Dixmier ([9], p. 1059) and Theorem A.

3.4. Proposition. *Any γ-closed subspace of a γ-reflexive space is γ-reflexive.*

Proof. Let X_0 be a γ-closed subspace of a γ-reflexive space $\langle X, \| \|, \| \|^*\rangle$. Thus the ball S is compact for the topology $\sigma(X, \Xi^*)$. The unit ball of $\langle X_0, \| \|\rangle$ is equal to $X_0 \cap S$, and since it is closed and convex, it is (by Mazur's theorem, [11], p. 80) closed for the topology $\sigma(X, \Xi^*)$, whence, as a subset of S, it is compact for $\sigma(X, \Xi^*)$. Let Ξ_0^* be the space conjugate to $\langle X_0, \| \|^*\rangle$; by the Hahn-Banach theorem the topology $\sigma(X_0, \Xi_0^*)$ is identical with the topology induced by $\sigma(X, \Xi^*)$ on X_0. Thus $X_0 \cap S$ is compact for the topology $\sigma(X_0, \Xi_0^*)$ and our proposition follows by 3.2 and 3.3.

3.5. Proposition. *A space which is γ-conjugate to a γ-reflexive two-norm space is γ-reflexive.*

Proof. Let $\langle X, \| \|, \| \|^*\rangle$ be γ-reflexive. Then the topologies $\sigma(\Xi^*, X)$ and $\sigma(\Xi^*, \mathfrak{X}^{(\gamma)})$ are equivalent. The ball Σ^* is compact for the topology $\sigma(\Xi^*, X)$ by the Alaoglu-Bourbaki theorem ([1], p. 255) and the desired conclusion follows by the application of 3.2 and 3.3.

3.6. Proposition. *Let the space $\langle X, \| \|, \| \|^*\rangle$ be normal and γ-complete, and let $\langle \Xi^*, \| \|^*, \| \|\rangle$ be γ-reflexive. Then $\langle X, \| \|, \| \|^*\rangle$ is also γ-reflexive.*

Proof. The space $\langle\mathfrak{X}^{(\gamma)}, \| \|, \| \|^*\rangle$ is γ-reflexive by Proposition 3.5 and $\langle X, \| \|, \| \|^*\rangle$ may be identified with a subspace of the γ-biconjugate space, that subspace being γ-complete by hypothesis. Thus Proposition 3.6 follows from Proposition 3.4.

3.7. Theorem. *Let $\langle X, \| \|, \| \|^*\rangle$ be a two-norm space, then the following conditions are equivalent:*

1° *$\langle X, \| \|, \| \|^*\rangle$ is γ-reflexive and $\Xi_\gamma = \Xi$,*

2° *$\langle X, \| \|\rangle$ is reflexive.*

In particular, *the reflexivity of $\langle X, \| \|\rangle$ implies conditions* (n) *and* (n_1) *for any norm $\| \|^*$ satisfying* (n_0).

Proof. At first, we shall prove that the reflexivity of $\langle X, \| \|\rangle$ implies $\Xi_\gamma = \Xi$. The set Ξ_γ is closed in $\langle \Xi, \| \|\rangle$ (see [15], p. 57) and it is total with respect to X (since Ξ^* is total). By the Hahn-Banach theorem and by the definition of reflexivity any closed total subset of the space Ξ conjugate to a reflexive Banach space must be equal to Ξ (see Dixmier [9], p. 1061).

Thus, $\Xi_\gamma = \Xi$ being proved, γ-convergence in $\langle X, \|\ \|, \|\ \|^*\rangle$ implies weak convergence, whence $\langle X, \|\ \|, \|\ \|^*\rangle$ must be normal. Let $\mathfrak{x}$ be a functional linear on $\langle \Xi_\gamma, \|\ \|\rangle$; by the reflexivity of $\langle X, \|\ \|\rangle$ and by $\Xi_\gamma = \Xi$, $\mathfrak{x}$ is of the form $\mathfrak{x}(\xi) = \xi(x)$, with an $x \in X$, which means that $\langle X, \|\ \|, \|\ \|^*\rangle$ is γ-reflexive.

Now, let us assume that $\langle X, \|\ \|, \|\ \|^*\rangle$ is γ-reflexive and that $\Xi_\gamma = \Xi$. Then the spaces conjugate to $\langle \Xi, \|\ \|\rangle$ and to $\langle \Xi_\gamma, \|\ \|\rangle$ are identical, whence $\langle X, \|\ \|\rangle$ must be reflexive.

4. Completion of two-norm spaces. Since the γ-complete spaces reveal important properties, the question arises naturally whether it is possible to embed a two-norm space into a γ-complete space, with the preservation of both norms. We may require also that every γ-linear functional be uniquely extensible to the new space. We shall show that it can always be done, moreover, there is only one such extension (within isomorphisms). The usual Cantor method of completion is not appropriate in our case, for $\gamma(\gamma(A)) \neq \gamma(A)$ in general.

4.1. THEOREM. *Let $\langle X, \|\ \|, \|\ \|^*\rangle$ be a normal two-norm space. There exists a normal, γ-complete two-norm space $\langle X^c, \|\ \|, \|\ \|^*\rangle$ containing $\langle X, \|\ \|, \|\ \|^*\rangle$ as a subspace. Every γ-linear functional ξ on X may be uniquely extended onto X^c with the preservation of the norm $\|\ \|$.*

Proof. Let $\mathfrak{X}_0$ be the γ-canonical image of X in $\mathfrak{X}^{(\gamma)}$. Since (by 2.1) the space $\mathfrak{X}^{(\gamma)}$ is γ-complete, the smallest γ-closed hull $\bar{\gamma}(\mathfrak{X}_0)$ spanned on $\mathfrak{X}_0$ (see 1.2) is also γ-complete. The set X identified with $\mathfrak{X}_0$ is a linear subspace of $\mathfrak{X}^{(\gamma)}$. Thus, the γ-canonical mapping being isometrical (with respect to $\|\ \|$ and $\|\ \|^*$ simultaneously), $\langle\bar{\gamma}(\mathfrak{X}_0), \|\ \|, \|\ \|^*\rangle$ is a normal γ-complete space containing a subspace equivalent to $\langle X, \|\ \|, \|\ \|^*\rangle$.

