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On generalized variations (I)

by

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0. Introduction. N. Wiener [10] introduces the notion of higher variations. These ideas have been developed by L. C. Young and E. R. Love (for basic treatises, see [11] and [4]), who consider p-th variations of a real or complex-valued function $x(t)$ of a real argument t, defined as follows:

$$V_p(x) = \Big[\sup_{\Pi} \sum_{i=1}^{m} |x(t_i) - x(t_{i-1})|^p\Big]^{1/p},$$

where Π denotes the partition $a = t_0 < t_1 < \ldots < t_m = b$ of the finite interval $\langle a, b\rangle$. In more recent papers [4] and [3] the concept of generalized absolute continuity with p-th power is introduced. The notion of p-th variations $V_p(x)$ has been generalized to that of M-th variations $V_M(x)$ of a function $x = x(t)$, defined by the formula given in section 1. We may mention a paper by L. C. Young [12] in which the author considers results related to the existence of Stieltjes integrals for functions of bounded M-th variations, with the second Helly theorem on the limit of Stieltjes integrals, some applications to the theory of Fourier series etc.

In section 1 of the present paper functions of bounded M-th variation are considered from a general point of view. Some results on the linearity of the set of functions of bounded M-th variation and on convergence in variation are given. Moreover, the first Helly extracting-theorem is proved. Section 2 is devoted to functions absolutely continuous with respect to a function M. Here the greater part of the theorems are generalizations of results obtained by E. R. Love [3]. Next, the equivalence of some definitions and approximation problems connected with convergence in variation are considered. Section 3 introduces the norm in the class of functions of bounded M-th variation and in the class of functions absolutely continuous with respect to a function M, making these classes Banach spaces. The connection between convergence in norm and convergence in variation is considered. Finally, in section 4, we prove a theorem on sequences of Stieltjes integrals and give the form of a linear functional over the space of functions absolutely continuous with respect to a function M, generalizing some results of [4] (see also [5]). The main results of sections 1-3 are given without proofs in our note [6]. The present paper is based on applications of methods introduced by W. Orlicz (see e. g. [8], [9]) for L^M-spaces to spaces of functions of bounded M-th variation and to spaces of functions absolutely continuous with respect to a function M. Our investigations are also connected with the general theory of modular spaces (see [7]).

We now give some notations used in this paper. We denote by $M(u)$ a continuous and non-decreasing function defined for $u \geqslant 0$, with $M(0) = 0$, $M(u) > 0$ for $u > 0$. We shall sometimes apply the following conditions:

(0) there exist $a > 0$ and $L > 0$ such that $M(u) \leqslant Lu$ for $0 < u \leqslant a$,

(o) $M(u)/u \to 0$ for $u \to 0$.

(∞) $M(u)/u \to \infty$ for $u \to \infty$,

(Δ_2) there exist $a > 0$ and $\varkappa > 0$ such that $M(2u) \leqslant \varkappa M(u)$ for $0 < u \leqslant a$.

(c) $M(u)$ is a convex function.

If $M(u)$ satisfies the conditions (c), (o) and (∞), then we denote by $N(u)$ the function *complementary* to $M(u)$, defined by the formula

$$N(v) = \max_{u \geqslant 0} [uv - M(u)],$$

satisfying also the conditions (c), (o) and (∞) (see [1], p. 14, Theorem 4). For the functions $M(u)$ and $N(u)$, the Young inequality

$$uv \leqslant M(u) + N(v), \qquad u, v \geqslant 0,$$

holds.

1. Functions of bounded variation. We consider complex-valued functions $x(t)$, defined in a finite closed interval $\langle a, b\rangle$. The value

$$V_M(x) = \sup_{\Pi} \sum_{i=1}^{m} M[|x(t_i) - x(t_{i-1})|],$$

where Π is a partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$ will be called the *M-th variation* of the function $x(t)$ in $\langle a, b\rangle$. The variation in a subinterval $\langle t', t''\rangle$ of the interval $\langle a, b\rangle$ will be denoted by $V_M(x; t', t'')$, where $V_M(x; t, t) = 0$; $V_M(x) = V_M(x; a, b)$ will always denote the variation in the fundamental interval $\langle a, b\rangle$. Moreover, we denote by $\mathcal{V}_M$ the class of all functions $x = x(t)$, $x(a) = 0$, of bounded M-th variation, i. e. $V_M(x) < \infty$, and by $\mathcal{V}_M^*$ the class of all functions $x = x(t)$ such that for a certain $k > 0$ (depending on x), $kx \in \mathcal{V}_M$. Obviously, $\mathcal{V}_M \subset \mathcal{V}_M^*$. We have

1.01. *If $M(u)$ satisfies (Δ_2), then $\mathcal{V}_M^* = \mathcal{V}_M$.*

This statement immediately follows from the boundedness of functions of class $\mathcal{V}_M^*$ and from the following lemma concerning the condition (Δ_2):

1.02. (Δ_2) *is equivalent to the following condition: For every $a' > 0$ there exists a $\varkappa(a') \geqslant 1$ such that $M(2u) \leqslant \varkappa(a')M(u)$ for $0 < u \leqslant a'$.*

This follows from the inequalities

$$M(u) \geqslant M(a)\frac{1}{\varkappa M(a/2)}M(u) = \frac{1}{\varkappa}\cdot\frac{M(a)}{M(2u)}\cdot\frac{M(u)}{M(a/2)}M(2u)$$
$$\geqslant \frac{1}{\varkappa}\cdot\frac{M(a)}{M(2a')}M(2u),$$

valid for $\frac{1}{2}a \leqslant u \leqslant a'$.

1.03. *If $M(u)$ satisfies (c), then the M-th variation of a monotonic function $x(t)$ in $\langle a, b\rangle$ is equal to $M[|x(b) - x(a)|]$.*

The inequality $M[|x(b) - x(a)|] \leqslant V_M(x)$ being obvious, we prove the reverse inequality. Since $M(u)$ satisfies (c), $M(u)/u$ is non-decreasing and we have, for an arbitrary partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$ such that $x(t_{i-1}) \neq x(t_i)$,

$$\sum_{i=1}^{m} M[|x(t_i) - x(t_{i-1})|] = \sum_{i=1}^{m} \frac{M[|x(t_i) - x(t_{i-1})|]}{|x(t_i) - x(t_{i-1})|}|x(t_i) - x(t_{i-1})|$$
$$\leqslant \frac{M[|x(b) - x(a)|]}{|x(b) - x(a)|}\sum_{i=1}^{m}|x(t_i) - x(t_{i-1})| = M[|x(b) - x(a)|].$$

Consequently, $V_M(x) \leqslant M[|x(b) - x(a)|]$.

1.1. Linearity. We now prove some theorems on the linearity of the spaces $\mathcal{V}_M$ and $\mathcal{V}_M^*$. First, we give some auxiliary inequalities.

1.11. *Let us assume that $M(u)$ satisfies (Δ_2), and $|x_i(t)| \leqslant K$ for $i = 1, 2, \ldots, n$. Further, let m denote the least non-negative integer satisfying the inequality $|c| \leqslant 2^m$. Then*

$$V_M(x_1 + x_2 + \ldots + x_n) \leqslant \varkappa^{n-1}((n-1)K)[V_M(x_1) + V_M(x_2) + \ldots + V_M(x_n)]$$

and

$$V_M(cx_1) \leqslant \varkappa^m(2^{m-1}K)V_M(x_1),$$

the function $\varkappa(a')$ being defined in 1.02.

The former inequality follows from the inequality

$$M(u_1+u_2+\ldots+u_n) \leqslant \varkappa^{n-1}((n-1)a')[M(u_1)+M(u_2)+\ldots+M(u_n)]$$

valid for $0 < u_i \leqslant a'$ and the latter is obtained by induction.

1.12. *If $M(u)$ satisfies* (c) *then*

$$V_M(x_1+x_2+\ldots+x_n) \leqslant \frac{1}{n}[V_M(nx_1)+V_M(nx_2)+\ldots+V_M(nx_n)].$$

Now we proceed to the problem of the linearity of the classes $\mathcal{V}_M$ and $\mathcal{V}_M^*$. Obviously, applying 1.11 and 1.12, we obtain the following result:

1.13. (c) *implies the linearity of $\mathcal{V}_M^*$, and* (Δ_2) *that of $\mathcal{V}_M$.*

It will also be proved that if $\mathcal{V}_M$ is linear, condition (Δ_2) holds. The proof of this statement requires two auxiliary lemmas.

1.14. *The convergence of the series $\sum M_1(u_n)$ implies the convergence of $\sum M_2(u_n)$ for every sequence of non-negative numbers u_n if and only if there exist numbers $a > 0$ and $b > 0$ such that $M_2(u) \leqslant bM_1(u)$ for $0 < u \leqslant a$.*

This lemma is proved in [1] (Theorem 1a, p. 5). For completeness, the proof will also be given here. Since the sufficiency is obvious, we only prove the necessity. Let us suppose that for each $a > 0$ and $b > 0$ there exists a number u such that $0 < u \leqslant a$ and $M_2(u) > bM_1(u)$. We put $M_1(a) = 1/n^2$, $b = n$ and denote the corresponding numbers u by u_n. Moreover, we denote by k_n the least positive integer such that $1/n^2 \leqslant k_nM_1(u_n) < 2/n^2$. Then, if we write $u_n' = u_m$ for $k_1+\ldots+k_{m-1} < n \leqslant k_1+\ldots+k_m$, where $m = 1, 2, \ldots$, $k_0 = 0$, we have $\sum M_1(u_n') < \infty$ and $\sum M_2(u_n') = \infty$.

1.15. *$\mathcal{V}_{M_1} \subset \mathcal{V}_{M_2}$, if and only if there exist $a > 0$ and $b > 0$ such that $M_2(u) \leqslant bM_1(u)$ for $0 < u \leqslant a$.*

S u f f i c i e n c y. First let us remark that, for every $a' > 0$, our condition implies the existence of a $b(a') > 0$ such that $M_2(u) \leqslant b(a')M_1(u)$ for $0 < u \leqslant a'$. The proof of this is similar to that of 1.02. Let us assume that $x = x(t)$ belongs to $\mathcal{V}_{M_1}$. The function $x(t)$ being bounded, $|x(t)| \leqslant K$, we have $M_2[|x(t')-x(t'')|] \leqslant b(2K)M_1[|x(t')-x(t'')|]$. Then, $V_{M_2}(x) \leqslant b(2K)V_{M_1}(x)$.

N e c e s s i t y. Suppose that $\mathcal{V}_{M_1} \subset \mathcal{V}_{M_2}$ and that for every $a > 0$ and $b > 0$ there exists a number u such that $M_2(u) > bM_1(u)$ and $0 < u \leqslant a$. Then 1.14 implies the existence of a sequence $u_n' \geqslant 0$ such that $\sum M_1(u_n') < \infty$ and $\sum M_2(u_n') = \infty$. Let w_n denote a fixed increasing sequence of points of the open interval (a, b). We put

$$x(t) = \begin{cases} u_n' & \text{for} \quad t = w_n, \\ 0 & \text{for} \quad t \neq w_n,\ n = 0, 1, \ldots \end{cases}$$

Then, if we choose an arbitrary partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$, we have

$$\sum_{i=1}^{m} M_1[|x(t_i)-x(t_{i-1})|] \leqslant 2\sum_{i=1}^{m} M_1(u_{l_i}')+\sum_{i=1}^{m} M_1(|u_{l_i}'-u_{l_{i-1}}'|) \leqslant 4\sum_{i=0}^{\infty} M_1(u_i').$$

Hence, $V_{M_1}(x) < \infty$. On the other hand, for the partition $a = t_0 < t_1 < \ldots < t_{2m} = b$, where $t_{2i+1} = w_i$, $i = 0, 1, \ldots, m-2$, and $t_{2i} = \frac{1}{2}(w_{i-1}+w_i)$, $i = 1, 2, \ldots, m-1$, we have

$$\sum_{i=1}^{2m-1} M_2[|x(t_i)-x(t_{i-1})|] \geqslant 2\sum_{i=0}^{m-2} M_2(u_i'),$$

whence $V_{M_2}(x) = \infty$.

