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# A new type of polynomials approximating a continuous or integrable function

by

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**1.** For a continuous function $f(x)$ defined on $\langle 0, 1\rangle$, the expression

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \tag{1}$$

is known as the Bernstein polynomial of order $n$ of the function $f(x)$. As is well-known, these elegant polynomials can be employed to give a simple constructive proof for the Weierstrass approximation theorem [5].

We shall now introduce a kind of polynomials of the form

$$P_n(f; x) = \frac{1}{\sqrt{n\pi}} \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\left[1-\left(\frac{k}{n}-x\right)^2\right]^n. \tag{2}$$

Evidently the polynomial $P_n(f; x)$ is just as simple as $B_n(f; x)$. Moreover, it has, in common with $B_n(f; x)$, the peculiarity of using only the values $f(k/n)$ $(k = 0, 1, \ldots, n)$ in its construction. As may be observed, the structural form of $P_n(f; x)$ is actually suggested by the Landau singular integral [4].

For a function $f(x)$ belonging to the space $L^p(0, 1)$ with $p \geqslant 1$ we may define

$$S_n(f; x) = \sqrt{\frac{n}{\pi}} \sum_{k=1}^{n}\left[1-\left(\frac{k}{n}-x\right)^2\right]^n \int_{(k-1)/n}^{k/n} f(t)\,dt. \tag{3}$$

This is again a polynomial in $x$. The similarity between (3) and (2) is apparently analogous to that between the Kantorovitch polynomial and (1) (see [2] or [3]).

**2.** We shall establish in this note a pair of theorems as follows:

THEOREM 1. *For a continuous function $f(x)$ defined on $\langle 0,1\rangle$, the relation*

$$\lim_{n\to\infty} P_n(f;x) = f(x) \tag{4}$$

*holds uniformly on $\eta \leqslant x \leqslant 1-\eta$, where $\eta$ is an arbitrary small fixed number with $0<\eta<\frac{1}{2}$.*

THEOREM 2. *The relation*

$$\lim_{n\to\infty} \int_0^1 |S_n(f;x)-f(x)|^p\,dx = 0 \tag{5}$$

*is true for every function $f(x)$ belonging to the space $L^p(0,1)$ with $p\geqslant 1$.*

The relation (4) clearly implies a new constructive proof of the Weierstrass theorem. As we shall see, Theorem 2 can be proved quite easily by means of a result of W. Orlicz [6].

LEMMA. *The relation*

$$\sigma_n = \sum_{k=0}^{n}\left[1-\left(\frac{k}{n}-x\right)^2\right]^n \frac{1}{\sqrt{n}} \to \sqrt{\pi} \qquad (n\to\infty) \tag{6}$$

*holds uniformly on $\eta\leqslant x\leqslant 1-\eta$.*

Let us split the summation $\sigma_n$ as follows: $\sigma_n = \sum' + \sum''$, where the summations $\sum'$ and $\sum''$ are taken for all values $k=0,1,\ldots,n$ which satisfy the conditions

$$\left|\frac{k}{n}-x\right| < \left(\frac{1}{n}\right)^{1/3}, \qquad \left|\frac{k}{n}-x\right| \geqslant \left(\frac{1}{n}\right)^{1/3}$$

respectively. Thus, using the fact that $(1-1/a)^a \uparrow e^{-1}$ $(a\to\infty)$, we obtain

$$\left|\sum{}''\right| \leqslant \sum_{k=0}^{n}\left[1-\left(\frac{1}{n}\right)^{2/3}\right]^n \frac{1}{\sqrt{n}} \leqslant \frac{n+1}{\sqrt{n}}\left(\frac{1}{e}\right)^{n^{1/3}} \to 0 \qquad (n\to\infty). \tag{7}$$

For the estimation of $\sum'$, we notice first that the condition $\left|\frac{k}{n}-x\right| < \left(\frac{1}{n}\right)^{1/3}$ implies

$$\left[1-\left(\frac{k}{n}-x\right)^2\right]^n = \left\{\exp\log\left[1-\left(\frac{k}{n}-x\right)^2\right]\right\}^n = \left\{1+O\left(\frac{1}{n^{1/3}}\right)\right\}\exp\left[-\left(\frac{k-nx}{\sqrt{n}}\right)^2\right],$$

the factor involved in the estimate $O(n^{-1/3})$ being independent of $x$.