Let ξ_0 ba any γ-linear functional on X, i. e. let $\xi_0 \in \Xi_\gamma$. Since $\mathfrak{X}^{(\gamma)}$ may be considered as the space conjugate to $\langle \Xi_\gamma, \|\ \|\rangle$ (see 2.2), the mapping

$$\bar{\xi}_0(\mathfrak{x}) = \mathfrak{x}(\xi_0) \qquad (\mathfrak{x} \in \bar{\gamma}(\mathfrak{X}_0))$$

extends in a natural way the functional ξ_0 onto $\bar{\gamma}(\mathfrak{X}_0)$. We shall prove that $\bar{\xi}_0$ is γ-linear on $\langle\bar{\gamma}(\mathfrak{X}_0), \|\ \|, \|\ \|^*\rangle$. Let $\mathfrak{x}_n \xrightarrow{\gamma} 0$. Then $\sup\{\mathfrak{x}_n(\xi): \xi \in \Xi_\gamma, \|\xi\| \leqslant 1, n = 1, 2, \ldots\} < \infty$ and $\sup\{\mathfrak{x}_n(\zeta): \zeta \in \Xi^*, \|\zeta\|^* \leqslant 1\} \to 0$ as $n \to \infty$, whence $\bar{\xi}(\mathfrak{x}_n) = \mathfrak{x}_n(\xi) \to 0$ for every $\xi \in \Xi^*$. Thus, the sequence $\mathfrak{x}_n$ is convergent to 0 in a dense subset of Ξ_γ and $\sup\|\mathfrak{x}_n\| < \infty$, whence $\mathfrak{x}_n(\xi) \to 0$ for all $\xi \in \Xi_\gamma$; in particular, $\bar{\xi}_0(\mathfrak{x}_n) = \mathfrak{x}_n(\xi_0) \to 0$.

The uniqueness of the extension $\xi \to \bar{\xi}$ follows by 1.2. Finally, $\|\bar{\xi}\| = \sup\{\mathfrak{x}(\xi): \mathfrak{x} \in \mathfrak{X}^{(\gamma)}, \|\mathfrak{x}\| \leqslant 1\} = \|\xi\|$.

The set X is not strictly contained in $\bar{\gamma}(\mathfrak{X}_0)$; however, we can define $X^c = X \cup [\bar{\gamma}(\mathfrak{X}_0) \setminus \mathfrak{X}_0]$. Then X^c, provided with linear operations and norms induced by $\bar{\gamma}(\mathfrak{X}_0)$, is the desired completion of $\langle X, \|\ \|, \|\ \|^*\rangle$.

$\langle X^c, \|\ \|, \|\ \|^*\rangle$ will be called the *γ-completion* of $\langle X, \|\ \|, \|\ \|^*\rangle$; it is easily seen that X^c is the smallest γ-closed set containing X. This condition, however, does not determine X^c uniquely, as can be seen from the following considerations.

Let us denote succesively: by l^1 — the space of all real sequences $x = \{x_n\}$ such that $\sum_{n=1}^{\infty}|x_n| < \infty$, with the norms

$$\|x\| = \sum_{n=1}^{\infty} |x_n|, \qquad \|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n|;$$

by $\mathfrak{M}$ — the set of Mazurkiewicz in l^1 (see [13] or [5], p. 133);

by x_0 — the element $\sum_{n=1}^{\infty} e_{2n-1}/2^n$ in l^1 (where e_n denotes the n-th unit vector in l^1); X — the set of all elements of form $y + tx_0$ with $y \in \mathfrak{M}$;

by λ — the functional defined on X by $\lambda(y+tx_0) = t$; $\langle X_1, \|\ \|, \|\ \|^*\rangle$ — the γ-completion of $\langle X, \|\ \|, \|\ \|^*\rangle$; $X_2 = \bar{\gamma}(X)$ in $\langle l^1, \|\ \|, \|\ \|^*\rangle$.

The functional λ is γ-linear on $\langle X, \|\ \|, \|\ \|^*\rangle$; however, it cannot be extended to the whole of X_2 with the preservation of γ-linearity (see [5], p. 133). Thus, by Theorem 4.1, the completions $X_1 = X^c$ and X_2 are essentially different; more precisely, we have proved that there exist normal two-norm spaces $\langle X, \|\ \|, \|\ \|^*\rangle$, $\langle X_1, \|\ \|, \|\ \|^*\rangle$ and $\langle X_2, \|\ \|, \|\ \|^*\rangle$ such that 1° $X \subset X_1$, $X \subset X_2$, 2° the norms $\|\ \|$ and $\|\ \|^*$ respectively are identical on X, $X \cap X_1$ and $X \cap X_2$, 3° the spaces $\langle X_1, \|\ \|, \|\ \|^*\rangle$ and $\langle X_2, \|\ \|, \|\ \|^*\rangle$ are γ-complete, 4° if X is considered as a subset of X_1, then $\bar{\gamma}(X) = X_1$, 5° if X is considered as a subset of X_2, then $\bar{\gamma}(X) = X_2$, 6° there exists no γ-γ-linear one-to-one mapping of $\langle X_1, \|\ \|, \|\ \|^*\rangle$ onto $\langle X_2, \|\ \|, \|\ \|^*\rangle$ equal on X to the identical mapping.

On the other hand, the γ-completion $\langle X^c, \|\ \|, \|\ \|^*\rangle$ is defined uniquely (within isomorphisms) by requiring that it be γ-complete and that the γ-linear functionals be extensible only in one manner on X^c, with preservation of the norm $\|\ \|$.