R e m a r k. It follows from 1.15 that there exists a function $M_1(u)$ such that $\mathcal{V}_M = \mathcal{V}_{M_1}$ and $M_1(u)/u \to \infty$ as $u \to \infty$. Moreover, if there exists $\lim\limits_{u\to 0+} M(u)/u > 0$, then $\mathcal{V}_M = \mathcal{V}_1$, where $\mathcal{V}_1$ is the class of all functions of bounded variation in the usual sense, equal to zero at the point a. Hence, the assumptions of the existence of $\lim\limits_{u\to\infty} M(u)/u$ and $\lim\limits_{u\to 0+} M(u)/u$ (implied for instance by (c)) are not essential generalizations of the assumptions (∞) or (o).

1.16. *The class $\mathcal{V}_M$ is linear if and only if (Δ_2) is satisfied.*

The sufficiency having been proved in 1.13, we now prove the necessity. Suppose that $\mathcal{V}_M$ is linear. Then $x \in \mathcal{V}_M$ implies $2x \in \mathcal{V}_M$. If we put $M_1(u) = M(u)$, $M_2(u) = M(2u)$, we obtain $\mathcal{V}_{M_1} \subset \mathcal{V}_{M_2}$ and 1.15 implies condition (Δ_2).

We now give three theorems on the connection between the variation of a function $x(t)$ in the whole interval $\langle a, b\rangle$ and the sum of variations in disjoint subintervals of $\langle a, b\rangle$ the sum of which is equal to $\langle a, b\rangle$.

1.17. *If $a < c < b$, then the inequality*

$$V_M(x; a, c)+V_M(x; c, b) \leqslant V_M(x; a, b)$$

holds. Moreover, if $M(u)$ satisfies (Δ_2) and $|x(t)| \leqslant K$, then

$$V_M(x; a, b) \leqslant \varkappa(2K)[V_M(x; a, c)+V_M(x; c, b)],$$

the function $\varkappa(a')$ *having been defined in* 1.02, *and if* $M(u)$ *satisfies* (c), *then*

$$V_M(x;a,b)\leqslant \tfrac{1}{2}[V_M(2x;a,c)+V_M(2x;c,b)].$$

The proofs of these inequalities will be omitted.

1.18. *Let us consider a fixed partition* $a=t_0<t_1<\ldots<t_m=b$ *of the interval* $\langle a,b\rangle$. *Suppose* $x(t_i)=0$ *for* $i=1,2,\ldots,m-1$.

(a) *If* $M(u)$ *satisfies* (Δ_2) *and* $x\in\mathcal{V}_M$, $|x(t)|\leqslant K$, *then*

$$V_M(x;a,b)\leqslant\varkappa(K)\sum_{i=1}^{m}V_M(x;t_{i-1},t_i).$$

(b) *If* $M(u)$ *satisfies* (c) *and* $kx\in\mathcal{V}_M$, *where* $k>0$, *then*

$$V_M(\tfrac{1}{2}kx;a,b)\leqslant\frac{1}{2}\sum_{i=1}^{m}V_M(kx;t_{i-1},t_i).$$

We give the proof for case (b) only. Case (a) is proved similarly. Denoting the given partition $a=t_0<t_1<\ldots<t_m=b$ by Π_0, let us choose an arbitrary partition Π: $a=\tau_0<\tau_1<\ldots<\tau_{m'}=b$ of the interval $\langle a,b\rangle$. We define two finite sequences of indices i_ν and μ_ν as follows: The positive integer i belongs to the increasing sequence i_ν, $\nu=1,2,\ldots,\mu$, if and only if in the interval (τ_{i-1},τ_i) there exist points of the partition Π_0. By $\mu_\nu+1$ we denote the number of points of the partition Π_0 belonging to the interval $(\tau_{i_\nu-1},\tau_{i_\nu})$. Thus, we have

$$\tau_{i_\nu-1}<t_\nu<t_{\nu+1}<\ldots<t_{\nu+\mu_\nu}<\tau_{i_\nu}.$$

Moreover, we denote by j_ν, $\nu=1,2,\ldots,m'-\mu$, the sequence of positive integers $1,2,\ldots,m'$ not belonging to the sequence i_ν. Thus, we have

$$\begin{aligned}\sum_{\nu=1}^{\mu}M[\tfrac{1}{2}k|x(\tau_{i_\nu})-x(\tau_{i_\nu-1})|]&\leqslant\sum_{\nu=1}^{\mu}M\{\tfrac{1}{2}k[|x(\tau_{i_\nu-1})|+|x(\tau_{i_\nu})|]\}\\&\leqslant\frac{1}{2}\sum_{\nu=1}^{\mu}\{M[k|x(\tau_{i_\nu-1})|]+M[k|x(\tau_{i_\nu})|]\}\\&\leqslant\frac{1}{2}\sum_{\nu=1}^{\mu}\{M[k|x(t_\nu)-x(\tau_{i_\nu-1})|]+\\&\quad+\sum_{\varrho=\nu+1}^{\nu+\mu_\nu}M[k|x(t_\varrho)-x(t_{\varrho-1})|]+\\&\quad+M[k|x(\tau_{i_\nu})-x(t_{\nu+\mu_\nu})|]\}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{i=1}^{m'}M[\tfrac{1}{2}k|x(\tau_i)-x(\tau_{i-1})|]&\leqslant\frac{1}{2}\sum_{\nu=1}^{\mu}\Big\{M[k|x(t_\nu)-x(t_{i_\nu-1})|]+\\&\quad+\sum_{\varrho=\nu+1}^{\nu+\mu_\nu}M[k|x(t_\varrho)-x(t_{\varrho-1})|)]+\\&\quad+M[k|x(\tau_{i_\nu})-x(t_{\nu+\mu_\nu})|]\Big\}+\\&\quad+\sum_{\nu=1}^{m'-\mu}M[\tfrac{1}{2}k|x(\tau_{j_\nu})-x(\tau_{j_\nu-1})|]\\&\leqslant\frac{1}{2}\sum_{i=1}^{m}V_M(kx;t_{i-1},t_i).\end{aligned}$$

The partition Π being arbitrary, we obtain the required inequality.

1.19. *If* $a=t_0<t_1<\ldots<t_m=b$ *and* $V_M(x;t_{i-1},t_i)<\infty$ *for* $i=1,2,\ldots,m$, *then* $V_M(x;a,b)<\infty$.

1.2. Convergence in variation. We now introduce in $\mathcal{V}_M^*$ the concept of convergence in variation. The sequence $x_n\in\mathcal{V}_M^*$ will be termed *convergent in variation to* $x\in\mathcal{V}_M^*$ if there exists a $k>0$ (not depending on n) such that $V_M[k(x_n-x)]\to 0$ for $n\to\infty$. Moreover, we shall say that a sequence $x_n\in\mathcal{V}_M^*$ satisfies the *Cauchy condition in variation* if there exists $k>0$ such that for every $\varepsilon>0$ there exists a K with the following property: for every $p,q>K$, $V_M[k(x_p-x_q)]<\varepsilon$. Finally, we shall say that $x_n\in\mathcal{V}_M^*$ is *bounded in variation*, if $V_M(kx_n)$ is bounded for $n=1,2,\ldots$, with a $k>0$ not depending on n. First we remark that

1.21. (a) *If a sequence* $x_n\in\mathcal{V}_M^*$ *is bounded in variation, then it is uniformly bounded, i. e.* $|x_n(t)|\leqslant K$ *in* $\langle a,b\rangle$,

(b) *If a sequence* $x_n\in\mathcal{V}_M$ *satisfies the Cauchy condition in variation or is convergent in variation to* $x\in\mathcal{V}_M^*$, *then it is uniformly convergent (to* $x(t)$*) in* $\langle a,b\rangle$.

The proof of this theorem will be left to the reader, as involving no difficulties.

1.22. *If* $M(u)$ *satisfies* (Δ_2) *or* (c), *then the class* $\mathcal{V}_M^*$ *is complete in variation, i. e. every sequence* $x_n\in\mathcal{V}_M^*$ *satisfying the Cauchy condition is convergent in variation to an* $x\in\mathcal{V}_M^*$. *Conversely, convergence in variation implies the Cauchy condition in variation.*

To prove this theorem, we remark that for a given $\varepsilon>0$ and an arbitrary partition $a=t_0<t_1<\ldots<t_m=b$ of the interval $\langle a,b\rangle$ we have

$$\begin{aligned}\sum_{i=1}^{m}M\{k|[x_p(t_i)-x_q(t_i)]-[x_p(t_{i-1})-x_q(t_{i-1})]|\}\\\leqslant V_M[k(x_p-x_q)]\leqslant\varepsilon\quad\text{for}\quad p,q>K.\end{aligned}$$

The sequence $x_n(t)$ being uniformly convergent to $x(t)$ in $\langle a, b\rangle$ (see 1.21), we obtain

$$\sum_{i=1}^{m} M\{k|[x_p(t_i)-x(t_i)]-[x_p(t_{i-1})-x(t_{i-1})]|\} \leqslant \varepsilon \quad \text{for} \quad p > K$$

and the definition of variation yields

$$V_M[k(x_p-x)] \leqslant \varepsilon \quad \text{for} \quad p > K.$$

The converse theorem follows from 1.11 and 1.12.

1.23. *Let us assume that $M(u)$ satisfies (Δ_2). Then $x_n \in \mathcal{V}_M$ is convergent in variation to $x \in \mathcal{V}_M$ if and only if $V_M(x_n - x) \to 0$ for $n \to \infty$. Moreover, the class $\mathcal{V}_M$ is a non-separable $\mathcal{L}^*$-space* (for terminology, see [2], p. 83 and 88) *complete in variation.*

To prove this theorem, we assume $V_M[k(x_n-x)] \to 0$. This ensures $k|x_n(t)-x(t)| \leqslant K$ uniformly in $\langle a, b\rangle$. If we put $1/k \leqslant 2^m$, where m is a positive integer, we obtain

$$V_M(x_n-x) \leqslant \varkappa^m(2^{m-1}K)\, V_M[k(x_n-x)]$$

(see 1.11) and $V_M(x_n-x) \to 0$. This implies that the class $\mathcal{V}_M$ is an $\mathcal{L}^*$-space. Non-separability of $\mathcal{V}_M$ follows from the fact that for the step-functions

$$x_{t_0}(t) = \begin{cases} 0 & \text{for} \quad a \leqslant t \leqslant t_0, \\ 1 & \text{for} \quad t_0 < t \leqslant b, \end{cases}$$

we have $V_M(x_{t_0'} - x_{t_0''}) = 2M(1)$ for $t_0' \neq t_0''$, the set of such functions being non-enumerable. The details of the proof of non-separability using condition (Δ_2) will be omitted. Completeness follows from 1.22.

1.24. *If $x_n(t) \to x(t)$ in $\langle a, b\rangle$, then*

$$V_M(x) \leqslant \varliminf_{n\to\infty} V_M(x_n).$$

Assuming $g = \varliminf_{n\to\infty} V_M(x_n) < \infty$ we choose a sequence $x_{n_k}(t)$ such that $V_M(x_{n_k}) \to g$. Given $\varepsilon > 0$, we obtain, for an arbitrary partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$,

$$\sum_{i=1}^{m} M[|x_{n_k}(t_i) - x_{n_k}(t_{i-1})|] \leqslant V_M(x_{n_k}) < g+\varepsilon \quad \text{for} \quad k > K,$$

and the convergence $x_{n_k}(t) \to x(t)$ yields $V_M(x) \leqslant g+\varepsilon$.

