STUDIA MATHEMATICA, T. XVIII. (1959)

# Determinant systems

by

R. SIKORSKI (Warszawa)

The purpose of this paper is to explain the algebraical structure of the theory of determinants in Banach spaces. This theory was developed recently in papers [1-10].

In the determinant theory of the linear equation

$$Ax = x_0$$

(in a space $X$) and the adjoint equation (in a conjugate space $\Xi$), written here in the form

$$\xi A = \xi_0,$$

some identities between the determinant and subdeterminants

$$D_0, D_1, D_2, \ldots$$

play an especially important part. These identities are assumed here as the basis of an axiomatic definition of $\{D_n\}$ in the case where $X$ and $\Xi$ are arbitrary linear spaces conjugate in a sense explained in § 1. The sequence $\{D_n\}$ is called a *determinant system* for $A$. The main problem investigated here is under what conditions $A$ has a determinant system $\{D_n\}$ and what is the connection between $A$ and $\{D_n\}$.

The answer is given by the main theorem (Corollary on p. 201) stating that $A$ has a determinant system $\{D_n\}$ if and only if $A$ is Fredholm, and then $\{D_n\}$ is determined by $A$ uniquely up to a constant factor $\neq 0$. A formula for $\{D_n\}$ is given which explains the connection between $\{D_n\}$ and solutions of the equations under examination. In the case where $A$ is inversible, this formula was earlier applied by Ruston [7] (in another formulation and in a special case).

Leżański's method of interpreting $D_n$ as a $2n$-linear functional on $\Xi^n \times X^n$ is used in this paper instead of the more complicated interpretation of Grothendieck and Ruston.

The spaces $\Xi$ and $X$ play everywhere a completely symmetric role. In order to preserve and to underline this symmetry, the endomorphism

$A$ in $X$ and its adjoint in $\Xi$ are here considered as a bilinear functional on $\Xi \times X$.

The first few sections of this paper are only a recapitulation of known facts from Linear Algebra in a form convenient for the further investigation. The definition of determinant systems is given in § 5, the main theorems — in § 6. The last section, § 7, contains a formula for determinant systems in Cartesian products.

No topology is supposed in linear spaces under consideration.

The dimension of linear subspaces of linear spaces is always understood in the algebraic sense.

**§ 1. Bilinear functionals.** We shall consider a fixed commutative algebraic field $F$ whose elements will be called *scalars* and denoted by $a, b, c, a$, with indices if necessary. Every mapping into $F$ will be called *functional.*

We shall also consider two fixed linear spaces $\Xi$ and $X$ (infinitely dimensional, in general) over the field $F$. The letters $\xi, \eta, \zeta$ (with indices if necessary) will always denote elements of $\Xi$, and the letters $x, y, z$ — elements of $X$.

We suppose that $\Xi$ and $X$ are *conjugate*, i. e. with every pair $(\xi, x) \in \Xi \times X$ there is associated a scalar denoted by $\xi x$, in such a way that:

($c_1$) the product $\xi x$ is a bilinear functional on $\Xi \times X$, i. e.

$$(c\xi)x = c\cdot(\xi x) = \xi(cx),$$

$$\xi(x_1+x_2) = \xi x_1+\xi x_2, \quad (\xi_1+\xi_2)x = \xi_1 x+\xi_2 x;$$

($c_2$) if $\xi x = 0$ for every $\xi \in \Xi$, then $x = 0$;

($c_2'$) if $\xi x = 0$ for every $x \in X$, then $\xi = 0$.

If $\xi x = 0$, then $\xi, x$ are said to be *orthogonal.*

Two subspaces $X_1, X_2$ of $X$ ($\Xi_1, \Xi_2$ of $\Xi$) are said to be *complementary* if every element $x \in X$ ($\xi \in \Xi$) can be uniquely represented in the form $x = x_1+x_2$ where $x_1 \in X_1$ and $x_2 \in X_2$ (in the form $\xi = \xi_1+\xi_2$ where $\xi_1 \in \Xi_1$ and $\xi_2 \in \Xi_2$).

If $A$ is a bilinear functional on $\Xi \times X$, the value of $A$ at the point $(\xi, x) \in \Xi \times X$ will be denoted by $\xi A x$.

$\mathfrak{A}$ will denote the class of all bilinear functionals on $\Xi \times X$ such that the following two conditions are satisfied:

(b) for every fixed $x \in X$ there exists a $y \in X$ such that $\xi A x = \xi y$ for every $\xi \in \Xi$;

(b$'$) for every fixed $\xi \in \Xi$, there exists an $\eta \in \Xi$ such that $\xi A x = \eta x$ for every $x \in X$.