4.2. PROPOSITION. *There exists a natural isomorphical embedding of the set X^c into the completion X^* of the space $\langle X, \|\ \|^*\rangle$. Hence X^c may be identified with a part of X^* and every functional linear on $\langle X, \|\ \|^*\rangle$ may be extended uniquely to X^c with the preservation of norm $\|\ \|^*$.*

Proof. Let us retain the notation of the proof of Theorem 4.1. The set $\mathfrak{X}_0$ equal to the γ-canonical map of X may be considered as a subset of $\mathfrak{X}^{(\gamma)}$ or of $\mathfrak{X}^*$, and

$$\mathfrak{X}_0 \subset \bar{\gamma}(\mathfrak{X}_0) \subset \mathfrak{X}^{(\gamma)} \subset \mathfrak{X}^*.$$

The canonical map of X^* in $\langle \mathfrak{X}^*, \| \|^* \rangle$ may be obtained as the closure of $\mathfrak{X}_0$ in $\langle \mathfrak{X}^*, \| \|^* \rangle$ and, obviously, this closure contains the set $\bar{\gamma}(\mathfrak{X}_0)$.

As an example let us consider the space $\langle \boldsymbol{L}^\infty, \| \|, \| \|^* \rangle$ of essentially bounded measurable functions defined on $[0, 1]$, with the norms

$$\|x\| = \operatorname*{ess\,sup}_{0 \leqslant t \leqslant 1} |x(t)|, \qquad \|x\|^* = \int_0^1 |x(t)|\,dt.$$

This two-norm space is equivalent to the γ-completion of the space $\langle \boldsymbol{C}, \| \|, \| \|^* \rangle$ of continuous functions, with norms defined by the above formulas.

Let $\langle X, \| \|, \| \|^* \rangle$ be a normal two-norm space. In [5] (p. 123) we dealt with the completion of the space $\langle X, \| \| \rangle$. For that completion, $\tilde{X}$, the norms $\| \|$ and $\| \|^*$ may be extended from $\langle X, \| \|, \| \|^* \rangle$ by passing to the limit, and the space $\langle \tilde{X}, \| \|, \| \|^* \rangle$ is also normal. This completion is not necessarily equal to the γ-completion, since the completeness of $\langle X, \| \| \rangle$ does not imply γ-completeness. On the contrary, the γ-completeness of $\langle X, \| \|, \| \|^* \rangle$ together with normality implies the completeness of $\langle X, \| \| \rangle$ (see [14], p. 1, [5], p. 122), whence $\tilde{X} \subset X^c$.

4.3. Proposition. *Let $\langle X, \| \|, \| \|^* \rangle$ be normal and let $\langle \tilde{X}, \| \| \rangle$ denote the completion of $\langle X, \| \| \rangle$. Then the γ-completion of $\langle X, \| \|, \| \|^* \rangle$ coincides with $\langle \tilde{X}, \| \|, \| \|^* \rangle$ if and only if every linear functional on $\langle X, \| \| \rangle$ has a unique extension to a functional linear on $\langle X^c, \| \| \rangle$.*

Proof. Necessity is trivial. Sufficiency follows by the Hahn-Banach theorem.

5. γ-separability, γ-compactness. A subset A of a two-norm space $\langle X, \| \|, \| \|^* \rangle$ will be termed *γ-dense* if $\gamma(A) = X$; $\langle X, \| \|, \| \|^* \rangle$ will be termed *γ-separable* if X contains a countable γ-dense subset.

5.1. Proposition. *The following conditions are equivalent:*

(a) *$\langle X, \| \|, \| \|^* \rangle$ is γ-separable,*

(b) *there exists a countable subset A of X such that $\bar{\gamma}(A) = X$,*

(c) *$\langle X, \| \|^* \rangle$ is separable.*

Proof. If $\gamma(A) = X$, then $\bar{\gamma}(A) = X$, too; if $\gamma(A) = X$, then, by 1.2, A is dense in $\langle X, \| \|^* \rangle$. If $\langle X, \| \|^* \rangle$ is separable, then $\langle S, \| \|^* \rangle$ is also separable, whence S and in turn X is γ-separable.

A two-norm space $\langle X, \| \|, \| \|^* \rangle$ will be termed *γ-compact* if every γ-bounded sequence contains a γ-convergent subsequence; γ-compactness is equivalent for normal spaces $\langle X, \| \|, \| \|^* \rangle$ to the compactness of the ball S with respect to $\| \|^*$. $\langle X, \| \|, \| \|^* \rangle$ will be called *γ-precompact* if every γ-bounded sequence contains a Cauchy subsequence for the convergence γ; this is equivalent to the precompactness of $\langle S, \| \|^* \rangle$. As stated in 5.6 a normal space is γ-precompact if and only if its γ-completion is γ-compact; this justifies the term "γ-precompact".

Any subspace of a γ-separable space is γ-separable; any subspace of a γ-precompact space is γ-precompact. A space is γ-compact if and only if it is γ-precompact and γ-complete. The separability of $\langle X, \| \| \rangle$ implies the γ-separability of $\langle X, \| \|, \| \|^* \rangle$; the γ-separability of $\langle X, \| \|, \| \|^* \rangle$ implies the γ-separability of its γ-completion. Any γ-precompact space is γ-separable.

Typical examples of γ-compact two-norm spaces are the following:

A. The space of all bounded complex-valued functions $x(z)$ analytic on the circle $|z| < 1$, with the norms

$$\|x\| = \sup\{|x(z)| : |z| < 1\}, \qquad \|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup\left\{|x(z)| : |z| = 1 - \frac{1}{n}\right\}.$$

B. The space X conjugate to a separable Banach space Z, with the norms

$$\|x\| = \sup\{x(z) : \|z\| \leqslant 1\}, \qquad \|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} |x(z_n)|,$$

where $z_1, z_2, \ldots$ is a fixed sequence dense in the unit ball in Z; γ-convergence in $\langle X, \| \|, \| \|^* \rangle$ is identical with weak convergence with respect to topology $\sigma(X, Z)$.