1.3. Helly's extracting theorem. *Every sequence $x_n \in \mathcal{V}_M^*$ bounded in variation (i. e. there exist constants $k > 0$ and $K > 0$ such that $V_M(kx_n) \leqslant K$*

for $n = 1, 2, \ldots$) includes a subsequence convergent to a function $x = x(t)$ of the class $\mathcal{V}_M^*$ pointwise in $\langle a, b\rangle$.

To prove this theorem we write $v_n(t) = V_M(kx_n; a, t)$ for $a \leqslant t \leqslant b$. The functions $v_n(t)$ are non-decreasing and bounded by K in $\langle a, b\rangle$. Thus, we conclude from the well-known Helly extracting theorem for sequences of monotonic functions that the sequence $v_n(t)$ includes a subsequence $v_{n_i}(t)$ convergent to a non-decreasing function $v(t)$ at every point t of the interval $\langle a, b\rangle$. Using the diagonal method we extract from the sequence of indices n_i a subsequence n_{i_j} such that $x_{n_{i_j}}(t)$ is convergent at every rational point of the interval $\langle a, b\rangle$ and at the points a, b. Writing $x_{n_{i_j}}(t) = x_j^*(t)$ and $v_{n_{i_j}}(t) = v_j^*(t)$, we obtain

$$v_j^*(t) \to v(t) \quad \text{for every} \quad t \in \langle a, b\rangle$$

and

$$x_j^*(t) \to x(t) \text{ for every rational } t \in (a, b) \text{ and for } t = a,\ t = b.$$

Now, let us assume that t_0 is a non-rational point of continuity of the function $v(t)$ in (a, b). We shall prove that the numerical sequence $x_j^*(t_0)$ is convergent. We take an arbitrary $\varepsilon > 0$ and choose a rational number $w \in (a, b)$, $w > t_0$, such that

$$0 \leqslant v(w) - v(t_0) < \tfrac{1}{3}M(\tfrac{1}{3}\varepsilon k),$$

and a J such that for $j > J$

$$|v_j^*(w) - v(w)| < \tfrac{1}{3}M(\tfrac{1}{3}\varepsilon k) \quad \text{and} \quad |v_j^*(t_0) - v(t_0)| < \tfrac{1}{3}M(\tfrac{1}{3}\varepsilon k).$$

Since, according to 1.17,

$$\begin{aligned} M[k|x_j^*(t_0) - x_j^*(w)|] &\leqslant V_M(kx_j^*; t_0, w) \leqslant v_j^*(w) - v_j^*(t_0) \\ &\leqslant |v_j^*(w) - v(w)| + [v(w) - v(t_0)] + |v(t_0) - v_j^*(t_0)| < M(\tfrac{1}{3}\varepsilon k) \end{aligned}$$

for $j > J$, we obtain

$$|x_j^*(t_0) - x_j^*(w)| < \tfrac{1}{3}\varepsilon \quad \text{for} \quad j > J.$$

We now choose a number P such that for $p, q > P$,

$$|x_p^*(w) - x_q^*(w)| < \tfrac{1}{3}\varepsilon$$

and put $J' = \max(J, P)$. We then obtain, for $p, q > J'$,

$$|x_p^*(t_0) - x_q^*(t_0)| < \varepsilon.$$

Thus, the sequence $x_j^*(t_0)$ is convergent. The set of points of discontinuity of the function $v(t)$ being at most enumerable, the diagonal method enables us to extract from the sequence $x_j^*(t)$ a subsequence convergent to a function $x(t)$ at every point of the interval $\langle a, b\rangle$. Evidently, $V_M(kx) \leqslant K$.

2. Absolutely continuous functions. A complex-valued function $x(t)$, defined in a finite closed interval $\langle a, b\rangle$, will be termed *absolutely continuous with respect to the function* $M(u)$ if the following condition is satisfied: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{m} M[|x(\beta_i)-x(a_i)|] < \varepsilon,$$

for all finite sets of non-overlapping intervals $(a_i, \beta_i) \subset \langle a, b\rangle$ $i = 1, 2, \ldots, m$, such that

$$\sum_{i=1}^{m} M(\beta_i - a_i) < \delta.$$

We denote by $\mathcal{AC}_M$ the class of all functions $x(t)$ absolutely continuous with respect to the function $M(u)$ and such that $x(a) = 0$. Moreover, we denote by $\mathcal{AC}_M^*$ the class of all functions $x = x(t)$ such that for a $k > 0$ (depending on x), $kx \in \mathcal{AC}_M$. It is easily seen that the functions of the class $\mathcal{AC}_M^*$ are continuous. Evidently, $\mathcal{AC}_M \subset \mathcal{AC}_M^*$. The following theorem is verified similarly to 1.01:

2.01. *If* $M(u)$ *satisfies* (Δ_2), *then* $\mathcal{AC}_M^* = \mathcal{AC}_M$.

2.1. Several equivalent definitions of absolute continuity. We now introduce the following conditions:

1′. *Given an arbitrary* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that for every partition* $a = t_0 < t_1 < \ldots < t_m = b$ *of the interval* $\langle a, b\rangle$ *with* $t_i - t_{i-1} < \delta$ *for* $i = 1, 2, \ldots, m$, *the inequality*

$$\sum_{i=1}^{m} M[|x(t_i)-x(t_{i-1})|] < \varepsilon$$

holds.

1*. *There exists a* $k > 0$ *such that* kx *satisfies condition* 1′.

2′. *For every* $\varepsilon > 0$ *there exists a partition* $a = t_0 < t_1 < \ldots < t_m = b$ *of the interval* $\langle a, b\rangle$ *such that the inequality*

$$\sum_{i=1}^{m} V_M(x; t_{i-1}, t_i) < \varepsilon$$

holds.

2*. *There exists a* $k > 0$ *such that* kx *satisfies condition* 2′.

In the following theorem the notation $A \overset{C}{\to} B$ will indicate that, assuming property C, condition A implies condition B.

2.11. *If* $x(a) = 0$, *the following implications hold:*

$$\mathcal{AC}_M \overset{(0)}{\to} 1', \quad 1' \to \mathcal{AC}_M, \quad \mathcal{AC}_M^* \overset{(0)}{\to} 1^*, \quad 1^* \to \mathcal{AC}_M^*,$$
$$1' \to 2', \quad 2' \overset{(\Delta_2)}{\to} \mathcal{AC}_M, \quad 1^* \to 2^*, \quad 2^* \overset{(c)}{\to} \mathcal{AC}_M^*.$$

We note that other characterizations of $\mathcal{AC}_M$ and $\mathcal{AC}_M^*$ are given in 2.22 and 2.41.

We prove the implications given in 2.11 successively.

$\mathcal{AC}_M \overset{(0)}{\to} 1'$. Given an $\varepsilon > 0$, we choose a number $\delta > 0$ such that for each finite set of non-overlapping subintervals (a_i, β_i) of the interval $\langle a, b\rangle$ the inequality

$$\sum_{i=1}^{m} M(\beta_i - a_i) < \delta \quad \text{implies} \quad \sum_{i=1}^{m} M[|x(\beta_i)-x(a_i)|] < \varepsilon.$$

Now we take a number $\delta' > 0$ such that for $0 < u < \delta'$, $M(u)/u < \delta/(b-a)$. Then, if we take a partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$ satisfying the condition $t_i - t_{i-1} < \delta$ for $i = 1, 2, \ldots, m$, we obtain

$$\sum_{i=1}^{m} M(t_i - t_{i-1}) = \sum_{i=1}^{m} \frac{M(t_i - t_{i-1})}{t_i - t_{i-1}} (t_i - t_{i-1}) < \frac{\delta}{b-a} \sum_{i=1}^{m} (t_i - t_{i-1}) = \delta.$$

Hence

$$\sum_{i=1}^{m} M[|x(t_i)-x(t_{i-1})|] < \varepsilon.$$

$1' \to \mathcal{AC}_M$. We choose a number $\delta > 0$ such that for every partition $a = t_0 < t_1 < \ldots < t_m = b$ of $\langle a, b\rangle$ with $t_i - t_{i-1} < \delta$ the inequality $\sum_1^m M[|x(t_i)-x(t_{i-1})|] < \varepsilon$ holds. Further, we take an arbitrary finite set of non-overlapping subintervals (a_i, β_i) of the interval $\langle a, b\rangle$ such that $\sum_1^{m'} M(\beta_i - a_i) < M(\delta)$. This implies $\beta_i - a_i < \delta$ for $i = 1, 2, \ldots, m'$. We take points $\beta_i = \tau_0^{(i)} < \tau_1^{(i)} < \ldots < \tau_{p_i}^{(i)} = a_{i+1}$ for $\beta_i < a_{i+1}$, $i = 0, 1, \ldots, m'$, where $\beta_0 = a$, $a_{m'+1} = b$, in such a way that $\tau_k^{(i)} - \tau_{k-1}^{(i)} < \delta$ for $i = 0, 1, \ldots, m'$; $k = 1, 2, \ldots, p_i$. Applying condition 1′ to the partition

$$a < \tau_1^{(0)} < \ldots < \tau_{p_0-1}^{(0)} < a_1 < \beta_1 < \tau_1^{(1)} < \ldots < \tau_{p_1-1}^{(1)} < a_2 < \ldots < b$$

we obtain

$$\sum_{i=1}^{m'} M[|x(\beta_i)-x(a_i)|] < \varepsilon.$$

The implications $\mathcal{AC}_M^* \overset{(\mathrm{o})}{\to} 1^*$ and $1^* \to \mathcal{AC}_M^*$ are obtained from those proved above by putting kx in place of x.

$1' \to 2'$ and $1^* \to 2^*$ being obvious, we now prove $2' \overset{(\Delta_2)}{\to} \mathcal{AC}_M$ and $2^* \overset{(\mathrm{o})}{\to} \mathcal{AC}_M^*$. Let us take an arbitrary $y = y(t)$ satisfying conditions $2'$, $y(a) = 0$ and an arbitrary $\eta > 0$. We choose a partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$ such that

$$\sum_{i=1}^{m} V_M(y; t_{i-1}, t_i) < \eta$$

and denote $\delta = \min(t_i - t_{i-1})$ where $i = 1, 2, \ldots, m$. Now let (α_ν, β_ν) (where $\nu = 1, 2, \ldots, m'$) be a finite set of non-overlapping subintervals of the interval $\langle a, b\rangle$ such that $\sum_1^{m'} M(\beta_\nu - \alpha_\nu) < M(\delta)$. This implies especially $\beta_\nu - \alpha_\nu < \delta$. There exists no interval (α_ν, β_ν) such that $\alpha_\nu \leqslant t_{i-1} < t_i \leqslant \beta_\nu$ for any index i, since this would imply $\delta \leqslant t_i - t_{i-1} \leqslant \beta_\nu - \alpha_i < \delta$. We now write $\tau_\nu = t_i$ if there exists an index i such that $\alpha_\nu \leqslant t_i < \beta_i$, and $\tau_\nu = \alpha_\nu$ if none exists. Then we have

$$\sum_{\nu=1}^{m'} M[|y(\beta_\nu) - y(\alpha_\nu)|] \leqslant \sum_{\nu=1}^{m'} M[|y(\beta_\nu) - y(\tau_\nu)| + |y(\tau_\nu) - y(\alpha_\nu)|].$$

Now let us assume that $M(u)$ satisfies (Δ_2), $|x(t)| \leqslant K$, and, for a given $\varepsilon > 0$, put $\eta = \varepsilon/\varkappa(2K)$ (see 1.02) and $y = x$. We then obtain

$$\sum_{\nu=1}^{m'} M[|x(\beta_\nu) - x(\alpha_\nu)|] \leqslant \varkappa(2K) \sum_{\nu=1}^{m'} \{M[|x(\beta_\nu) - x(\tau_\nu)|] + M[|x(\tau_\nu) - x(\alpha_\nu)|]\} \leqslant \varkappa(2K) \sum_{i=1}^{m} V_M(x; t_{i-1}, t_i) < \varepsilon.$$

Now we suppose that $M(u)$ satisfies (o), and put $\eta = 2\varepsilon$, $y = \frac{1}{2}kx$. Then convexity implies

$$\sum_{\nu=1}^{m'} M[\tfrac{1}{2}k|x(\beta_\nu) - x(\alpha_\nu)|] \leqslant \tfrac{1}{2} \sum_{i=1}^{m} V_M(kx; t_{i-1}, t_i) < \varepsilon.$$

As an example of the applications of the last theorem we now give an inclusion-theorem for absolutely continuous functions.