On the other hand, it follows from ($c_2$) that there exists at most one $y$ satisfying (b). The unique element $y$ satisfying (b) for a given $x$ will be denoted by $Ax$. By definition

$$\xi(Ax) = \xi A x \quad \text{for every} \quad \xi \in \Xi.$$

It follows from ($c_2'$) that there exists at most one $\eta$ satisfying (b$'$). The unique element $\eta$ satisfying (b$'$) for a given $\xi$ will be denoted by $\xi A$. By definition

$$(\xi A)x = \xi A x \quad \text{for every} \quad x \in X.$$

It immediately follows from the definition that $\mathfrak{A}$ is a linear space (over $F$) with the natural definition of addition and multiplication by scalar. For every $A \in \mathfrak{A}$,

$$A(cx) = c(Ax), \quad A(x_1+x_2) = Ax_1+Ax_2$$

and

$$(c\xi)A = c(\xi A), \quad (\xi_1+\xi_2)A = \xi_1 A+\xi_2 A,$$

i. e. the mapping $y = Ax$ is an endomorphism of $X$, and the mapping $\eta = \xi A$ is an endomorphism of $\Xi$.

For instance, if

(1) $$\xi I x = \xi x \quad \text{for every} \quad (\xi, x) \in \Xi \times X,$$

then $I \in \mathfrak{A}$ and $Ix = x$ for every $x \in X$ and $\xi I = \xi$ for every $\xi \in \Xi$.

Another example of an element $K$ in $\mathfrak{A}$ is given by the following formula:

$$\xi K x = \xi x_0 \cdot \xi_0 x$$

(i. e., $\xi K x$ is the product of the scalars $\xi x_0$ and $\xi_0 x$) where $x_0 \in X$ and $\xi_0 \in \Xi$ are fixed. This bilinear functional will be denoted by $x_0 \cdot \xi_0$ and called *one-dimensional* because the endomorphisms

$$Kx = x_0 \cdot \xi_0 x \quad \text{and} \quad \xi K = \xi x_0 \cdot \xi_0$$

map $X$ and $\Xi$ onto their at most one-dimensional subspaces spanned by $x_0$ and $\xi_0$ respectively.

The bilinear functional

$$K = \sum_{i=1}^{r} x_i \cdot \xi_i,$$

i. e. the bilinear functional $K$ defined by the formula

$$\xi K x = \sum_{i=1}^{m} \xi x_i \cdot \xi_i x,$$

also belongs to $\mathfrak{A}$ and is called *finitely dimensional* since the endomorphisms

$$Kx = \sum_{i=1}^{r} x_i \cdot \xi_i x \quad \text{and} \quad \xi K = \sum_{i=1}^{r} \xi x_i \cdot \xi_i$$

map $X$ and $\Xi$ onto their finitely dimensional subspaces spanned by $x_1, \ldots, x_r$ and $\xi_1, \ldots, \xi_r$ respectively.

If $A_1, A_2 \in \mathfrak{A}$, then

$$\xi A_1(A_2 x) = (\xi A_1) A_2 x \quad \text{for every } \xi \in \Xi \text{ and } x \in X,$$

and the left (or: right) side of this identity is a bilinear functional in $\mathfrak{A}$. We shall denote it by $A_1 A_2$. By definition

$$\xi A_1 A_2 x = \xi A_1 (A_2 x) = (\xi A_1) A_2 x.$$

It immediately follows from the definition that the endomorphism $y = (A_1 A_2)x$ is the superposition of the endomorphisms $y = A_1 z$ and $z = A_2 x$, i. e.

$$(A_1 A_2)x = A_1(A_2 x).$$

Analogously, the endomorphism $\eta = \xi(A_1 A_2)$ is the superposition of the endomorphisms $\eta = \zeta A_2$ and $\zeta = \xi A_1$, i. e.

$$\xi(A_1 A_2) = (\xi A_1) A_2.$$

Notice that if $K = \sum_{i=1}^{r} x_i \cdot \xi_i$ is finitely dimensional and $A \in \mathfrak{A}$, then

$$KA = \sum_{i=1}^{r} x_i \cdot \xi_i A \quad \text{and} \quad AK = \sum_{i=1}^{r} Ax_i \cdot \xi_i.$$

Thus $KA$ and $AK$ are finitely dimensional.