C. The space of real functions satisfying the condition of Lipschitz in $\langle 0, 1 \rangle$, provided with the norms

$$\|x\| = |x(0)| + \sup\left\{\frac{|x(t+h) - x(t)|}{h} : 0 \leqslant t \leqslant 1, 0 \leqslant h \leqslant 1 - t\right\},$$

$$\|x\|^* = \sup_{0 \leqslant t \leqslant 1} |x(t)|.$$

D. Let $\langle X, \| \| \rangle$ be a Banach space and let T be a one-to-one linear completely continuous operation from $\langle X, \| \| \rangle$ into another Banach space $\langle Y, \| \| \rangle$, and let $\|x\|^* = \|Tx\|$. Then $\langle X, \| \|, \| \|^* \rangle$ is γ-compact.

E. Let $\langle X, \| \| \rangle$ be a separable Banach space with a Schauder basis $e_1, e_2, \ldots$ and let

$$\|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} |\xi_n(x)|$$

where $\xi_1, \xi_2, \ldots$ denote the functionals bi-orthogonal to $e_1, e_2, \ldots$, i. e. such that $\xi_m\left(\sum_{n=1}^{\infty} t_n e_n\right) = t_m$. Then $\langle X, \|\ \|, \|\ \|^*\rangle$ is γ-precompact.

F. Let $\langle X, \|\ \|\rangle$ be a Banach space, $\langle \Xi, \|\ \|\rangle$ its conjugate space, let a subset A of Ξ be a compact basis in the sense of W. Orlicz and V. Pták ([15], p. 63), i. e. a total (6) and (strongly) compact subset of $\langle \Xi, \|\ \|\rangle$, and let

$$\|x\|^* = \sup\{|\xi(x)| : \xi \in A\}.$$

Then $\langle X, \|\ \|, \|\ \|^*\rangle$ is γ-precompact ([15], p. 64).

5.2. Theorem. *Any normal γ-compact two-norm space is γ-reflexive.*

Proof. This follows by Theorem 3.2, since the topology of the norm $\|\ \|^*$ is finer on S than the topology $\sigma(X, \Xi_\gamma)$.

The converse statement is false, as shown by the example $\langle \boldsymbol{L}^2, \|\ \|_2, \|\ \|_1\rangle$, where $\|x\|_p = \left(\int_0^1 |x(t)|^p dt\right)^{1/p}$ for $p = 1, 2$.

The next theorem is an analogue of a theorem of Banach ([6], p. 189), and the proof is quite similar to that of Banach.

5.3. Theorem. *The γ-separability of the space $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$ implies the γ-separability of $\langle X, \|\ \|, \|\ \|^*\rangle$.*

Proof. Let $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$ be γ-separable. Then, by 5.1, Ξ^* is separable with respect to $\|\ \|$. Let $\zeta_1, \zeta_2, \ldots$ be a sequence dense in $\langle \Omega, \|\ \|\rangle$, where $\Omega = \{\xi : \xi \in \Xi^*, \|\xi\| = 1\}$, and let $x_1, x_2, \ldots$ be a sequence of elements of X such that $\|x_n\| = 1$, and $\zeta_n(x_n) > \frac{1}{2}$ for $n = 1, 2, \ldots$

We shall prove that the smallest linear set Y spanned on the elements x_n is dense in $\langle X, \|\ \|^*\rangle$. Indeed, assuming the contrary, there would exist a functional $\xi \in \Xi^*$ such that $\|\xi\| = 1$ and $\xi(x) = 0$ for all $x \in Y$. Then

$$\|\zeta_n - \xi\| \geqslant |\zeta_n(x_n) - \xi(x_n)| = |\zeta_n(x_n)| > \tfrac{1}{2} \quad \text{for} \quad n = 1, 2, \ldots,$$

which is impossible. Thus $\langle X, \|\ \|^*\rangle$ is separable, whence $\langle X, \|\ \|, \|\ \|^*\rangle$ is γ-separable.

5.4. Proposition. *Let $\langle X, \|\ \|, \|\ \|^*\rangle$ be normal. Then the space $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$ is γ-separable if and only if the space $\langle \Xi_\gamma, \|\ \|\rangle$ is separable.*

This follows by Proposition 5.1 and by Theorem A.

5.5. Theorem. *The space $\langle X, \|\ \|, \|\ \|^*\rangle$ is γ-precompact if and only if the space $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$ is γ-compact.*

Proof. Let us assume that $\langle X, \|\ \|, \|\ \|^*\rangle$ is not γ-precompact. Then there exist $\varepsilon > 0$ and a sequence x_n such that $\frac{1}{2} \leqslant \|x_n\| \leqslant 1$ and $\|x_n - x_m\|^* \geqslant 3\varepsilon$ for $n \neq m$. We shall define a subsequence $y_k = x_{n_k}$ by induction.

(6) A set $\Omega \subset \Xi$ is called *total* if $\sup\{|\xi(x)| : \xi \in \Gamma\} > 0$ for every $x \neq 0$.

Let $y_1 = x_1$ and let us assume $y_1, \ldots, y_k$ to be already defined. The set Y_k of the linear combinations of $y_1, \ldots, y_k$ is finitely dimensional, whence $\langle Y_k \cap S^*, \|\ \|^*\rangle$ is compact. For every k the set

$$E_k = \{n : d^*(x_n, Y_k) < \varepsilon\} \quad \text{where} \quad d^*(A, B) = \inf\{\|x-y\|^* : x \in A, y \in B\}$$

is finite. Indeed, for every $n \in E_k$ there exists $z_n \in Y_k$ such that $\|z_n - x_n\|^* < \varepsilon$. Then for $n \in E_k$ and $m \in E_k$

$$\|z_n\|^* \leqslant \|z_n - x_n\|^* + \|x_n\|^* < \varepsilon + \|x_n\| \leqslant 1 + \varepsilon,$$

which means that $z_n \in Y_k \cap (1+\varepsilon) S^*$, and

$$\|z_n - z_m\|^* \geqslant \|x_n - x_m\|^* - (\|x_n - z_n\|^* + \|x_m - z_m\|^*) > \varepsilon.$$

Since $Y \cap (1+\varepsilon) S^*$ is compact (with respect to $\|\ \|^*$), E_k is finite.