2.12. *Denote by $\mathcal{CV}_M$ and $\mathcal{CV}_M^*$ the classes of all continuous functions belonging to $\mathcal{V}_M$ and $\mathcal{V}_M^*$, respectively.*

(a) *If $M(u)$ satisfies* (0), *then $\mathcal{AC}_M \subset \mathcal{CV}_M$ and $\mathcal{AC}_M^* \subset \mathcal{CV}_M^*$.*

(b) *If $M_2(u)/M_1(u) \to 0$ for $u \to 0$ and $M_2(u)$ satisfies* (o), *then $\mathcal{CV}_{M_1} \subset \mathcal{AC}_{M_2}$ and $\mathcal{CV}_{M_1}^* \subset \mathcal{AC}_{M_2}^*$.*

(c) *If $M_1(u)$ satisfies* (0) *and $M_2(u)/M_1(u) \to 0$ for $u \to 0$, then $\mathcal{AC}_{M_1} \subset \mathcal{AC}_{M_2}$ and $\mathcal{AC}_{M_1}^* \subset \mathcal{AC}_{M_2}^*$.*

It is easily seen that the inclusions for non-starred classes imply the same inclusions for starred classes. First, we prove (a) and (b) for non-starred classes. Assumption (0) implies the following condition: for every $a' > 0$ there exists an $L(a')$ such that $M(u) \leqslant L(a')u$ for $0 < u \leqslant a'$. We denote $L(b-a) = L'$ and choose a number $\delta > 0$ such that for every finite set of non-overlapping subintervals (α_i, β_i), $i = 1, 2, \ldots, m'$, of the interval $\langle a, b\rangle$, the inequality $\sum_1^{m'} M(\beta_i - \alpha_i) < \delta$ implies $\sum_1^{m'} M[|x(\beta_i) - x(\alpha_i)|] < 1$. Now, we fix a partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$ such that $t_i - t_{i-1} < \delta/L'$ for $i = 1, 2, \ldots, m$ and take an arbitrary partition $t_{i-1} = \tau_0 < \tau_1 < \ldots < \tau_p = t_i$ of $\langle t_{i-1}, t_i\rangle$. We have

$$\sum_{\nu=1}^{p} M(\tau_\nu - \tau_{\nu-1}) \leqslant L' \sum_{\nu=1}^{p} (\tau_\nu - \tau_{\nu-1}) = L'(t_i - t_{i-1}) < \delta.$$

Hence

$$\sum_{\nu=1}^{p} M[|x(\tau_\nu) - x(\tau_{\nu-1})|] < 1$$

and

$$V_M(x; t_{i-1}, t_i) \leqslant 1.$$

Applying 1.19 we obtain $V_M(x; a, b) < \infty$, i. e. $x \in \mathcal{V}_M$.

Now let us assume the condition $M_2(u)/M_1(u) \to 0$ for $u \to 0$ to be satisfied. Assume $x \in \mathcal{CV}_{M_1}$. On fixing an $\varepsilon > 0$ we choose a number $\delta' > 0$ such that $M_2(u)/M_1(u) < \varepsilon/V_{M_1}(x)$ for $0 < u \leqslant \delta'$. We now take a number $\eta > 0$ such that for $|h| < \eta$, $|x(t+h) - x(t)| < \delta'$ and put $\delta = M_2(\eta)$. Then for each set of non-overlapping subintervals (α_i, β_i), $i = 1, 2, \ldots, m$, of the interval $\langle a, b\rangle$ such that $\sum_1^m M_2(\beta_i - \alpha_i) < \delta$, we have $\beta_i - \alpha_i < \eta$ and, consequently, $|x(\beta_i) - x(\alpha_i)| < \delta'$. Hence

$$\sum_{i=1}^{m} M_2[|x(\beta_i) - x(\alpha_i)|] = \sum_{i=1}^{m} \frac{M_2[|x(\beta_i) - x(\alpha_i)|]}{M_1[|x(\beta_i) - x(\alpha_i)|]} M_1[|x(\beta_i) - x(\alpha_i)|] \leqslant \frac{\varepsilon}{V_{M_1}(x)} \sum_{i=1}^{m} M_1[|x(\beta_i) - x(\alpha_i)|] \leqslant \varepsilon$$

for $x(\alpha_i) \neq x(\beta_i)$. Thus, the function $x(t)$ belongs to $\mathcal{AC}_{M_2}$.

Property (c) follows from (a) and (b).

2.2. Approximation by step-functions. We now give three theorems on approximation in variation of absolute continuous functions by step-functions.

2.21. *We assume that $M(u)$ satisfies conditions* (o) *and* (Δ_2) *or* (o) *and* (c) *and* $x \in \mathcal{AC}_M^*$. *Then the following property holds:*

(+) *for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$, satisfying the condition $t_i - t_{i-1} < \delta$ for $i = 1, 2, \ldots, m$ and for every $t_{i-1} \leqslant \tau_i \leqslant t_i$, the step-function*

$$s(t) = \begin{cases} x(\tau_i) & \text{for} \quad t_{i-1} \leqslant t < t_i, \\ x(\tau_m) \text{ or } x(b) & \text{for} \quad t = b \end{cases}$$

satisfies the inequality $V_m[k'(x-s)] < \varepsilon$, where $k' > 0$ depends on x only.

We restrict the proof to the case of conditions (o) and (c). As follows from 2.11, there exists a $\delta > 0$ such that for each partition $a = t_0' < t_1' < \ldots < t_{m'}' = b$ such that $t_i' - t_{i-1}' < \delta$ for $i = 1, 2, \ldots, m'$,

$$\sum_1^{m'} M[k|x(t_i') - x(t_{i-1}')|] < 2\varepsilon.$$

We choose an arbitrary partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$, such that $t_i - t_{i-1} < \frac{1}{2}\delta$ for $i = 1, 2, \ldots, m$, and take $t_{i-1} \leqslant \tau_i \leqslant t_i$, where $i = 1, 2, \ldots, m$. We then consider the partition $a = \tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_m \leqslant \tau_{m+1} = b$ of $\langle a, b\rangle$. Since $\tau_i - \tau_{i-1} < \delta$ for $i = 1, 2, \ldots, m+1$, it follows that

$$\sum_{i=1}^{m+1} V_M(kx; \tau_{i-1}, \tau_i) \leqslant 2\varepsilon.$$

Evidently, $V_M(ks; \tau_{i-1}, \tau_i) \leqslant V_M(kx; \tau_{i-1}, \tau_i)$ for $i = 1, 2, \ldots, m+1$. Applying 1.18 to the function $\frac{1}{2}(x-s)$, we obtain

$$\begin{aligned} V_M[\tfrac{1}{2}k(x-s)] &\leqslant \tfrac{1}{2}\sum_{i=1}^{m+1} V_M[\tfrac{1}{2}k(x-s); \tau_{i-1}, \tau_i] \\ &\leqslant \tfrac{1}{4}\sum_{i=1}^{m+1} [V_M(kx; \tau_{i-1}, \tau_i) + V_M(ks; \tau_{i-1}, \tau_i)] \\ &\leqslant \tfrac{1}{2}\sum_{i=1}^{m+1} V_M(kx; \tau_{i-1}, \tau_i) \leqslant \varepsilon. \end{aligned}$$

2.22. *Let us assume that $M(u)$ satisfies* (o) *and* (c). *A function $x(t)$ defined in $\langle a, b\rangle$, $x(a) = 0$, belongs to the class $\mathcal{AC}_M^*$ if and only if the property* (+) *given in 2.21 holds. The same statement is true if we assume* (Δ_2) *instead of* (c), *and put $\mathcal{AC}_M$ in place of $\mathcal{AC}_M^*$* (cf. also [5]).

The necessity follows from 2.21. To prove the sufficiency we choose two partitions of the interval $\langle a, b\rangle$, Π: $a = t_0 < t_1 < \ldots < t_m = b$ and Π': $a = t_0' < t_1' < \ldots < t_k' = b$, both satisfying (+) with a given $\varepsilon > 0$ and such that $t_i \neq t_j'$ for $i = 1, 2, \ldots, m-1$; $j = 1, 2, \ldots, k-1$ and $\tau_k' = b$. Moreover, we denote by $s_\Pi(t)$ and $s_{\Pi'}(t)$ certain step-functions corresponding, according to (+), to the partitions Π and Π', respectively. If we take numbers $\alpha_i < \beta_i$ such that $\langle \alpha_i, \beta_i\rangle \subset (t_{j-1}', t_j')$ and $t_i = \frac{1}{2}(\alpha_i + \beta_i)$ for $i = 1, 2, \ldots, m-1$ and $\langle \alpha_m, \beta_m\rangle \subset (t_{k-1}', b\rangle$ with $\beta_m = b$, $\beta_0 = a$, we obtain, according to 1.17,

$$\sum_{i=1}^{m} V_M(k'x; \alpha_i, \beta_i) = \sum_{i=1}^{m} V_M[k'(x - s_{\Pi'}); \alpha_i, \beta_i] \leqslant V_M[k'(x - s_{\Pi'})] < \varepsilon$$

and

$$\sum_{i=1}^{m} V_M(k'x; \beta_{i-1}, \alpha_i) = \sum_{i=1}^{m} V_M[k'(x - s_\Pi); \beta_{i-1}, \alpha_i] \leqslant V_M[k'(x - s_\Pi)] < \varepsilon,$$

the functions $s_\Pi(t)$ and $s_{\Pi'}(t)$ being constants within the intervals $\langle \alpha_i, \beta_i\rangle$ and $\langle \beta_{i-1}, \alpha_i\rangle$, respectively. Hence

$$\sum_{i=1}^{m} [V_M(k'x; \beta_{i-1}, \alpha_i) + V_M(k'x; \alpha_i, \beta_i)] < 2\varepsilon$$

and our theorem follows from 2.11.

2.23. *Assuming* (o) *and* (Δ_2) *or* (o) *and* (c), *the set $\mathcal{W}$ of all step-functions with rational complex values having in (a, b) rational points of discontinuity only is dense in variation in $\mathcal{AC}_M^*$, i. e. for each $x \in \mathcal{AC}_M^*$ there exists a $k'' > 0$ such that for every $\varepsilon > 0$ there exists a function $s' \in \mathcal{W}$ such that $V_M[k''(x - s')] < \varepsilon$.*

To prove this theorem, it is sufficient to choose, for a given $x \in \mathcal{AC}_M^*$ and a given $\varepsilon > 0$, a function s as in 2.21 with rational t_i and $\tau_i = t_i$ and to take such rational complex numbers w_i, $i = 0, 1, \ldots, m$, that

$$M[k'|s(t) - w_i|] < \frac{\varepsilon}{m+1} \quad \text{for} \quad t_{i-1} \leqslant t < t_i,$$

$$M[k'|s(b) - w_0|] < \frac{\varepsilon}{m+1},$$

and

$$s'(t) = \begin{cases} w_i & \text{for} \quad t_{i-1} \leqslant t < t_i, \\ w_0 & \text{for} \quad t = b. \end{cases}$$

Indeed, given an arbitrary partition $a = \tau_0' < \tau_1' < \ldots < \tau_{m'}' = b$, we have

$$\sum_{i=1}^{m'} M[\tfrac{1}{2}k'|s(\tau_i') - s'(\tau_i') - s(\tau_{i-1}') + s'(\tau_{i-1}')|] \leqslant 2\sum_{i=0}^{m} M[k'|s(t_i) - s'(t_i)|] < 2\varepsilon,$$

i. e.

$$V_M[\tfrac{1}{2}k'(x-s')] \leqslant 2\varepsilon.$$

Applying 1.11 or 1.12, we easily obtain the required inequality.