In particular, if $K' = \sum_{j=1}^{r'} x'_j \cdot \xi'_j$, then

$$KK' = \sum_{i=1}^{r} \sum_{j=1}^{r'} x_i \cdot \xi_i x'_j \cdot \xi'_j = \sum_{i=1}^{r} \sum_{j=1}^{r'} (\xi_i x'_j) x_i \cdot \xi'_j. \tag{2}$$

The linear space $\mathfrak{A}$ with the multiplication $A_1 A_2$ defined above is a linear (non-commutative) ring. The functional $I$ (see (1)) is the unit of $\mathfrak{A}$, i. e. $IA = AI = A$ for every $A \in \mathfrak{A}$. Therefore $I$ will be called the *unit on* $\Xi \times X$.

The following three conditions are equivalent for every $A$ in $\mathfrak{A}$:

($i_0$) $A$ has an inverse $A^{-1}$ in $\mathfrak{A}$ (i. e. there exists an element $A^{-1} \in \mathfrak{A}$ such that $AA^{-1} = A^{-1}A = I$);

($i_1$) the endomorphism $y = Ax$ is a one-to-one mapping of $X$ onto $X$, and the endomorphism $\eta = \xi A$ is a one-to-one mapping of $\Xi$ onto $\Xi$;



($i_2$) the endomorphism $y = Ax$ maps $X$ onto $X$, and the endomorphism $\eta = \xi A$ maps $\Xi$ onto $\Xi$.

A bilinear functional $P \in \mathfrak{A}$ is said to be a *projection* if $PP = P$, i. e, if the endomorphisms $y = Px$ and $\eta = \xi P$ are projections in the ordinary sense. We recall that $y = Px$ is then the projection onto the subspace $X_1 = [Px: x \in X]$ along the subspace $X_2 = [x: Px = 0]$, and $X_1, X_2$ are complementary. Analogously, $\eta = \xi P$ is the projection onto the subspace $\Xi_1 = [\xi P: \xi \in \Xi]$ along the subspace $\Xi_2 = [\xi: \xi P = 0]$, and $\Xi_1, \Xi_2$ are complementary.

For instance, if $x_1, \ldots, x_r$ and $\xi_1, \ldots, \xi_r$ are normed biorthogonal, i. e. $\xi_i x_j = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker symbol, then the bilinear functional $P \in \mathfrak{A}$ defined by the formula

$$P = I - \sum_{i=1}^{r} x_i \cdot \xi_i$$

is a projection, called an *r-dimensional projection*. In fact, we then have:

$$Px = x \text{ if } x \text{ is orthogonal to } \xi_1, \ldots, \xi_r,$$

$$Px = 0 \text{ if } x \text{ is a linear combination of } x_1, \ldots, x_r,$$

i. e. the endomorphism $y = Px$ is a projection of $X$ onto the subspace orthogonal to $\xi_1, \ldots, \xi_r$ and along the $r$-dimensional subspace spanned by $x_1, \ldots, x_r$. Analogously,

$$\xi P = \xi \text{ if } \xi \text{ is orthogonal to } x_1, \ldots, x_r,$$

$$\xi P = 0 \text{ if } \xi \text{ is a linear combination of } \xi_1, \ldots, \xi_r,$$

i. e. the endomorphism $\eta = \xi P$ is a projection of $\Xi$ onto the subspace orthogonal to $x_1, \ldots, x_r$ and along the $r$-dimensional subspace spanned by $\xi_1, \ldots, \xi_r$.

The functional $I \in \mathfrak{A}$ is the only 0-dimensional projection.

**§ 2. Quasi-inverse.** Let $\mathfrak{A}$ be any algebraic ring (non-commutative, in general). An element $B \in \mathfrak{A}$ is said to be a *quasi-inverse* of an element $A \in \mathfrak{A}$ if

$$ABA = A \quad \text{and} \quad BAB = B. \tag{3}$$

It immediately follows from the definition that if $B$ is a quasi-inverse of $A$, then $A$ is also a quasi-inverse of $B$.

We shall consider only the case where $\mathfrak{A}$ has a unit element denoted by $I$.

Observe that if $A$ has a right (left) inverse $B$, then $B$ is a quasi-inverse of $A$, and then every quasi-inverse of $A$ is a right (left) inverse of $A$. Consequently, if $A$ has the inverse $A^{-1}$, then $A^{-1}$ is the only quasi-inverse of $A$.

If

$$(4)\qquad \begin{matrix} A, L, \\ B, \mathcal{L} \end{matrix}$$

are elements in $\mathfrak{A}$ such that

$$(5)\qquad A\mathcal{L} = 0 = \mathcal{L}A, \quad BL = 0 = LB,$$

$$(6)\qquad AB+L\mathcal{L} = I, \quad BA+\mathcal{L}L = I,$$

then $B$ is a quasi-inverse of $A$, $\mathcal{L}$ is a quasi-inverse of $L$, and

$$(7)\qquad (A+L)(B+\mathcal{L}) = I = (B+\mathcal{L})(A+L),$$

i. e. $B+\mathcal{L} = (A+L)^{-1}$.