For every $x$ $(\eta\leqslant x\leqslant 1-\eta)$ we see that the condition $\left|\frac{k}{n}-x\right| < \left(\frac{1}{n}\right)^{1/3}$ is precisely equivalent to

$$\Delta:\ -n^{1/6} < \frac{k-nx}{\sqrt{n}} < n^{1/6}.$$

Hence for $n\to\infty$ we have

$$\lim_{n\to\infty}\sum_{\Delta}{}' = \lim_{n\to\infty}\sum_{\Delta}{}' \exp\left(-\left(\frac{k-nx}{\sqrt{n}}\right)^2\right)\left\{1+O\left(\left(\frac{1}{n}\right)^{1/3}\right)\right\}\frac{1}{\sqrt{n}} \tag{8}$$

$$= \int_{-\infty}^{\infty} \exp(-t^2)\,dt = \sqrt{\pi}.$$

To see that (8) holds uniformly in $x$ $(\eta\leqslant x\leqslant 1-\eta)$, one needs only to replace the condition $\Delta$ by the following:

$$\delta:\ -n^{1/6} < \frac{[k-nx]}{\sqrt{n}} < n^{1/6}$$

where $[a]$ denotes, as usual, the integral part of the real number $a$, i. e. $a-1<[a]\leqslant a$. An easy calculation gives, for $n$ large enough,

$$\left|\exp\left(-\left(\frac{k-nx}{\sqrt{n}}\right)^2\right) - \exp\left(-\left(\frac{[k-nx]}{\sqrt{n}}\right)^2\right)\right| \leqslant 3\left(\frac{1}{n}\right)^{1/3}\exp\left(-\left(\frac{[k-nx]}{\sqrt{n}}\right)^2\right).$$

Hence the following relation holds uniformly:

$$\lim_{n\to\infty}\sum_{\Delta}{}' = \lim_{n\to\infty}\sum_{\delta}{}' \exp\left(-\left(\frac{\nu}{\sqrt{n}}\right)^2\right)\frac{1}{\sqrt{n}} = \sqrt{\pi}, \tag{9}$$

where the condition $\delta$ means that $\nu$ ranges over all those integers for which $-n^{1/6} < \nu/\sqrt{n} < n^{1/6}$.

It is now easy to prove Theorem 1. Let $|f(x)|<M$ and denote by $\omega_f(\varepsilon)$ as usual the modulus of continuity of $f(x)$. Clearly we have

$$\left|P_n(f;x) - \frac{\sigma_n}{\sqrt{\pi}}f(x)\right| \leqslant \frac{1}{\sqrt{\pi}}\sum_{k=0}^{n}\left|f\left(\frac{k}{n}\right)-f(x)\right|\left[1-\left(\frac{k}{n}-x\right)^2\right]^n\frac{1}{\sqrt{n}}$$

$$\leqslant \omega_f\left(\frac{1}{n^{1/3}}\right)\frac{1}{\sqrt{\pi}}\sum{}' + 2M\frac{1}{\sqrt{\pi}}\sum{}'',$$

where the right-hand side converges to zero uniformly, in view of (7) and (9). Hence our theorem is proved by the lemma.