2.3. Completeness and separability in variation. Now we shall consider the problems of linearity, completeness and separability in variation of the classes $\mathscr{AC}_M$ and $\mathscr{AC}_M^*$. The following theorem may be proved similarly to 1.13:

2.31. (c) *implies the linearity of* $\mathscr{AC}_M^*$ *and* (Δ_2) *the linearity of* $\mathscr{AC}_M$.

We now proceed to theorems dealing with completeness and separability in variation of spaces $\mathscr{AC}_M$ and $\mathscr{AC}_M^*$.

2.32. *If* $M(u)$ *satisfies* (Δ_2) *or* (c), *then the class* $\mathscr{AC}_M^*$ *is complete in variation, i. e. every sequence* $x_n \in \mathscr{AC}_M^*$ *satisfying the Cauchy condition in variation is convergent in variation to an* $x \in \mathscr{AC}_M^*$.

The proof of this theorem is analogous to that of 1.22.

2.33. *If* $M(u)$ *satisfies the conditions* (o) *and* (Δ_2) *or* (o) *and* (c), *then the class* $\mathscr{AC}_M^*$ *is separable in variation.*

To prove the separability, let us take the set $\mathcal{W}$: $s_1'(t), s_2'(t), \ldots$ defined in 2.23. Now, on fixing two positive integers m and n and a positive rational number k, we choose a function $x_{n,m}^{(k)} \in \mathscr{AC}_M^*$ such that

$$V_M[k(s_n' - x_{n,m}^{(k)})] < \frac{1}{m},$$

if such a function exists. Denoting by $\mathcal{W}_1$ the set of all such functions $x_{n,m}^{(k)}(t)$, we shall prove that, given an element $x \in \mathscr{AC}_M^*$, there exists a subsequence of the sequence $x_{n,m}^{(k)}(t)$ convergent to x in variation. Put, in 2.23, $\varepsilon = 1/m$ and choose the positive number k'' rational. There exists an n such that $V_M[k''(x - s_n')] < 1/m$. It follows from the definition of the set $\mathcal{W}_1$ that there exists a function $x_{n,m}^{(k)}(t)$ such that $V_M[k''(s_n' - x_{n,m}^{(k)})] < 1/m$. Supposing (c), we obtain from 1.12

$$V_M[\tfrac{1}{2}k''(x - x_{n,m}^{(k)})] \leqslant \tfrac{1}{2}\{V_M[k''(x-s_n')] + V_M[k''(s_n' - x_{n,m}^{(k)})]\} < \frac{1}{m}.$$

In the case of (Δ_2) we apply 1.11 instead of 1.12.

2.4. Absolute continuity and the variation of the translated function; approximation by Steklov functions. Here we shall assume the function $x = x(t)$ to be periodic with period $b-a$. This assumption will be made in order to simplify the calculations; our theorems hold also for non-periodic functions. Further, we write $x_h(t) = x(t+h)$, and term x_h the *translated function* to x. We shall prove the following theorem:

2.41. *Let us assume that* $M(u)$ *satisfies* (o) *and* (c), *and let* $x = x(t)$, $x(a) = 0$, *be a measurable and periodic function with period* $b-a$. *Then* $x \in \mathscr{AC}_M^*$ *if and only if the translated functions* x_h *converge to* x *in variation as* $h \to 0+$, *i. e. if there exists a* $k' > 0$ (*depending on* x) *such that* $V_M[k'(x_h - x)] \to 0$ *for* $h \to 0+$.

The necessity follows from the following theorem:

2.42. *If* $M(u)$ *satisfies* (o) *and* (Δ_2) *or* (o) *and* (c) *and* $x \in \mathscr{AC}_M^*$ *is periodic with period* $b-a$, *then* $V_M[k'(x_h - x)] \to 0$ *for* $h \to 0+$, *where* k' *is a positive constant depending on* x *only.*

To prove 2.42 in the case of conditions (o) and (c), we fix a number $\varepsilon > 0$ and choose a step-function $s(t)$ as in 2.21 with $\tau_i = t_{i-1}$:

$$s(t) = \begin{cases} x(t_{i-1}) & \text{for} \quad t_{i-1} \leqslant t < t_i,\ i = 1, 2, \ldots, m, \\ x(b) & \text{for} \quad t = b \end{cases}$$

such that $V_M[k(x-s)] < \varepsilon$ with a $k > 0$ and $\sum_1^m V_M(kx; t_{i-1}, t_i) < \varepsilon$ (see 2.11, $\mathscr{AC}_M^* \overset{(o)}{\to} 1^*$). If we put $x(t+h) = x_h(t)$ and $s(t+h) = s_h(t)$ (with a periodical continuation of $s(t)$ for t not belonging to $\langle a, b\rangle$), we have, according to 1.17,

$$V_M[\tfrac{1}{2}k(x_h - s_h)] \leqslant V_M[k(x-s)] < \varepsilon.$$

Now let us take a number h such that $0 < h < \delta = \min(t_i - t_{i-1})$, where $i = 1, 2, \ldots, m$. Then $s(t_i + h) - s(t_i) = 0$ for $i = 1, 2, \ldots, m-1$, and 1.18 implies

$$V_M[\tfrac{1}{8}k(s_h - s)] \leqslant \tfrac{1}{2}\sum_{i=1}^{m} V_M[\tfrac{1}{4}k(s_h - s; t_{i-1}, t_i)]$$
$$\leqslant \tfrac{1}{2}\sum_{i=1}^{m}[V_M(\tfrac{1}{2}ks_h; t_{i-1}, t_i) + V_M(\tfrac{1}{2}ks; t_{i-1}, t_i)]$$
$$\leqslant \sum_{i=1}^{m} V_M(ks; t_{i-1}, t_i) \leqslant \sum_{i=1}^{m} V_M(kx; t_{i-1}, t_i) < \varepsilon.$$

Hence 1.12 implies

$$V_M[\tfrac{1}{24}k(x_h-x)] \leqslant \tfrac{1}{3}\{V_M[\tfrac{1}{8}k(x_h-s_h)]+V_M[\tfrac{1}{8}k(s_h-s)]+V_M[\tfrac{1}{8}k(s-x)]\} < \varepsilon$$

for $0 < h < \delta$. The proof in the case of conditions (o) and (Δ_2) will be left to the reader.

To prove the sufficiency of 2.41 we need the following lemmas; to formulate them we must introduce the following definitions:

(*) *The function $x(t)$ defined in $\langle a, b\rangle$ satisfies condition (*) if it is measurable, periodic with period $b-a$ and if there exists a constant $k' > 0$, depending only on x such that $V_M[k'(x_h-x)] \to 0$ for $h \to 0+$, where $x_h(t) = x(t+h)$.*

2.43. *If $M(u) \to \infty$ as $u \to \infty$, then every function $x(t)$ satisfying condition (*) is bounded.*

Condition (*) yields $V_M[k'(x_h-x)] \leqslant K$ for certain $k' > 0$, $K > 0$ and $0 \leqslant h \leqslant 2\varepsilon$. We choose a positive number n such that $M(k'n) > K$. First, we prove the function $x(t)$ to be bounded in a neighbourhood of the point a. Let us assume $\overline{\lim\limits_{t\to a+}}|x(t)| = \infty$. Choose a function $\varphi(t)$, continuous in $\langle a, a+2\varepsilon\rangle$ and such that $x(t) = \varphi(t)$ in a set $E \subset \langle a, a+2\varepsilon\rangle$ of measure $mE > \frac{5}{4}\varepsilon$, $\varphi(a) = x(a)$. There exists a $\delta > 0$ such that, for $|t'-t''| < \delta$, $|\varphi(t')-\varphi(t'')| < \varepsilon$. We choose a number $t_0 \in \langle a, a+\min(\delta, \frac{1}{2}\varepsilon)\rangle$ such that $|x(t_0)-x(a)| > n+\varepsilon$. We now apply the following well-known lemma:

If we denote by E_t the set of all numbers h such that $t+h \in E$, where $E \subset \langle a, a+2\varepsilon\rangle$, $mE > \frac{5}{4}\varepsilon$, then $m(E_t \cap E) > 0$ for $0 \leqslant t < \frac{1}{2}\varepsilon$. Hence there exists a positive number $h < 2\varepsilon-(t_0-a)$, $a+h \in E_{t_0-a} \cap E$, such that both equalities $x(t_0+h) = \varphi(t_0+h)$ and $x(a+h) = \varphi(a+h)$ hold simultaneously. Then we obtain $|x(t_0+h)-x(a+h)| = |\varphi(t_0+h)-\varphi(a+h)| < \varepsilon$ and

$$\begin{aligned} K \geqslant V_M[k'(x_h-x)] &\geqslant M[k'|x(t_0+h)-x(a+h)-x(t_0)+x(a)|] \\ &\geqslant M[k'||x(t_0)-x(a)|-|x(t_0+h)-x(a+h)||] \geqslant M(k'n), \end{aligned}$$

which contradicts our assumption; hence there exist numbers $\delta_1 > 0$ and $K_1 > 0$ such that $|x(t)| \leqslant K_1$ for $a \leqslant t \leqslant a+\delta_1$.

Now, it follows from our assumptions that $|x(t+h)-x(t)-x(a+h)+x(a)| \leqslant K_2$ for $0 \leqslant h \leqslant \varepsilon$ and for a constant K_2 not depending on t; hence $|x(t+h)| \leqslant |x(t)|+|x(a)|+K_1+K_2$ for $0 \leqslant h \leqslant \min(\delta_1, \varepsilon)$. We choose a partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$ such that $t_i-t_{i-1} \leqslant \min(\delta_1, \varepsilon)$ for $i = 1, 2, \ldots, m$ and write $K' = \max\limits_i|x(t_i)|+|x(a)|+K_1+K_2$. Then for $t \in \langle t_i, t_{i+1}\rangle$, $|x(t)| = |x[t_i+(t-t_i)]| \leqslant K'$.

Remark. It is easily seen that under the assumptions of 2.43 $x(t)$ is continuous. The proof is analogous to that of the boundedness of $x(t)$ given above.

2.44. *Let us assume that $M(u)$ satisfies (c), and $x(t)$ satisfies condition (*). Then the Steklov functions $x_n^s(t)$ of the function $x(t)$, defined by the formula*

$$x_n^s(t) = n\int_t^{t+1/n} x(\tau)\,d\tau$$

are convergent in variation to $x(t)$.