In fact, (7) follows immediately from (5) and (6). Equalities (3) can be obtained from (6) by the right or left multiplication by $A$ or $B$, on account of (5). In the same way we obtain the analogues of (3) for $L, \mathcal{L}$.

Observe that hypotheses (5), (6) remain unchanged if we commute rows or columns in the matrix (4).

Suppose now that $A, L, \mathcal{L}$ are elements in $\mathfrak{A}$ such that

$$(8)\qquad A\mathcal{L} = 0 = \mathcal{L}A,$$

and that $\mathcal{L}$ is a quasi inverse of $L$, i. e.

$$(9)\qquad L\mathcal{L}L = L, \quad \mathcal{L}L\mathcal{L} = \mathcal{L}.$$

Then the elements

$$(10)\qquad A_0 = A+L, \quad P = I-L\mathcal{L}, \quad Q = I-\mathcal{L}L$$

satisfy the equations

$$(11)\qquad A = PA_0 = A_0Q,$$

$$(12)\qquad PP = P, \quad QQ = Q.$$

Moreover, if the inverse $A_0^{-1}$ exists, then the element

$$(13)\qquad B = A_0^{-1}-\mathcal{L}$$

is a quasi-inverse of $A$. More exactly, elements (4) then satisfy conditions (5) and (6).



The proof can be obtained by a simple calculation. Only the proof of the second part of (5) is a little more difficult and will be quoted here. We have

$$BL = A_0^{-1}L-\mathcal{L}L = A_0^{-1}(L-A_0\mathcal{L}L) = A_0^{-1}\big(L-(A+L)\mathcal{L}L\big) = 0$$

on account of (8), (9). By the same method we prove $LB = 0$.

In the sequel of this paper, $\mathfrak{A}$ will always be the ring defined in § 1 with the unit $I$ defined by (1).

Let $A, B$ be any bilinear functionals in $\mathfrak{A}$. Let

$$X_1 = [Ax: x\in X], \qquad \Xi_1 = [\xi A: \xi\in\Xi],$$
$$X_0 = [Bx: x\in X], \qquad \Xi_0 = [\xi B: \xi\in\Xi].$$

Then the following three conditions are equivalent:

($q_0$) $B$ is a quasi-inverse of $A$ (i. e. (3) hold);

(q) $ABy = y$ for every $y\in X_1$ and $BAx = x$ for every $x\in X_0$;

(q′) $\eta = \eta BA$ for every $\eta\in\Xi_1$ and $\xi = \xi AB$ for every $\xi\in\Xi_0$.

The condition (q) can also be formulated as follows: the mapping $x = By$, considered as a mapping defined only on $X_1$, is the inverse of the mapping $y = Ax$, considered as a mapping defined only on $X_0$. Similarly, (q′) can be formulated in the form: the mapping $\xi = \eta B$, considered as a mapping defined only on $\Xi_1$, is the inverse of the mapping $\eta = \xi A$, considered as a mapping defined only on $\Xi_0$.

Suppose now that (4) are bilinear functionals in $\mathfrak{A}$ such that conditions (5), (6) are satisfied. To formulate their fundamental property it will be convenient to denote them as follows:

$$\begin{matrix} A_{0,0}, & A_{0,1}, \\ A_{1,0}, & A_{1,1}. \end{matrix}$$

Conditions (5), (6) can be now written in the form:

$$(14)\qquad A_{i,j}A_{1-i,1-j} = 0,$$

$$(15)\qquad A_{i,j}A_{i,1-j}+A_{1-i,j}A_{1-i,1-j} = I, \quad A_{i,1-j}A_{i,j}+A_{1-i,1-j}A_{1-i,j} = I$$

where $i, j = 0, 1$. Let

$$X_{i,j}^0 = [x\colon A_{i,j}x = 0], \qquad X_{i,j}^1 = [A_{i,j}x\colon x\in X],$$
$$\Xi_{i,j}^0 = [\xi\colon \xi A_{i,j} = 0], \qquad \Xi_{i,j}^1 = [\xi A_{i,j}\colon \xi\in\Xi]$$

for $i, j = 0, 1$.