To prove Theorem 2, it suffices to verify that there is a constant $M$ such that

$$\int_0^1 K_n(x,t)\,dt \leqslant M, \qquad \int_0^1 K_n(x,t)\,dx \leqslant M, \tag{10}$$

where $K_n(x,t)$ is a positive kernel defined by $K_n(x,0) = \sqrt{\frac{n}{\pi}}(1-x^2)^n$ and

$$K_n(x,t) = \sqrt{\frac{n}{\pi}}\left[1-\left(\frac{k}{n}-x\right)^2\right]^n \qquad \left(\frac{k-1}{n} < t \leqslant \frac{k}{n}\right),$$

with $k = 1, 2, \ldots, n$; $0 \leqslant x \leqslant 1$. Evidently, by the lemma we have, for $n$ large,

$$\int_0^1 K_n(x,t)\,dt = \frac{1}{\sqrt{n\pi}}\sum_{k=1}^{n}\left[1-\left(\frac{k}{n}-x\right)^2\right]^n \sim 1.$$

Moreover, for each $t$ or $k$, making use of Laplace's asymptotic formula [7] for an integral containing a large parameter, we easily get the following estimate:

$$\int_0^1 K_n(x,t)\,dx = \sqrt{\frac{n}{\pi}}\int_0^1\left[1-\left(\frac{k}{n}-x\right)^2\right]^n dx$$
$$\leqslant \sqrt{\frac{n}{\pi}}\int_{-1}^{1}(1-x^2)^n dx \sim 1 \qquad (n\to\infty).$$

Hence, for $n$ large enough, we may always choose $M = 2$. Finally, by applying the lemma and using the uniform continuity of $f(x)$, it is easily shown that $|S_n(f;x) - P_n(f;x)|$ tends to zero uniformly for any continuous function $f(x)$. Consequently (5) is valid for all continuous functions defined on $\langle 0,1\rangle$, which obviously form an everywhere dense set in $L^p(0,1)$. Hence Theorem 2 is established by a general theorem of Orlicz.

**3.** It is known that Chlodovsky [1] had generalized the Bernstein polynomials to the case of the unbounded interval $(0,\infty)$, so that the modified polynomials can be used to approximate a continuous function defined on $(0,\infty)$. In fact, the same idea does also apply to the polynomials $P_n(f;x)$. For instance, we may define

$$T_n(f;x) = \frac{1}{\sqrt{n\pi}}\sum_{\nu=0}^{n} f\left(\frac{\nu}{n^{3/4}}\right)\left[1-\left(\frac{\nu}{n}-\frac{x}{n^{1/4}}\right)^2\right]^n, \tag{11}$$



and it can be shown that *the relation*

$$\lim_{n\to\infty} T_n(f;x) = f(x) \qquad (0 < x < \infty) \tag{12}$$

*is true for any continuous function $f(x)$ defined and bounded on $(0,\infty)$.*

The proof of (12) consists in splitting the summation on the right-hand side of (11) as $\sum' + \sum''$, where the summations $\sum'$ and $\sum''$ are taken for all values $\nu$ satisfying the conditions

$$\left|\frac{\nu}{n^{3/4}} - x\right| < \left(\frac{1}{n}\right)^{1/6}, \qquad \left|\frac{\nu}{n^{3/4}} - x\right| \geqslant \left(\frac{1}{n}\right)^{1/6}$$

respectively. The whole procedure of the proof is quite similar to that of proving (6) and (4). We therefore omit its details here.

As is easily seen, Chlodovsky's generalization of Bernstein polynomials cannot be modified in order to be capable of approximating bounded continuous functions defined on the whole interval $(-\infty,\infty)$. However, if we modify $T_n(f;x)$ to the form

$$T_n^*(f;x) = \frac{1}{\sqrt{n\pi}}\sum_{\nu=-n}^{n} f\left(\frac{\nu}{n^{3/4}}\right)\left[1-\left(\frac{\nu}{n}-\frac{x}{n^{1/4}}\right)^2\right]^n, \tag{13}$$

we can in fact establish the following

Theorem 3. *For any continuous function $f(x)$ defined and bounded on the interval $(-\infty,\infty)$ we have*

$$\lim_{n\to\infty} T_n^*(f;x) = f(x) \qquad (-\infty < x < \infty). \tag{14}$$

This theorem can be proved by considering the cases $x > 0$, $x < 0$ and $x = 0$ separately. The case $x = 0$ can be verified independently, and the case $x < 0$ can be transformed to that of $x > 0$ by a substitution $x = -y$. For the case $x > 0$ we may express

$$T_n^*(f;x) = \frac{1}{\sqrt{n\pi}}\left(\sum_{\nu=-n}^{-1} + \sum_{\nu=0}^{n}\right) = \frac{1}{\sqrt{n\pi}}\sum_{\nu=-n}^{-1} + T_n(f;x).$$