We take an arbitrary partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$. Writing $D(t,\tau) = x(t+\tau)-x(t)$, we apply the Jessen inequality for integrals

$$M\left\{\left|\frac{\int_0^1 F(\tau)p(\tau)\,d\tau}{\int_0^1 p(\tau)\,d\tau}\right|\right\} \leqslant \frac{\int_0^1 M[|F(\tau)|]p(\tau)\,d\tau}{\int_0^1 p(\tau)\,d\tau}$$

with

$$p(\tau) = \begin{cases} n & \text{for} \quad 0 \leqslant \tau \leqslant 1/n, \\ 0 & \text{for} \quad \tau > 1/n, \end{cases}$$

$$F(\tau) = \begin{cases} k'|D(t_i,\tau)-D(t_{i-1},\tau)| & \text{for} \quad 0 \leqslant \tau \leqslant 1/n, \\ 0 & \text{for} \quad \tau > 1/n, \end{cases}$$

obtaining

$$M\left[nk'\int_0^{1/n}|D(t_i,\tau)-D(t_{i-1},\tau)|\,d\tau\right] \leqslant n\int_0^{1/n} M[k'|D(t_i,\tau)-D(t_{i-1},\tau)|]\,d\tau.$$

Since

$$x_n^s(t)-x(t) = n\int_0^{1/n} D(t,\tau)\,d\tau,$$

it follows that

$$\begin{aligned} &\sum_{i=1}^m M[k'|x_n^s(t_i)-x(t_i)-x_n^s(t_{i-1})+x(t_{i-1})|] \\ &\qquad = \sum_{i=1}^m M\left\{nk'\left|\int_0^{1/n}[D(t_i,\tau)-D(t_{i-1},\tau)]\,d\tau\right|\right\} \\ &\qquad \leqslant n\int_0^{1/n}\sum_{i=1}^m M[k'|D(t_i,\tau)-D(t_{i-1},\tau)|]\,d\tau. \end{aligned}$$

However,
$$\sum_{i=1}^{m} M[k'|D(t_i,\tau)-D(t_{i-1},\tau)|] \leqslant V_M[k'(x_\tau - x)] < \varepsilon$$
for sufficiently small τ. Thus, $V_M[k'(x_n^s - x)] \leqslant \varepsilon$ for sufficiently large n.

Now we are able to prove the sufficiency of 2.41.

2.45. *If $M(u)$ satisfies* (c), *$x(t)$ satisfies condition* (*), *and* $x(a)=0$, *then $x \in \mathscr{AC}_M^*$.*

Since the Steklov functions $x_n^s(t)$ of a bounded function $x(t)$ satisfy the Lipschitz condition, 2.43 implies that $x_n^s(t)-x_n^s(a)$ belongs to $\mathscr{AC}_M^*$. Further, since the sequence x_n^s converges in variation to x, 1.22 implies that $x_n^s(t)-x_n^s(a)$ satisfies the Cauchy condition in variation and it follows from 2.32 that $x \in \mathscr{AC}_M^*$.

Theorem 2.41 and lemma 2.44 imply the following approximation theorem for Steklov functions:

2.46. *If $M(u)$ satisfies* (o) *and* (c), *then for every periodic function $x \in \mathscr{AC}_M^*$ with period $b-a$ the Steklov functions of x converge in M-variation to x.*

2.5. Approximation by singular integrals. Now we shall consider the problem of approximation in variation of periodic functions $x \in \mathscr{AC}_M^*$ by singular integrals of the form
$$I_n(t) = \int_a^b K_n(\tau)x(t+\tau)\,d\tau.$$

2.51. *If $K_n(t) \geqslant 0$ and $\int_a^b K_n(t)\,dt = \vartheta_n$, $M(u)$ satisfies* (c) *and $x(t)$ is periodic with period $b-a$, then*
$$V_M(I_n) \leqslant V_M(2\vartheta_n x).$$

To prove this theorem we apply the Jessen inequality (see the proof of 2.44), obtaining for a given partition $a=t_0<t_1<\ldots<t_m=b$ of the interval $\langle a, b\rangle$:
$$\sum_{i=1}^{m} M[|I_n(t_i)-I_n(t_{i-1})|] \leqslant \sum_{i=1}^{m} M\Big[\int_a^b |x(t_i+\tau)-x(t_{i-1}+\tau)|K_n(\tau)\,d\tau\Big]$$
$$\leqslant \int_a^b \frac{K_n(\tau)}{\vartheta_n} \sum_{i=1}^{m} M[\vartheta_n|x(t_i+\tau)-x(t_{i-1}+\tau)|]\,d\tau$$
$$\leqslant V_M(2\vartheta_n x),$$
whence $V_M(I_n) \leqslant V_M(2\vartheta_n x)$.

2.52. *If $K_n(t) \geqslant 0$, $\int_a^b K_n(t)\,dt \to 1$ as $n\to\infty$ and $\int_{a+\delta}^{b-\delta} K_n(t)\,dt \to 0$ as $n \to \infty$ for each $0<\delta<\frac{1}{2}(b-a)$, $M(u)$ satisfies* (o) *and* (c) *and $x \in \mathscr{AC}_M^*$ is periodic with period $b-a$, then $I_n(t)$ converges to $x(t)$ in M-variation.*

We have
$$I_n(t)-x(t) = \left(1-\frac{1}{\vartheta_n}\right)I_n(t) + \int_a^b \frac{K_n(\tau)}{\vartheta_n}[x(t+\tau)-x(t)]\,d\tau,$$
where $\vartheta_n = \int_a^b K_n(t)\,dt$. Moreover, 2.51 implies
$$V_M\left[\frac{k}{2}\left(1-\frac{1}{\vartheta_n}\right)I_n\right] \leqslant V_M[k(\vartheta_n-1)x] \leqslant |\vartheta_n-1|\,V_M(kx)$$
for $|\vartheta_n-1|<1$.

Let us choose a number $0<\delta<\frac{1}{2}(b-a)$. As in the proof of 2.51, we have
$$\sum_{i=1}^{m} M\left\{\frac{k}{2}\left|\int_a^b \frac{K_n(\tau)}{\vartheta_n}[x(t_i+\tau)-x(t_i)]\,d\tau - \int_a^b \frac{K_n(\tau)}{\vartheta_n}[x(t_{i-1}+\tau)-x(t_{i-1})]\,d\tau\right|\right\} \leqslant J_n(a,b),$$
where
$$J_n(a,b) = \int_a^b \frac{K_n(\tau)}{\vartheta_n} \sum_{i=1}^{m} M[\tfrac{1}{2}k|x(t_i+\tau)-x(t_i)-x(t_{i-1}+\tau)+x(t_{i-1})|]\,d\tau.$$

We write $J_n(a,b) = J_n(a,a+\delta)+J_n(a+\delta,b-\delta)+J_n(b-\delta,b)$. The following inequality holds:
$$J_n(a+\delta,b-\delta) \leqslant \tfrac{1}{2}V_M(kx)\frac{1}{\vartheta_n}\int_{a+\delta}^{b-\delta} K_n(\tau)\,d\tau.$$
Moreover,
$$J_n(a,a+\delta) \leqslant \sup_{0<\tau\leqslant\delta} V_M[\tfrac{1}{2}k(x_\tau - x)],$$
and similarly
$$J_n(b-\delta,b) \leqslant \sup_{0<\tau\leqslant\delta} V_M[\tfrac{1}{2}k(x_\tau - x)].$$
Hence
$$J_n(a,b) \leqslant 2\sup_{0<\tau\leqslant\delta} V_M[\tfrac{1}{2}k(x_\tau - x)] + \frac{1}{2\vartheta_n}V_M(kx)\int_{a+\delta}^{b-\delta} K_n(\tau)\,d\tau.$$

Applying 2.42, let us now fix a $\delta > 0$ such that

$$\sup_{0<\tau\leqslant\delta} V_M[\tfrac{1}{2}k(x_\tau - x)] < \tfrac{1}{4}\varepsilon$$

and a number N such that, for $n > N$,

$$\int_{a+\delta}^{b-\delta} K_n(\tau)\,d\tau < \frac{2\vartheta_n\varepsilon}{V_M(kx)}$$

and

$$|\vartheta_n - 1| < \min\left\{\frac{\varepsilon}{2V_M(kx)}, 1\right\}.$$

Then $J_n(a, b) < \frac{3}{2}\varepsilon$ and

$$V_M\left\{\frac{k}{2}\int_a^b \frac{K_n(\tau)}{\vartheta_n}[x(t+\tau) - x(t)]\,d\tau\right\} \leqslant \frac{3}{2}\varepsilon.$$

Thus 1.12 implies

$$V_M[\tfrac{1}{4}k(I_n - x)] < \varepsilon \quad \text{for} \quad n > N.$$

Theorem 2.52 may be applied to singular integrals of Fejér, de la Vallée Poussin, etc.

3. The norm for functions of bounded variation. Here we shall always be assuming that the function $M(u)$ satisfies condition (c). We introduce in $\mathcal{V}_M^*$ the *norm* $\|x\|_M$ (or shortly $\|x\|$) as the infimum of the set of numbers $k > 0$ such that $V_M(x/k) \leqslant 1$. It is easily seen that

3.01. *If* $M(u) = u^p$ *for* $p \geqslant 1$, *then*

$$\|x\| = \left[\sup_\Pi \sum_{i=1}^m |x(t_i) - x(t_{i-1})|^p\right]^{1/p}.$$

3.02. *If* $\|x\| \neq 0$, *then* $V_M(x/\|x\|) \leqslant 1$. *Moreover*, $V_M(x) \leqslant \|x\|$ *for* $\|x\| \leqslant 1$.

3.03. *The norm* $\|x\|$ *satisfies the usual properties of the B-norm*: a) $\|x\| = 0$ *if and only if* $x(t) \equiv 0$, b) $\|x+y\| \leqslant \|x\| + \|y\|$, c) $\|ax\| = |a|\|x\|$.

The triangle-inequality b) follows from the inequalities:

$$\sum_{i=1}^m M\left[\frac{|x(t_i) + y(t_i) - x(t_{i-1}) - y(t_{i-1})|}{\|x\| + \|y\|}\right]$$
$$\leqslant \frac{\|x\|}{\|x\|+\|y\|}\sum_{i=1}^m M\left[\frac{1}{\|x\|}|x(t_i) - x(t_{i-1})|\right] + $$
$$+ \frac{\|y\|}{\|x\|+\|y\|}\sum_{i=1}^m M\left[\frac{1}{\|y\|}|y(t_i) - y(t_{i-1})|\right] \leqslant 1,$$

whence $V_M[(x+y)/(\|x\|+\|y\|)] \leqslant 1$ and $\|x+y\| \leqslant \|x\| + \|y\|$. The properties a) and c) are trivial.

3.1. Norm-convergence and convergence in variation. We have the following connections between norm-convergence and convergence in variation.

3.11. *If* $M(u)$ *satisfies* (c), *then norm-convergence implies convergence in variation to the same limit in* $\mathcal{V}_M^*$. *When* $M(u)$ *satisfies* (c) *and* (Δ_2), *norm-convergence and convergence in variation are equivalent in* $\mathcal{V}_M^*$.

The first part of this theorem follows from 3.02, and the second from 1.11.

3.2. Completeness and separability. The following theorem holds:

3.21. *If* (c) *is assumed, the class* $\mathcal{V}_M^*$ *with norm* $\|x\|$ *and the usual definitions of addition and scalar-multiplication of elements is a complete and non-separable Banach space. Moreover, if we assume* (0) *and* (c), $\mathcal{AC}_M^*$ *is a complete subspace of* $\mathcal{V}_M^*$. *If we assume* (o), (c) *and* (Δ_2), $\mathcal{AC}_M^*$ *is separable.*

First we prove the completeness of $\mathcal{V}_M^*$. We fix a number $\varepsilon > 0$ and choose n_0 such that for $m, n > n_0$, $\|x_m - x_n\| < \varepsilon$, i. e.

$$V_M[(x_m - x_n)/\varepsilon] \leqslant 1 \quad \text{and} \quad \sum_{i=1}^m M[|x_m(t_i) - x_n(t_i) - x_m(t_{i-1}) + x_n(t_{i-1})|/\varepsilon] \leqslant 1$$

for an arbitrary partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$. It follows that $x_n(t)$ is uniformly convergent to a function $x(t)$ in $\langle a, b\rangle$ and that $V_M[(x_n - x)/\varepsilon] \leqslant 1$ for $n > n_0$. Hence $\|x_n - x\| \leqslant \varepsilon$ for $n > n_0$, where $x \in \mathcal{V}_M^*$, $\mathcal{V}_M^*$ being linear.