Suppose that (14), (15) hold. Then the following equalities express the fundamental property of $A_{i,j}$:

$$(16)\qquad X_{i,j}^{1-k} = X_{1-i,1-j}^{k} \quad\text{and}\quad \Xi_{i,j}^{1-k} = \Xi_{1-i,1-j}^{k} \quad \text{for } i, j, k = 0,1.$$

By symmetry, it suffices to prove only that

$$X_{i,j}^0 = X_{1-i,1-j}^1.$$

Suppose $x \in X_{i,j}^0$, i. e. $A_{i,j}x = 0$. Then, multiplying the second equality (15) by $x$, we obtain $x = A_{1-i,1-j}(A_{1-i,j}x)$, which implies $x \in X_{1-i,1-j}^1$. Suppose now that $x \in X_{1-i,1-j}$, i. e. $x = A_{1-i,1-j}y$ $(y \in X)$. Then, multiplying (14) by $y$, we infer that $A_{i,j}x = 0$, i. e. $x \in X_{i,j}^0$.

The following theorem immediately results from the above consideration (see (7), $(\mathrm{q}_0)$, (q), (q′) and (16)):

THEOREM I. *Suppose that* (4) *are bilinear functionals in* $\mathfrak{A}$ *satisfying conditions* (5) *and* (6). *Then* $B$ *is a quasi-inverse of* $A$ *and*:

$(\mathrm{e}_0)$ $x$ *satisfies the equation*

$$Ax = 0$$

*if and only if* $x$ *is of the form* $x = \mathit{L}y$ *where* $y \in X$;

$(\mathrm{e}_0')$ $\xi$ *satisfies the equation*

$$\xi A = 0$$

*if and only if* $\xi$ *is of the form* $\xi = \eta L$ *where* $\eta \in \Xi$;

(e) *the equation*

$$Ax = x_0 \tag{17}$$

*has a solution* $x$ *if and only if* $Lx_0 = 0$; *then* $x = Bx_0$ *is the only solution of* (17) *such that* $Lx = 0$;

(e′) *the equation*

$$\xi A = \xi_0 \tag{17'}$$

*has a solution* $\xi$ *if and only if* $\xi_0 L = 0$; *then* $\xi = \xi_0 B$ *is the only solution of* (17′) *such that* $\xi L = 0$.

**§ 3. Fredholm functionals.** A subspace $X_1$ of $X$ is said to be of *codimension* $x$ if there exist $r$ linearly independent elements $\zeta_1, \ldots, \zeta_r \in \Xi$ such that $X_1$ is the set of all elements $x$ orthogonal to $\zeta_1, \ldots, \zeta_r$, i. e.

$$X_1 = [x\colon\ \zeta_i x = 0 \text{ for } i = 1, \ldots, r].$$

Then there exists an $r$-dimensional subspace $X_2$ complementary to $X_1$. Every complementary subspace $X_2$ is spanned by some elements $y_1, \ldots, y_r$ such that

$$\zeta_i y_j = \delta_{i,j}, \tag{18}$$

and every element $x \in X$ can then be uniquely represented in the form

$$x = x' + a_1 y_1 + \ldots + a_r y_r \quad \text{where} \quad x' \in X_1.$$



The whole space $X$ is of codimension 0.

By symmetry, a subspace $\Xi_1$ of $\Xi$ is said to be of *codimension* $r$ if there exist $r$ linearly independent elements $z_1, \ldots, z_r \in X$ such that $\Xi_1$ is the set of all elements $\xi$ orthogonal to $z_1, \ldots, z_r$, i. e.

$$\Xi_1 = [\xi\colon\ \xi z_i = 0 \text{ for } i = 1, \ldots, r].$$

Then there exists an $r$-dimensional subspace $\Xi_2$ complementary to $\Xi_1$. Every complementary subspace $\Xi_2$ is spanned by some elements $\eta_1, \ldots, \eta_r$ such that

$$\eta_i z_j = \delta_{i,j}, \tag{18'}$$

and every element $\xi \in \Xi$ can then be uniquely represented in the form

$$\xi = \xi' + a_1\eta_1 + \ldots + a_r\eta_r \quad \text{where} \quad \xi' \in \Xi'.$$

The whole space $\Xi$ is of codimension 0.

A bilinear functional $A \in \mathfrak{A}$ is said to be *Fredholm of order* $r$ if

(f) the subspace $X_1 = [Ax\colon x \in X]$ is of codimension $r$,

and

(f′) the subspace $\Xi_1 = [\xi A\colon \xi \in X]$ is of codimension $r$.