Let s_k be the least index not belonging to E_k, and let $y_{k+1} = x_{s_k}$. Thus, $d^*(y_{k+1}, Y_y) \geqslant \varepsilon$ for every k, whence

$$\sup\{a : \|a_1 y_1 + \ldots + a_k y_k + a y_{k+1}\|^* \leqslant 1\} = \frac{1}{\varepsilon} \sup\{a : d^*(Y_k, a y_k) \leqslant \varepsilon\} \leqslant \frac{1}{\varepsilon},$$

which means that the functional

$$\eta_{k+2}(a_1 y_1 + \ldots + a_{k+1} y_{k+1}) = a_{k+1}$$

considered on $\langle Y_{k+1}, \|\ \|^*\rangle$ has the norm $\leqslant 1/\varepsilon$. Let ζ_k be the Hahn-Banach extension of $\varepsilon\eta_k$ on $\langle X, \|\ \|^*\rangle$. Then $\|\zeta_k\|^* \leqslant 1$, $\zeta_k(y_k) = \varepsilon$ and $\zeta_{k+1}(x) = 0$ for $x \in Y_k$, whence

$$\|\zeta_k - \zeta_m\| \geqslant \frac{1}{\|y_k\|} |\zeta_k(y_k) - \zeta_m(y_k)| \geqslant \varepsilon \quad \text{for} \quad m > k.$$

Thus, $\langle \Xi^*, \|\ \|\rangle$ is not precompact.

Now let us assume that $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$ is not γ-compact. Then $\langle \Sigma^*, \|\ \|\rangle$ is not precompact, since $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$ is normal and γ-complete. Arguing as above we can choose a sequence $\zeta_1, \zeta_2, \ldots$ of functionals so that $\|\zeta_n\|^* \leqslant 1$ and

$$\left\| \zeta_n - \sum_{k=1}^{n-1} a_k \zeta_k \right\| > \varepsilon > 0$$

for arbitrary $a_1, a_2, \ldots, a_n$ and $n = 2, 3, \ldots$ By a theorem of Banach ([6], p. 119) there exists, for every fixed $n \geqslant 2$, an element $x_n \in X$ such that $\zeta_n(x_n) = \varepsilon$, $\|x_n\| = 1$ and $\zeta_m(x_n) = 0$ for $m = 1, 2, \ldots, n-1$. This implies

$$\|x_m - x_n\|^* \geqslant \zeta_m(x_m - x_n) = \zeta_m(x_m) = \varepsilon$$

for $m < n$, which means that $\langle S, \|\ \|^*\rangle$ is not precompact.

5.6. Proposition. *Let $\langle X, \| \|, \| \|^* \rangle$ be normal. Then $\langle X, \| \|, \| \|^* \rangle$ is γ-precompact if and only if its γ-completion $\langle X^c, \| \|, \| \|^* \rangle$ is γ-compact.*

Proof. This follows by Theorem 5.5, since the spaces $\langle X, \| \|, \| \|^* \rangle$ and $\langle X^c, \| \|, \| \|^* \rangle$ have the same γ-conjugate space.

5.7. Proposition. *Let $\langle X, \| \|^* \rangle$ be prereflexive* (7) *and separable. Then all γ-conjugate spaces to $\langle X, \| \|, \| \|^* \rangle$ are γ-separable.*

Proof. The space $\langle X, \| \|^* \rangle$ being prereflexive, $\langle \Xi^*, \| \|^* \rangle$ and $\langle \mathfrak{X}^*, \| \|^* \rangle$ are reflexive. Thus the separability of $\langle X, \| \|^* \rangle$ implies the same for $\langle \Xi^*, \| \|^* \rangle$. By the reflexivity of $\langle \Xi^*, \| \|^* \rangle$, by 2.1 and by Theorem 3.7, the space $\langle \Xi^*, \| \|^*, \| \| \rangle$ is γ-reflexive, and, by Proposition 3.5, $\langle \mathfrak{X}^{(\nu)}, \| \|, \| \|^* \rangle$ and its γ-conjugate spaces are γ-reflexive. Thus, $\langle \Xi^*, \| \|^*, \| \| \rangle$ being γ-separable, all odd γ-conjugate spaces are γ-separable, whence, by Theorem 5.3, the even γ-conjugate spaces are also γ-separable.

One-to-one γ-γ-linear operations from a γ-complete two-norm space onto another have not, in general, the Banach inversion property (a trivial example: identical operation considered as an operation from $\langle X, \| \|, \| \| \rangle$ onto $\langle X, \| \|, \| \|^* \rangle$). However, the inverse operation is γ-γ-continuous if the first space $\langle X, \| \|, \| \|^* \rangle$ is γ-compact. More precisely:

5.8. Theorem. *Let the space $\langle X, \| \|, \| \|^* \rangle$ be γ-compact and let $\langle X, \| \| \rangle$ and $\langle Y, \| \| \rangle$ be complete; let U be a γ-γ-linear one-to-one mapping of $\langle X, \| \|, \| \|^* \rangle$ onto $\langle Y, \| \|, \| \|^* \rangle$. Then U is an isomorphism between $\langle X, \| \| \rangle$ and $\langle Y, \| \| \rangle$, and the inverse mapping U^{-1} is also γ-γ-linear.*

Proof. By Proposition 1.1 and by the Banach inversion theorem ([6], p. 41), $\|x_n - x_0\| \to 0$ is equivalent to $\|U(x_n) - U(x_0)\| \to 0$.