The non-separability of $\mathcal{V}_M^*$ follows from the inequality $\|x_{t_0'} - x_{t_0''}\| > \min[M(1), 1]$, holding for $a < t_0' < t_0'' < b$, where x_{t_0} is defined as in the proof of 1.23.

The completeness of $\mathcal{AC}_M^*$ follows from that of $\mathcal{V}_M^*$ and from 3.02 and 2.32. The separability of $\mathcal{AC}_M^*$ follows from 3.11 and 2.33.

3.3. Second definition of the norm. Here we shall assume that $M(u)$ satisfies conditions (o), (∞) and (c) and denote by $N(u)$ the function, complementary to $M(u)$, defined in the introduction. We introduce in $\mathcal{V}_M^*$ a second *norm* $\|x\|_M^0$ (or shortly $\|x\|^0$) as the supremum of the set of numbers

$$\sum_{i=1}^m [x(t_i) - x(t_{i-1})]\beta_i,$$

whenever $a = t_0 < t_1 < \ldots < t_m = b$, $\sum_1^m N(|\beta_i|) \leqslant 1$ and $m = 1, 2, \ldots$

The following theorem establishes the equivalence of the norms $\|x\|$ and $\|x\|^0$.

3.31. *Let us assume* (o), (∞) *and* (c); *then the inequalities*
$$\|x\| \leqslant \|x\|^0 \leqslant 2\|x\|$$
hold for each $x \in \mathcal{V}_M^*$.

This theorem results immediately from the first part of the following lemma:

3.32. *Let us assume* (o), (∞) *and* (c), *and denote by* $N(u)$ *the function complementary to* $M(u)$, *and introduce norms in the class* $\mathcal{A}_M$ *of all vectors* $a = (a_1, \ldots, a_m)$ (m *fixed*) *by the formulas*

$\|a\|_M$ = *infimum of the set of numbers* $k > 0$ *such that* $\sum_1^m M\left(\frac{|a_i|}{k}\right) \leqslant 1$,

$\|a\|_M^0$ = *supremum of the set of numbers* $\sum_1^m a_i\beta_i$ *whenever* $\sum_1^m N(|\beta_i|) \leqslant 1$.

Then $\mathcal{A}_M$ *is a Banach space under each of these norms,*
$$\|a\|_M \leqslant \|a\|_M^0 \leqslant 2\|a\|_M,$$
and $F(a) = \sum_1^m a_i b_i$ *is a linear functional over* $\mathcal{A}_M$ *such that the norm*
$$\|F\| = \sup_{\|a\|_M^0 \leqslant 1} |F(a)|$$
satisfies the inequalities
$$\|F\| \leqslant \|b\|_N^0 \leqslant 2\|F\|.$$

This lemma being known, we give its proof only for completeness. The Young inequality $uv \leqslant M(u)+N(v)$ implies
$$\frac{1}{\|a\|_M}\sum_{i=1}^m a_i\beta_i \leqslant \sum_{i=1}^m M\left(\frac{|a_i|}{\|a\|_M}\right)+\sum_{i=1}^m N(|\beta_i|) \leqslant 2 \quad \text{for} \quad \sum_{i=1}^m N(|\beta_i|) \leqslant 1.$$
Hence
$$\|a\|_M^0 = \sup \sum_1^m a_i\beta_i \leqslant 2\|a\|_M,$$
where the supremum is taken over all vectors $(\beta_1, \ldots, \beta_m)$ such that $\sum_1^m N(|\beta_i|) \leqslant 1$.

Now let us choose, for an $\|a\|_M^0 \neq 0$, numbers $\beta_1, \ldots, \beta_m$ such that
$$\frac{1}{\|a\|_M^0}\sum_{i=1}^m a_i\beta_i = \sum_{i=1}^m M\left(\frac{|a_i|}{\|a\|_M^0}\right)+\sum_{i=1}^m N(|\beta_i|).$$

It is easily verified that such numbers $\beta_1, \ldots, \beta_m$ exist (see e. g. [1], p. 15). We now prove that $\sum_1^m N(|\beta_i|) \leqslant 1$. Let us suppose $\varrho = \sum_1^m N(|\beta_i|) > 1$. The convexity of $N(u)$ implies
$$\sum_{i=1}^m N\left(\frac{|\beta_i|}{\varrho}\right) \leqslant \frac{1}{\varrho}\sum_{i=1}^m N(|\beta_i|),$$
whence
$$\sum_{i=1}^m a_i\frac{\beta_i}{\varrho} \leqslant \|a\|_M^0$$
and
$$\sum_{i=1}^m M\left(\frac{|a_i|}{\|a\|_M^0}\right)+\varrho \leqslant \varrho.$$
This implies $a_i = 0$ for $i = 1, 2, \ldots, m$ and $\|a\|_M^0 = 0$, which contradicts our assumptions. Thus $\sum_1^m N(|\beta_i|) \leqslant 1$ and $\sum_1^m a_i\beta_i \leqslant \|a\|_M^0$. This implies
$$\sum_{i=1}^m M\left(\frac{|a_i|}{\|a\|_M^0}\right) \leqslant 1-\sum_{i=1}^m N(|\beta_i|) \leqslant 1$$
and $\|a\|_M \leqslant \|a\|_M^0$.

Now we consider the functional $F(a) = \sum_1^m a_i b_i$. The definition of $\|b\|_N^0$ yields
$$\sum_{i=1}^m \frac{a_i b_i}{\|a\|_M} \leqslant \|b\|_N^0$$
and
$$\sum_{i=1}^m a_i b_i \leqslant \|b\|_N^0\|a\|_M \leqslant \|b\|_N^0\|a\|_M^0.$$
Therefore $F(a)$ is linear, and $\|F\| \leqslant \|b\|_N^0$. Now it is easily seen that for $\sum_1^m M(|a_i|) \leqslant 1$, $\|a\|_M^0 \leqslant 2$. Hence $\sum_1^m a_i b_i \leqslant 2\|F\|$ for $\sum_1^m M(|a_i|) \leqslant 1$ and the definition of $\|b\|_N^0$ yields $\|b\|_N^0 \leqslant 2\|F\|$.

From 1.24 and the definition of the norm follows the theorem

3.33. *If* $x_n(t) \to x(t)$ *in* $\langle a, b\rangle$ *and* $x_n \in \mathcal{V}_M^*$, $M(u)$ *satisfying* (c), *then* $\|x\| \leqslant \varliminf_{n\to\infty}\|x_n\|$. *If, moreover,* $M(u)$ *satisfies* (o) *and* (∞), *then* $\|x\|^0 \leqslant \varliminf_{n\to\infty}\|x_n\|^0$.

Now we give some remarks dealing with weak convergence in $\mathcal{V}^*_M$. Put $\xi_t(x) = x(t)$,

$$\eta(x) = \sum_{i=1}^{m} [\xi_{t_i}(x) - \xi_{t_{i-1}}(x)] b_i$$

with $\sum_1^m N(|b_i|) \leqslant 1$, $a = t_0 < t_1 < \ldots < t_m = b$, $m = 1, 2, \ldots$, and denote by Ξ_0 the set of all linear functionals over $\mathcal{V}^*_M$ of the form $\eta(x)$.

3.34. *If we assume* (c), (o) *and* (∞), *then* $\|\eta\| \leqslant 2$ *for each* $\eta \in \Xi_0$ *and* $\sup_{\eta \in \Xi_0} |\eta(x)| \geqslant \|x\|$, *i. e. the set* Ξ_0 *of linear functionals over* $\mathcal{V}^*_M$ *is fundamental.*

This theorem easily follows from 3.31.

3.35. *If* $M(u)$ *satisfies* (c), (o) *and* (∞), *then every sequence of elements of* $\mathcal{V}^*_M$ *bounded with respect to the norm contains a subsequence weakly convergent with respect to the fundamental set of linear functionals* Ξ_0. *Moreover, weak convergence of a subsequence* $x_n \in \mathcal{V}^*_M$ *with respect to the fundamental set* Ξ_0 *implies*

$$\|x\| \leqslant \varliminf_{n\to\infty} \|x_n\| \quad \textit{and} \quad \|x\|^0 \leqslant \varliminf_{n\to\infty} \|x_n\|^0.$$

Theorem 3.35 is obtained by applying 1.3 and 3.33.

4. Linear functionals. Here we shall prove some theorems on Stieltjes integrals and give the form of the linear functional over $\mathcal{AC}_M$; the notion of the Stieltjes integral will be used in the sense of Riemann-Stieltjes.

4.1. Sequences of Stieltjes integrals. First, we prove the following theorem:

4.11. *Let us assume that* $M(u)$ *satisfies* (c), (o) *and* (∞), *and that* $y(t)$ *is continuous in* $\langle a, b\rangle$, $y(a) = 0$. *If* $|\int_a^b x\,dy| \leqslant K\|x\|_M$ *for each step-function* $x(t)$ *or for each polygonal function* $x(t)$, *where* $x(a) = 0$, *then* $\|y\|_N \leqslant 8K$, *and thus* $V_N(y/8K) \leqslant 1$, $N(u)$ *being the function complementary to* $M(u)$.

To prove this theorem we first assume the inequality $|\int_a^b x\,dy| \leqslant K\|x\|_M$ to be satisfied for each step-function $x(t)$ vanishing at a. We choose an arbitrary partition $a = t_0 < t_1 < \ldots < t_m = b$ of the interval $\langle a, b\rangle$ and a step-function $x(t) = a_i$ for $t_{i-1} < t \leqslant t_i$, $i = 1, 2, \ldots, m$, $x(a) = a_0 = 0$. We have

$$\|x\|_M^0 = \sup \sum_i (a_{n_i} - a_{n_{i-1}})\beta_i \leqslant \sup \Big(\sum_i |a_{n_i}|\beta_i + \sum_i |a_{n_{i-1}}|\beta_i \Big),$$

where the supremum is taken over all $\beta_1, \beta_2, \ldots, \beta_{m'}$ such that $\sum_1^{m'} N(|\beta_i|) \leqslant 1$ and all subsequences n_i of the finite sequence $0, 1, 2, \ldots, m$, where $n_0 = 0$, $n_{m'} = m$. Hence

$$\|x\|_M^0 \leqslant 2 \sup \sum_1^m a_i \beta_i',$$

where β_i' are such that $\sum_1^m N(|\beta_i'|) \leqslant 1$. Since

$$\int_a^b x\,dy = \sum_{i=1}^m a_i [y(t_i) - y(t_{i-1})],$$

we obtain

$$\Big|\sum_{i=1}^m a_i [y(t_i) - y(t_{i-1})]\Big| \leqslant K\|x\|_M \leqslant K\|x\|_M^0 \leqslant 2K \sup \sum_{i=1}^m a_i \beta_i',$$

where $\sum_1^m N(|\beta_i'|) \leqslant 1$. Now we apply the second part of lemma 3.32 with $b_i = y(t_i) - y(t_{i-1})$ and $F(a) = \sum_1^m a_i b_i$. We obtain $\|b\|_N^0 \leqslant 2\|F\| \leqslant 4K$, whence

$$\sum_{i=1}^m a_i [y(t_i) - y(t_{i-1})] \leqslant 4K$$

for arbitrary a_i such that $\sum_1^m M(|a_i|) \leqslant 1$. This yields $\|y\|_N^0 \leqslant 4K$. Theorem 3.31 implies $\|y\|_N \leqslant 4K$ and we obtain $V_M(y/4K) \leqslant 1$.