Assuming that $|f| < M$, clearly we have, for $n$ large,

$$\left|\frac{1}{\sqrt{n\pi}}\sum_{\nu=-n}^{-1}\right| = \frac{1}{\sqrt{n\pi}}\left|\sum_{k=1}^{n} f\left(\frac{-k}{n^{3/4}}\right)\left[1-\left(\frac{k}{n}+\frac{x}{n^{1/4}}\right)^2\right]^n\right|$$
$$\leqslant \frac{M}{\sqrt{n\pi}}\sum_{k=1}^{n}\left[1-\left(\frac{x}{n^{1/4}}\right)^2\right]^n = \frac{Mn}{\sqrt{n\pi}}(1-n^{-1/2}x^2)^n \to 0.$$

Hence we get $\lim T_n^*(f;x) = \lim T_n(f;x) = f(x)$ for $x > 0$.

Finally, it may be worth mentioning that our new polynomials $P_n(f;x)$, $T_n(f;x)$ etc. can also be generalized to the cases of a complex variable and of several variables. Further investigation of these polynomials is being accomplished in a joint paper of the author with L. P. Hsu, which will appear elsewhere.

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# On modular spaces

by

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In the present paper the authors investigate functionals $\varrho(x)$ defined in a real linear space $X$, which are called modulars. An $F$-norm will be introduced in certain subspaces of the space $X$. In the second part of this paper some examples of modulars are considered.

**1.** First, the following definition of a modular and a pseudomodular will be given:

**1.01.** Given a linear space $X$, a functional $\varrho(x)$ defined on $X$ with values $-\infty < \varrho(x) \leqslant \infty$ will be called a *modular* if the following conditions hold:

A.1. $\varrho(x) = 0$ *if and only if* $x = 0$,

A.2. $\varrho(-x) = \varrho(x)$,

A.3. $\varrho(\alpha x+\beta y) \leqslant \varrho(x)+\varrho(y)$ *for every* $\alpha, \beta \geqslant 0$, $\alpha+\beta = 1$.

If $\varrho(x)$ satisfies the condition $\varrho(0) = 0$ instead of A.1, then $\varrho(x)$ will be called a *pseudomodular.*

**1.02.** We now give some simple properties of pseudomodulars.

*Let us assume $\varrho(x)$ to be a pseudomodular on $X$. Then*

(a) $\varrho(x) \geqslant 0$,

(b) $\varrho(ax)$ *is a non-decreasing function of* $a \geqslant 0$ *for each* $x \in X$,

(c) $\varrho(\sum_{i=1}^{n} a_i x_i) \leqslant \sum_{i=1}^{n} \varrho(x_i)$ *for* $a_i \geqslant 0$, $\sum_{i=1}^{n} a_i = 1$.

*Moreover, if $X_\varrho$ denotes the set of $x \in X$ such that $\varrho(x) < \infty$, the set $X_\varrho$ is convex and symmetric with respect to* 0.

The properties (a) and (b) easily follow from A.3 and A.2; (c) is obtained by induction as follows:

$$\varrho\left(\sum_{i=1}^{n} a_i x_i\right) = \varrho\left(\sum_{i=1}^{n-1} a_i \frac{\sum_{i=1}^{n-1} a_i x_i}{\sum_{i=1}^{n-1} a_i} + a_n x_n\right) \leqslant \varrho\left(\frac{\sum_{i=1}^{n-1} a_i x_i}{\sum_{i=1}^{n-1} a_i}\right) + \varrho(x_n) \leqslant \sum_{i=1}^{n} \varrho(x_i).$$