In other words, $A \in \mathfrak{A}$ is Fredholm of order $r$ if:

$(\mathrm{f}_1)$ the equation $Ax = 0$ has exactly $r$ linearly independent solutions $z_1, \ldots, z_r$;

$(\mathrm{f}_1')$ the equation $\xi A = 0$ has exactly $r$ linearly independent solutions $\zeta_1, \ldots, \zeta_r$;

$(\mathrm{f}_2)$ the equation $Ax = x_0$ has a solution $x$ if and only if $\zeta_i x_0 = 0$ for $i = 1, \ldots, r$;

$(\mathrm{f}_2')$ the equation $\xi A = \xi_0$ has a solution $\xi$ if and only if $\xi_0 z_r = 0$ for $i = 1, \ldots, r$.

In particular, $A$ is Fredholm of order 0 if and only if $y = Ax$ is a one-to-one mapping of $X$ onto itself and $\eta = \xi A$ is a one-to-one mapping of $\Xi$ onto itself, i. e. if $A$ has an inverse $A^{-1}$ in the ring $\mathfrak{A}$ (see $(\mathrm{i}_0)$, $(\mathrm{i}_1)$). For instance, $I$ is Fredholm of order 0.

If the spaces $X$ and $\Xi$ are finitely dimensional, then they have the same dimension and, by a known theorem of Linear Algebra, every bilinear functional on $\Xi \times X$ is Fredholm.

If $X = X' \times X''$ and $\Xi = \Xi' \times \Xi''$, where $\Xi'$, $X'$ are conjugate and $\Xi''$, $X''$ are conjugate, then so are $\Xi$, $X$. If $X'$, $X''$ are bilinear functionals on $\Xi' \times X$, $\Xi'' \times X''$ respectively, then

$$\xi A x = \xi' A' x' + \xi'' A'' x'' \quad \text{for} \quad \xi = (\xi', \xi'') \in \Xi \quad \text{and} \quad x = (x', x'') \in X \tag{19}$$

is a bilinear functional on $\Xi\times X$. We shall denote the functional $A$ by $(A', A'')$. If $A'$ and $A''$ are Fredholm of orders $r'$ and $r''$ respectively, then $A=(A', A'')$ is Fredholm of order $r=r'+r''$.

Let $\Xi$, $X$ be some conjugate spaces. Every bilinear functional $X$ of the form $A=I-K$ where $K$ is finitely dimensional, is Fredholm. In fact, $X$ and $\Xi$ can then be represented in the form $X=X'\times X''$, $\Xi=\Xi'\times\Xi''$ where $\Xi'$, $X'$ are conjugate, $\Xi''$, $X''$ are conjugate, $X'$, $\Xi'$ are finitely dimensional, and $A=(A', I'')$, $I''$ being the unit on $\Xi''$, $X''$.

If $A_1$ has the inverse $A_1^{-1}$, and $A_2$ is Fredholm, then $A_1A_2$ and $A_2A_1$ are Fredholm (of the same order as $A_2$).

Consequently, every bilinear functional of the form

$$(I-K)A_0 \quad \text{or} \quad A_0(I-K),$$

where $A_0$ has the inverse and $K$ is finitely dimensional, is Fredholm.

Conversely, if $A$ is Fredholm, then $A$ can be represented in the form

$$A=(I-K_1)A_0=A_0(I-K_2) \tag{20}$$

where $A_0$ has the inverse $A_0^{-1}$, and $K_1, K_2$ are finitely dimensional. Moreover, we can always assume that

$$P=I-K_1 \quad \text{and} \quad Q=I-K_2$$

are $r$-dimensional projections ($r$ = the order of $A$).

In fact, let $z_1,\ldots,z_r$ and $\zeta_1,\ldots,\zeta_r$ satisfy $(\mathrm{f}_1)$ and $(\mathrm{f}_1')$, and let $y_1,\ldots,y_r$ and $\eta_1,\ldots,\eta_r$ satisfy (18) and (18'). Let

$$L=\sum_{i=1}^{r} y_i\cdot\eta_i, \qquad \L=\sum_{i=1}^{r} z_i\cdot\zeta_i,$$

$$K_1=L\L=\sum_{i=1}^{r} y_i\cdot\zeta_i, \qquad K_2=\L L=\sum_{i=1}^{r} z_i\cdot\eta_i$$

(see (2), (18), (18')). Then $A, L, \L$ satisfy (8) and (9). Consequently (see (10), (11), (12)), $P$ and $Q$ are projections and the bilinear functional

$$A_0=A+L$$

satisfies (20). $A_0$ has the inverse $A_0^{-1}$ since (see $(\mathrm{i}_0)$, $(\mathrm{i}_2)$) the endomorphism $y=A_0x$ maps $X$ onto itself, and $\eta=\xi A_0$ maps $\Xi$ onto itself. In fact, if $y=x'+a_1y_1+\ldots+a_ry_r$ where $x'\in X_1$, i. e. $x'=Ax$ for some $x\in X$, then $y=A_0\big((I-K_2)x+\sum_{i=1}^{r}a_iz_i\big)$. The proof of an analogous statement for $\eta=\xi A_0$ is similar.