Now, let $y_n = U(x_n) \xrightarrow{\gamma} 0$ and let x_n' be any subsequence of x_n. Then $\sup_{n=1,2,\ldots} \|y_n\| < \infty$, whence $\sup_{n=1,2,\ldots} \|x_n'\| < \infty$. By the γ-compactness of the space $\langle X, \| \|, \| \|^* \rangle$, there exists a subsequence x_n'' of x_n' and an element x_0 such that $x_n'' \xrightarrow{\gamma} x_0$. This implies $U(x_n'') \to U(x_0)$, $U(x_0) = 0$. Since U is one-to-one, $U(x_0) = 0$ gives $x_0 = 0$; x_n' being an arbitrary subsequence of x_n, we infer that $x_n \xrightarrow{\gamma} 0$.

5.9. Proposition. *Let $\langle X, \| \|, \| \|^* \rangle$ be γ-compact, let U be a γ-γ-linear operation from $\langle X, \| \|, \| \|^* \rangle$ onto $\langle Y, \| \|, \| \|^* \rangle$, and let $\langle X, \| \| \rangle$ and $\langle Y, \| \| \rangle$ be complete. Then $\langle Y, \| \|, \| \|^* \rangle$ is γ-compact.*

Proof. Let y_n be a γ-bounded sequence in Y. Then $\sup_{n=1,2,\ldots} \|y_n\| = M < \infty$. By a theorem of Banach ([6], p. 40) there exists a number k such that $U(kS) \supset \{y : y \in Y, \|y\| \leqslant M\}$. Thus, there exist elements x_n of X such that $\|x_n\| \leqslant k$ and $U(x_n) = y_n$ for $n = 1, 2, \ldots$ Since $\langle X, \| \|, \| \|^* \rangle$ is γ-compact, a subsequence x_{n_k} of x_n is γ-convergent to an element $x_0 \in X$. Then $y_{n_k} = U(x_{n_k}) \xrightarrow{\gamma} U(x_0)$.

Let us remark that the assumption of the completeness of $\langle Y, \| \| \rangle$ is indispensable in 5.8 as well as in 5.9. Example: a γ-compact space $\langle X, \| \|, \| \|^* \rangle$ and the identical operation from $\langle X, \| \|, \| \|^* \rangle$ onto $\langle X, \| \|^*, \| \|^* \rangle$; if X is infinitely dimensional, $\langle X, \| \|^*, \| \|^* \rangle$ is not γ-precompact ([6], p. 84).

(7) A linear normed space is called *prereflexive* if its completion is reflexive.

5.10. Theorem. *A two-norm space $\langle X, \| \|, \| \|^* \rangle$ is γ-precompact if and only if it is γ-separable and if γ-convergence in $\langle \Xi^*, \| \|^*, \| \| \rangle$ is equivalent to convergence with respect to the topology $\sigma(\Xi^*, X)$.*

Proof. Let $\langle X, \| \|, \| \|^* \rangle$ be γ-precompact. Then it is γ-separable, whence, by 5.1, there exists a sequence x_n dense in $\langle S, \| \|^* \rangle$. Let us write

$$\|\xi\|_1 = \sum_{n=1}^{\infty} \frac{1}{2^n} |\xi(x_n)|;$$

then $\|\xi\|_1 \leqslant \|\xi\| \leqslant \|\xi\|^*$ for $\xi \in \Xi^*$. Since γ-convergence in $\langle \Xi^*, \| \|^*, \| \|_1 \rangle$ is equivalent to convergence with respect to $\sigma(\Xi^*, X)$, the space $\langle \Xi^*, \| \|^*, \| \|_1 \rangle$ is γ-compact. On the other hand, $\langle \Xi^*, \| \|^*, \| \| \rangle$ is also γ-compact (by Theorem 5.5), whence, by Theorem 5.8, the γ-convergences in $\langle \Xi^*, \| \|^*, \| \| \rangle$ and in $\langle \Xi^*, \| \|, \| \|_1 \rangle$ are equivalent.

Now let us assume that $\langle X, \| \|, \| \|^* \rangle$ is γ-separable and that γ-convergence in $\langle \Xi^*, \| \|^*, \| \| \rangle$ is equivalent to weak convergence. Then $\langle \Xi^*, \| \|^*, \| \| \rangle$ is γ-compact, whence, by Theorem 5.5, $\langle X, \| \|, \| \|^* \rangle$ is γ-precompact.

5.11. Proposition. *Let $\langle X, \| \| \rangle$ be a Banach space. The following conditions are equivalent.:*

(a) *there exists a coarser norm $\| \|^*$ such that $\langle X, \| \|, \| \|^* \rangle$ is γ-precompact,*

(b) *there exists a total sequence of linear functionals on $\langle X, \| \| \rangle$.*

If $\langle X, \| \| \rangle$ is separable, then (a) *and* (b) *are always satisfied.*

Proof. If $\langle X, \| \|, \| \|^* \rangle$ is γ-precompact, then, by Theorem 5.5, $\langle \Xi^*, \| \| \rangle$ is separable, and any sequence η_n dense in Ξ^* is total. The converse implication follows by a theorem of W. Orlicz and V. Pták ([15], p. 63-64).

Now let $\langle X, \| \| \rangle$ be separable, let $x_1, x_2, \ldots$ be a sequence dense in the set $A = \{x \in X, \|x\| = 1\}$ and let $\xi_1, \xi_2, \ldots$ be functionals of Ξ such that $\|\xi_n\| = 1$ and $\xi_n(x_n) = 1$ for $n = 1, 2, \ldots$ Obviously, the sequence ξ_n is total. Moreover,

$$\|x\| = \sup_{n=1,2,\ldots} |\xi_n(x)| \quad \text{for every } x \in X.$$

6. The case $\Xi_\gamma = \Xi$. As shown in [5], p. 127, we have $\Xi_\gamma = \Xi^*$ for normal two-norm spaces if and only if the norms $\|\ \|$ and $\|\ \|^*$ are equivalent. On the other hand, the second extreme case $\Xi = \Xi_\gamma$ may occur in non-trivial cases (in particular, Theorem 3.7 deals with such spaces). Therefore it seems to be worth while to examine more precisely the spaces satisfying this condition.