Now let us assume the inequality $|\int_a^b x\,dy| \leqslant K\|x\|_M$ to be valid for each polygonal function $x(t)$. Then 4.11 results from the following lemma:

4.12. *If* $M(u)$ *satisfies condition* (o), $y(t)$ *is continuous in* $\langle a, b\rangle$ *and the inequality* $|\int_a^b x\,dy| \leqslant K\|x\|_M$ *holds for each polygonal function* $x(t)$, $x(a) = 0$, *then the inequality*

$$\Big|\int_a^b x\,dy\Big| \leqslant 2K\|x'\|_M + |x(a)| \cdot |y(b) - y(a)|$$

holds for each step-function $x(t)$. *Here* $x'(t) = x(t) - x(a)$.

Take an arbitrary partition $a = t_0 < t_1 < \ldots < t_m = b$ and put $x(t) = a_i$ for $t_{i-1} < t \leqslant t_i$, where $i = 1, 2, \ldots, m$, $x(a) = a_0 = 0$. Write

$$x_n(t) = \begin{cases} a_i & \text{for} \quad t_{i-1} + \frac{1}{n} \leqslant t \leqslant t_i, \; i = 1, 2, \ldots, m, \\ n(a_{i+1} - a_i)(t - t_i) + a_i & \text{for} \quad t_i < t < t_i + \frac{1}{n}, \; i = 0, 1, \ldots, m-1, \\ 0 & \text{for} \quad t = a, \end{cases}$$

where the integer n is so large that $t_i - t_{i-1} > 1/n$ for $i = 1, 2, \ldots, m$. We have $\left|\int_a^b x_n dy\right| \leqslant K\|x_n\|_M$. However, 1.03 yields $V_M(x) = M[|x(b) - x(a)|]$ for monotonic $x(t)$. Hence it is easily seen that $V_M(x_n) \leqslant 2V_M(x)$, and this yields $\|x_n\|_M \leqslant 2\|x\|_M$. Hence we obtain $\left|\int_a^b x_n dy\right| \leqslant 2K\|x\|_M$. It is sufficient to prove $\int_a^b x_n dy \to \int_a^b x dy$. But

$$\int_a^b x_n dy - \int_a^b x dy$$
$$= \sum_{i=1}^{m} \left\{ (a_i - a_{i+1}) \left[y\left(t_i + \frac{1}{n}\right) - y(t_i) \right] - (a_i - a_{i+1}) y\left(t_i + \frac{1}{n}\right) + \right.$$
$$\left. + n(a_i - a_{i+1}) \int_{t_i}^{t_i + 1/n} y\, dt \right\}.$$

If, for the step-function $x(t)$, $x(a) \neq 0$, it is sufficient to apply our inequality to the function $x'(t)$.

It should be noted that 4.11 implies the following theorem for sequences of linear functionals:

4.13. *If $M(u)$ satisfies the conditions* (c), (o) *and* (∞), *$y_n(t)$ are continuous in $\langle a, b\rangle$, $y_n(a) = 0$, $\xi_n(x) = \int_a^b x\, dy_n$ are linear functionals over $\mathcal{AC}_M$ and $|\xi_n(x)| = O(1)$ for every $x \in \mathcal{AC}_M$, then $\|y_n\|_N$ is bounded.*

4.14. It will be noted that, if we put in 4.11 $\sum_1^m a_i[y(t_i) - y(t_{i-1})]$ instead of $\int_a^b x\, dy$, a_i being the values of the step-function $x(t)$, the assumption of the continuity of $y(t)$ is in the first part of 4.11 (in the case of the step-functions) superfluous.

4.2. The general form of linear functional over $\mathcal{AC}_M$. We shall need the following two lemmas:

4.21. *Given a step-function $x(t)$ in $\langle a, b\rangle$, let us assume that the function $y^*(t)$, having at each point $t \in \langle a, b\rangle$ a left-hand and a right-hand limit, is continuous at all points of discontinuity of $x(t)$. If $y(t) = y^*(t-0)$ in (a, b) and $y(a) = y^*(a)$, $y(b) = y^*(b)$, then the integrals $\int_a^b x\, dy$ and $\int_a^b x\, dy^*$ exist and are equal, and $V_M(y) \leqslant V_M(y^*)$.*

4.22. *We assume that the function $y(t)$, given in $\langle a, b\rangle$, is one-sidely continuous in (a, b) and that the set E is dense in $\langle a, b\rangle$ and $a, b \in E$. If for each sequence of partitions Π_n: $a = t_0^{(n)} < t_1^{(n)} < \ldots < t_{m_n}^{(n)} = b$ of the interval $\langle a, b\rangle$ such that $\max_i (t_i^{(n)} - t_{i-1}^{(n)}) \to 0$ when $n \to \infty$ and $t_i^{(n)} \in E$, where $i = 0, 1, \ldots, m_n$, $n = 1, 2, \ldots$, and for arbitrary $\tau_{i-1}^{(n)} \in \langle t_{i-1}^{(n)}, t_i^{(n)}\rangle$ the sums*

$$S(\Pi_n) = \sum_{i=1}^{m_n} x(\tau_i^{(n)})[y(t_i^{(n)}) - y(t_{i-1}^{(n)})]$$

are convergent to the same limit, $x(t)$ being a continuous function in $\langle a, b\rangle$, then the Stieltjes integral $\int_a^b x\, dy$ exists.

The proofs of these two lemmas will be left to the reader.

4.23. *Let us assume that $M(u)$ satisfies* (c), (o), (Δ_2) *and* (∞), *and that $\xi(x)$ is a linear functional over $\mathcal{AC}_M$. Then there exists one and only one function $y \in \mathcal{V}_N^*$ (N being complementary to M), left-sidely continuous in (a, b), and such that*

$$\xi(x) = \int_a^b x\, dy$$

for each $x \in \mathcal{AC}_M$.

The proof of this theorem can be performed by the method of the extension of $\xi(x)$ from $\mathcal{AC}_M$ to $\mathcal{V}_M$ or by use of Helly's extraction theorem. Since the latter method is constructive and gives the form of the approximating functionals, we prefer it here. Moreover, for simplicity we give the proof in the case of periodic functions $x(t)$ only, i. e. we establish the form of linear functionals over the subspace of all functions $x \in \mathcal{AC}_M$ such that $x(a) = x(b) = 0$, applying approximation by the Fejér means. Instead of the latter, other means may be applied in the proof. In the general case the proof would require a generalization of 2.5 to the non-periodical case. The proof by the method of extension applying theorems 2.22 and 4.14 will be left to the reader.

We remark first that $|\xi(x)| \leqslant \|\xi\| \cdot \|x\|_M$ for each $x \in \mathcal{AC}_M$. Especially $|\xi(\sigma_n')| \leqslant \|\xi\| \cdot \|\sigma_n'\|$, where $\sigma_n(t)$ is the n-th Fejér mean of $x(t)$, $\sigma_n'(t)$

$= \sigma_n(t)-\sigma_n(a)$. However, 2.51 yields $V_M(\sigma_n) \leqslant V_M(2x)$. Consequently, $V_M(\sigma_n/2\|x\|_M) \leqslant V_M(x/\|x\|_M) \leqslant 1$ and $\|\sigma_n'\|_M \leqslant 2\|x\|_M$. We obtain $|\xi(\sigma_n')| \leqslant 2\|\xi\|\cdot\|x\|_M$. Choose such continuous functions $y_n(t)$, $y_n(a)=0$, that $\xi(\sigma_n') = \int\limits_a^b x\,dy_n$. Then

$$(\cdot) \qquad \Big|\int\limits_a^b x\,dy_n\Big| \leqslant 2\|\xi\|\cdot\|x\|_M \quad \text{for} \quad x \in \mathcal{AC}_M.$$

Now let us assume that $x(t)$ belongs to $\mathcal{AC}_M$. 2.52 yields $V_M(\sigma_n-x)\to 0$. Then $\|\sigma_n'-x\|_M \to 0$, according to 3.11. This yields $\xi(\sigma_n')\to\xi(x)$ and

$$\int\limits_a^b x\,dy_n \to \xi(x) \quad \text{for} \quad x\in\mathcal{AC}_M.$$

Now we have to prove that there exist a sequence of indices n_i and a function $y\in\mathcal{V}_N^*$ such that the integral $\int\limits_a^b x\,dy$ exists for each $x\in\mathcal{AC}_M$ and

$$(:) \qquad \int\limits_a^b x\,dy_{n_i} \to \int\limits_a^b x\,dy \quad \text{for} \quad x\in\mathcal{AC}_M.$$

To prove this fact, let us first note that, according to 4.11 and $(\cdot)$, $V_N(y_n/16\|\xi\|)\leqslant 1$ for $n=1,2,\ldots$ The extraction theorem 1.3 implies the existence of a sequence of indices n_i and of a function $y^*(t)$ with $V_N(y^*/16\|\xi\|)\leqslant 1$ such that $y_{n_i}(t)\to y^*(t)$ for each $t\in\langle a,b\rangle$. Denoting by $s(t)$ a step-function in $\langle a,b\rangle$ with points of discontinuity being points of continuity of $y^*(t)$, it is easily seen that $\int\limits_a^b s\,dy_{n_i}\to\int\limits_a^b s\,dy^*$. Lemma 4.21 implies $\int\limits_a^b s\,dy_{n_i}\to\int\limits_a^b s\,dy$ with $y(t)=y^*(t-0)$ for $a<t<b$, $y(a)=0$, $y(b)=y^*(b)$, $V_N(y/16\|\xi\|)\leqslant 1$ and thus, $\|y\|_N\leqslant 16\|\xi\|$. Since, according to $(\cdot)$ and 4.12, $\Big|\int\limits_a^b s\,dy_{n_i}\Big|\leqslant 4\|\xi\|\cdot\|s'\|_M+|s(a)y_{n_i}(b)|$, we obtain

$$(\therefore) \qquad \Big|\int\limits_a^b s\,dy\Big| \leqslant 4\|\xi\|\cdot\|s'\|_M+|s(a)y(b)|.$$

Now let us take a sequence of partitions Π_n: $a=t_0^{(n)}<t_1^{(n)}<\ldots$ $\ldots<t_{m_n}^{(n)}=b$ of the interval $\langle a,b\rangle$ such that $\max\limits_i(t_i^{(n)}-t_{i-1}^{(n)})\to 0$ when $n\to\infty$ and $t_i^{(n)}$, where $i=1,2,\ldots,m_n-1$ are points of continuity of $y(t)$, and choose arbitrarily $\tau_i^{(n)}\in\langle t_{i-1}^{(n)},t_i^{(n)}\rangle$. Then the Riemann sums of the Stieltjes integral $\int\limits_a^b x\,dy$ are equal to

$$S(\Pi_n)=\sum_{i=1}^{m_n} x(\tau_i^{(n)})[y(t_i^{(n)})-y(t_{i-1}^{(n)})]=\int\limits_a^b s_n\,dy,$$

where

$$s_n(t)=\begin{cases} x(\tau_i^{(n)}) & \text{for} \quad t_{i-1}^{(n)}\leqslant t<t_i^{(n)},\\ x(\tau_{m_n}^{(n)}) & \text{for} \quad t=b.\end{cases}$$

Thus

$$|S(\Pi_n)-S(\Pi_m)|=\Big|\int\limits_a^b(s_n-s_m)\,dy\Big|\leqslant 4\|\xi\|\cdot\|s_n'-s_m'\|_M+|[s_n(a)-s_m(a)]y(b)|,$$

according to $(\therefore)$. However, 2.22 and 3.11 imply $\|s_n'-x\|_M\to 0$. Moreover, $s_n(a)\to 0$. Then the sequence $S(\Pi_n)$ is convergent and lemma 4.22 implies the existence of $\int\limits_a^b x\,dy$.

The easy proof of the convergence $\int\limits_a^b x\,dy_{n_i}\to\int\limits_a^b x\,dy$ will be omitted.

It will be noted that our proof also implies the convergence $\int\limits_a^b x\,dy_n\to$ $\to\int\limits_a^b x\,dy=\xi(x)$.

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