Every Fredholm functional $A\in\mathfrak{A}$ has a quasi-inverse, viz.

$$B=A_0^{-1}-\L \tag{21}$$

is a quasi-inverse of $A$ (see (13)).

If $A_1, A_2$ are Fredholm, so is $A_1A_2$. In fact, $A_1=B_1(I-K_1)$, $A_2=B_2(I-K_2)$, where $B_1, B_2$ have inverses and $K_1, K_2$ are finitely dimensional. Hence $A_1A_2=B_1(I-K_1)B_2(I-K_2)$. However, $(I-K_1)B_2$ is Fredholm, i. e. $(I-K_1)B_2=B_3(I-K_3)$, where $B_3$ has an inverse and $K_3$ is finitely dimensional. Hence $A_1A_2=A_0(I-K)$ where $A_0=B_1B_3$ has an inverse and $K=K_2+K_3-K_3K_2$ is finitely dimensional.

Example. Let $X'=\Xi''=(c)$ = the space of all convergent sequences and $X''=\Xi'=(l)$ = the space of all absolutely convergent series. Then $\Xi'$, $X'$ and $\Xi''$, $X''$ are conjugate and, consequently, so are $\Xi=\Xi'\times\Xi''$, $X=X'\times X''$. The bilinear functional

$$\xi'A'x'=\sum_{n=1}^{\infty}\alpha_{n+1}a_n \quad \text{for} \quad \xi'=(\alpha_n)\in\Xi' \quad \text{and} \quad x=(a_n)\in X'$$

is not Fredholm on $\Xi'\times X'$, and the bilinear functional

$$\xi''A''x''=\sum_{n=1}^{\infty}\alpha_n a_{n+1} \quad \text{for} \quad \xi''=(\alpha_n)\in\Xi'' \text{ and } x''=(a_n)\in X''$$

is not Fredholm on $\Xi''\times X''$; however, the bilinear functional $A=(A', A'')$ (see (19)) is Fredholm on $\Xi\times X$. The bilinear functionals $A_1=(A', I'')$ and $A_2=(I', A'')$ (where $I'$ and $I''$ are units on $\Xi'\times X'$ and $\Xi''\times X''$ respectively) are not Fredholm but $A_1A_2=(A', A'')$ is Fredholm. The order of $A=(A', A'')$ is 1. The order of $A^n=AA\ldots A$ ($n$ times) is $n$.

**§ 4. Relativization to a subring.** It is sometimes convenient to consider only a special subring $\mathfrak{B}$ instead of the whole ring $\mathfrak{A}$. We shall always assume that $\mathfrak{B}$ satisfies the following conditions:

$(\mathrm{r}_1)$ $I$ belongs to $\mathfrak{B}$; every finite dimensional $K\in\mathfrak{A}$ also belongs to $\mathfrak{B}$;

$(\mathrm{r}_2)$ if $A\in\mathfrak{B}$ has an inverse $A^{-1}$ in $\mathfrak{A}$, then $A^{-1}$ also belongs to $\mathfrak{B}$.

Thus the notions of inverses in $\mathfrak{A}$ and $\mathfrak{B}$ coincide, and, for every $A\in\mathfrak{B}$, each of the conditions $(\mathrm{i}_0)$, $(\mathrm{i}_1)$, $(\mathrm{i}_2)$ is equivalent to the existence of the inverse of $A$ in $\mathfrak{B}$.

Observe that if $A\in\mathfrak{B}$ is Fredholm and $B$ is a quasi-inverse of $A$, then $B\in\mathfrak{B}$ also. This immediately follows from (21). Observe that $B\in\mathfrak{B}$ also in the case where $\mathfrak{B}$ is only an algebraical subring (non-linear, in general) satisfying $(\mathrm{r}_2)$ and the following condition:

$(\mathrm{r}_0)$ $I$ belongs to $\mathfrak{B}$; if $K\in\mathfrak{A}$ is finitely dimensional and $A\in\mathfrak{B}$, then $A+K\in\mathfrak{B}$.