Any two-norm space $\langle X, \|\ \|, \|\ \|^*\rangle$ satisfying $\Xi_\gamma = \Xi$ will be termed *saturated.*

6.1. PROPOSITION. *Any saturated two-norm space is normal.*

Proof. Let $x_n \epsilon X$, let $\|x_n\| \leqslant M$ for $n = 1, 2, \ldots$, and let $\|x_n - x_0\|^* \to 0$ as $n \to \infty$. Then x_n converges weakly to x_0, whence $\|x_0\| \leqslant M$.

6.2. PROPOSITION. *The following conditions are equivalent:*

(a) $\Xi_\gamma = \Xi$,

(b) Ξ^* *is dense in* $\langle \Xi, \|\ \|\rangle$,

(c) *any γ-convergent sequence is weakly convergent with respect to the topology* $\sigma(X, \Xi)$,

(d) *for every $\xi \epsilon \Xi$ and for every $\varepsilon > 0$ there exists a constant K such that*

$$\xi(x) \leqslant \varepsilon + K\|x\|^* \quad \textit{for all } x \epsilon S,$$

(e) *the set $\Xi_\gamma \cap \Sigma$ is closed for the topology* $\sigma(\Xi, X)$,

(f) *every convex and closed subset of $\langle X, \|\ \|\rangle$ is γ-closed,*

(g) *every linear closed subset of $\langle X, \|\ \|\rangle$ is γ-closed.*

Proof. The equivalence (a) $\Longleftrightarrow$ (c) is trivial; (a) $\Longleftrightarrow$ (b) follows by Theorem A and by 6.1.

(b) $\Longleftrightarrow$ (d). Let $\xi \epsilon \Xi$, let $\varepsilon > 0$ and let $\zeta \epsilon \Xi^*$ be such that $\|\zeta - \xi\| < \varepsilon$; then

$$\zeta(x) \geqslant \zeta(x) - \varepsilon \quad \text{for any } x \epsilon S,$$

$$\zeta(x) \leqslant K\|x\|^* \quad \text{for any } x \epsilon X.$$

These inequalities imply (d). Conversely, let (d) be satisfied. Choose elements $x_1, \ldots, x_n$ of S and positive numbers $t_1, \ldots, t_n$ arbitrarily. Setting $\vartheta_i = t_i(t_1 + \ldots + t_n)^{-1}$ we infer that $\|\sum_{i=1}^n \vartheta_i x_i\| \leqslant 1$ and, by condition (d),

$$\sum_{i=1}^n t_i[\xi(x_i) - \varepsilon] = (t_1 + \ldots + t_n)\Big[\xi\Big(\sum_{i=1}^n \vartheta_i x_i\Big) - \varepsilon\Big]$$

$$\leqslant (t_1 + \ldots + t_n) K \Big\|\sum_{i=1}^n \vartheta_i x_i\Big\|^* = K\Big\|\sum_{i=1}^n t_i x_i\Big\|^*,$$

which implies, by a theorem of Mazur and Orlicz ([12], p. 147), the existence of a distributive functional ζ satisfying the inequality $\zeta(x) \geqslant \xi(x) - \varepsilon$ for all $x \epsilon S$ and the inequality $\zeta(x) \leqslant K\|x\|^*$ for all $x \epsilon S$. Thus $\zeta \epsilon \Xi^*$ and $\|\zeta - \xi\| < \varepsilon$ which means that Ξ^* is dense in $\langle \Xi, \|\ \|\rangle$.

The implication (a) $\Longrightarrow$ (e) being trivial, we shall prove that (e) $\Rightarrow$ (a). The set $\Xi_\gamma \cap r\Sigma$ is closed for every $r > 0$, whence by a theorem of Bourbaki ([7], p. 129), the set Ξ_γ is closed for the topology $\sigma(\Xi, X)$. The set Ξ_γ is total, for Ξ^* is total, whence Ξ_γ is dense in Ξ for the topology $\sigma(\Xi, X)$ (Dixmier [9], p. 1061) and $\Xi_\gamma = \Xi$.

(c) $\Longrightarrow$ (f). Let A be convex and closed in $\langle X, \|\ \|\rangle$ and let $x_n \epsilon A$, $x_n \xrightarrow{\gamma} x_0$. By (c), x_n converges weakly to x_0 in $\langle X, \|\ \|\rangle$, whence, by Mazur's theorem ([11], p. 80), $x_0 \epsilon A$.

(f) $\Longrightarrow$ (g). Trivial.

(g) $\Longrightarrow$ (a). Let $\xi \epsilon \Xi$; then the set $\{x: \xi(x) = 0\}$ is closed in $\langle X, \|\ \|\rangle$, whence it is γ-closed. By theorem 3.2 of [5] the functional ξ is γ-linear.

6.3. PROPOSITION. *Any subspace of a saturated two-norm space is saturated.*

In particular, *if $\langle \mathfrak{X}^{(\gamma)}, \|\ \|, \|\ \|^*\rangle$ is saturated, so is $\langle X, \|\ \|, \|\ \|^*\rangle$.*

Proof. This follows by 6.2.

The converse of the second part of Proposition 6.3 is not true.

Indeed, let $\boldsymbol{J}$ denote the space of James; it is composed of the sequences $x = \{x_\nu\}$ convergent to zero and such that

$$\|x\| = \sup_{n=1,2,\ldots} \ \sup_{p_1 < p_2 < \ldots < p_{2n+1}} \Big[\sum_{i=1}^n (x_{p_{2i-1}} - x_{p_{2i}})^2 + (x_{p_{2n+1}})^2\Big]^{1/2} < \infty.$$

James proved ([10], p. 523) that $\langle \boldsymbol{J}, \|\ \|\rangle$ is a separable, non-reflexive Banach space isomorphic to its second conjugate, and that the functionals $\eta_n(x) = x_n$ $(n = 1, 2, \ldots)$ compose a basis for the space conjugate to $\langle \boldsymbol{J}, \|\ \|\rangle$.

Now, let us introduce in $\boldsymbol{J}$ the starred norm

$$\|x\|^* = \Big(\sum_{n=1}^\infty \frac{1}{2^n}|x_n|^2\Big)^{1/2},$$

then $\|x\|^* \leqslant \sup_{n=1,2,\ldots} |x_n| \leqslant \|x\|$. The space $\langle \boldsymbol{J}, \|\ \|, \|\ \|^*\rangle$ is normal but not γ-complete. Since the functionals η_n are γ-linear, the space $\langle \boldsymbol{J}, \|\ \|, \|\ \|^*\rangle$ is saturated. The γ-completion $\langle \boldsymbol{J}^c, \|\ \|, \|\ \|^*\rangle$ of $\langle \boldsymbol{J}, \|\ \|, \|\ \|^*\rangle$ is easily seen to consist of all convergent sequences for which $\|x\| < \infty$; this space is shown by James to be the biconjugate space of $\langle \boldsymbol{J}, \|\ \|\rangle$. The functional $\eta_\infty(x) = \lim_{n\to\infty} x_n$ is linear on $\langle \boldsymbol{J}^c, \|\ \|\rangle$; it is, however, not γ-linear. Indeed, let $x^{(n)} = \{0, \ldots, 0, 1, 1, \ldots\}$ (n noughts), whence $\|x^{(n)}\| = \sqrt{2}$ and

$\|x^{(n)}\|^* = (\sum_{\nu=n+1}^{\infty} 2^{-\nu})^{1/2} \to 0$; on the other hand, $\eta_\infty(x^{(n)}) = 1$ does not tend to $\eta_\infty(0)$. This example implies the following

6.4. Proposition. *Neither the γ-biconjugate space nor the γ-completion of a saturated space needs to be saturated.*

6.5. Proposition. *Let $\| \|^*$ be a coarser norm in the space l^1 (with $\|x\| = \sum_{\nu=1}^{\infty}|x_\nu|$), such that $\langle X, \| \|, \| \|^*\rangle$ is saturated. Then $\| \|$ and $\| \|^*$ are equivalent.*

Proof. This follows by Proposition 6.2, by the theorem of Schur ([6], p. 137) and by Theorem 1.2 of [3].

6.6. Proposition. *Let the γ-completion $\langle X^c, \| \|, \| \|^*\rangle$ be saturated (this is the case in particular when $\langle \mathfrak{X}^{(\gamma)}, \| \|, \| \|^*\rangle$ is saturated); then the space $\langle X^c, \| \|\rangle$ is equal to the completion $\langle \tilde{X}, \| \|\rangle$ of $\langle X, \| \|\rangle$. If, moreover, $\langle X, \| \|\rangle$ is complete, then $\langle X, \| \|, \| \|^*\rangle$ is γ-complete.*

Proof. Since $\langle X^c, \| \|, \| \|^*\rangle$ is saturated, each linear functional is uniquely extensible from $\langle X, \| \|\rangle$ to $\langle X^c, \| \|\rangle$, by Theorem 4.1. From Proposition 4.3 it follows that $X^c = \tilde{X}$.

6.7. Theorem. *Let $\langle X, \| \|, \| \|^*\rangle$ be γ-compact and saturated. Then $\langle X, \| \|\rangle$ is reflexive and separable, and γ-convergence in $\langle X, \| \|, \| \|^*\rangle$ is equivalent to weak convergence in $\langle X, \| \|\rangle$.*

Proof. The reflexivity of $\langle X, \| \|\rangle$ follows by Theorem 5.2 and Theorem 3.7. Next, by Theorem 5.5, the space $\langle \Xi^*, \| \|^*, \| \|\rangle$ is γ-compact, whence it is γ-separable. By Proposition 5.4, $\langle \Xi_\gamma, \| \|\rangle$ is separable, whence, by $\Xi = \Xi_\gamma$, $\langle \Xi, \| \|\rangle$ is separable, and by a theorem of Banach ([6], p. 189) $\langle X, \| \|\rangle$ is also separable. Finally, let $\xi_1, \xi_2, \ldots$ be a sequence dense in $\langle \Sigma, \| \|\rangle$ and let $\|x\|_1^* = \sum_{n=1}^{\infty} 2^{-n}|\xi_n(x)|$. Evidently, γ-convergence in $\langle X, \| \|, \| \|^*\rangle$ is equivalent to weak convergence. By 6.2, γ-convergence in $\langle X, \| \|, \| \|^*\rangle$ implies weak convergence; the converse follows by the application of Theorem 5.8.

6.8. Proposition. *Let $\langle X, \| \|\rangle$ be reflexive and not separable. Then there exists no coarser norm $\| \|^*$ such that γ-convergence in $\langle X, \| \|, \| \|^*\rangle$ is equivalent to weak convergence in $\langle X, \| \|\rangle$.*

Proof. The reflexivity of $\langle X, \| \|\rangle$ implies that every bounded sequence of elements of X contains a weakly convergent subsequence (Pettis [16]). Suppose that there exists a norm $\| \|^*$ coarser than $\| \|$ such that weak convergence in $\langle X, \| \|\rangle$ is equivalent to γ-convergence in $\langle X, \| \|, \| \|^*\rangle$; this implies the γ-compactness of $\langle X, \| \|, \| \|^*\rangle$ and, in turn, by Theorem 6.7, the separability of $\langle X, \| \|\rangle$.

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