**§ 5. Determinant systems.** Let $\mathfrak{B}$ satisfy the conditions $(\mathrm{r}_1)$, $(\mathrm{r}_2)$ mentioned in § 4, and let $A \in \mathfrak{B}$. By a *determinant system* for $A$ in $\mathfrak{B}$ we shall understand every infinite sequence

$$D_0, D_1, D_2, \dots \tag{22}$$

such that:

$(\mathrm{d}_1)$ $D_n$ is $2n$-linear functional on $\underbrace{\Xi \times \Xi \times \dots \times \Xi}_{n\,\text{times}} \times \underbrace{X \times X \times \dots \times X}_{n\,\text{times}}$, the value of $D_n$ at a point $(\xi_1, \dots, \xi_n, x_1, \dots, x_n)$ of this Cartesian product being denoted by $D_n\begin{pmatrix}\xi_1, \dots, \xi_n\\ x_1, \dots, x_n\end{pmatrix}$; in particular $D_0$ is a scalar;

$(\mathrm{d}_2)$ for $n = 2, 3, \dots$, $D_n\begin{pmatrix}\xi_1, \dots, \xi_n\\ x_1, \dots, x_n\end{pmatrix}$ is skew symmetric in $\xi_1, \dots, \xi_n$ and in $x_1, \dots, x_n$, i. e. for every permutation $\mathfrak{p} = (p_1, \dots, p_n)$ of numbers $1, \dots, n$

$$D_n\begin{pmatrix}\xi_{p_1}, \dots, \xi_{p_n}\\ x_1, \dots, x_n\end{pmatrix} = \operatorname{sgn}\mathfrak{p} \cdot D_n\begin{pmatrix}\xi_1, \dots, \xi_n\\ x_1, \dots, x_n\end{pmatrix} = D_n\begin{pmatrix}\xi_1, \dots, \xi_n\\ x_{p_1}, \dots, x_{p_n}\end{pmatrix}$$

where $\operatorname{sgn}\mathfrak{p} = 1$ if the permutation $\mathfrak{p}$ is even, and $\operatorname{sgn}\mathfrak{p} = -1$ if $\mathfrak{p}$ is odd;

$(\mathrm{d}_3)$ for $n = 1, 2, \dots$, $D_n\begin{pmatrix}\xi_1, \dots, \xi_n\\ x_1, \dots, x_n\end{pmatrix}$, interpreted as a function of $\xi_i$ and $x_j$ only, belongs to $\mathfrak{B}$;

$(\mathrm{d}_4)$ there exists an integer $r \geqslant 0$ such that $D_r$ does not vanish identically;

$(\mathrm{d}_5)$ the following identities hold for $n = 0, 1, 2, \dots$:

$$(\mathrm{D}_n)\quad D_{n+1}\begin{pmatrix}\xi_0 A, \xi_1, \dots, \xi_n\\ x_0, x_1, \dots, x_n\end{pmatrix} = \sum_{i=0}^{n} (-1)^i \xi_0 x_i \cdot D_n\begin{pmatrix}\xi_1, \dots\dots\dots\dots, \xi_n\\ x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n\end{pmatrix},$$

$$(\mathrm{D}'_n)\quad D_{n+1}\begin{pmatrix}\xi_0, \xi_1, \dots, \xi_n\\ Ax_0, x_1, \dots, x_n\end{pmatrix} = \sum_{i=0}^{n} (-1)^i \xi_i x_0 \cdot D_n\begin{pmatrix}\xi_0, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n\\ x_1, \dots\dots\dots\dots, x_n\end{pmatrix}.$$

It follows immediately from $(\mathrm{d}_2)$ that

$$D_n\begin{pmatrix}\xi_1, \dots, \xi_n\\ x_1, \dots, x_n\end{pmatrix} = 0$$

if either $\xi_i = \xi_j$ or $x_i = x_j$ for $i \neq j$.

The least integer $r$ such that $D_r$ does not vanish identically is called the *order* of the determinant system (22).

If $X$ and $\Xi$ are $m$-dimensional spaces over $F$ with the usual definition of the product $\xi x$, then every $A \in \mathfrak{A}$ has a determinant system. Viz. fix a coordinate system in $X$ and the corresponding coordinate system in $\Xi$.



Then, in the summation notation from the Tensor Calculus, if $x = (a^1, \dots, a^m)$ and $\xi = (a_1, \dots, a_m)$, then $\xi x = a_i a^i$, and there exists a scalar square matrix $\{c^i_j\}$ such that $\xi A x = a_i c^i_j a^j$. Let $D_0$ be the algebraical determinant of $\{c^i_j\}$. If $0 < n \leqslant m$, $i_1, \dots, i_n$ are different positive integers $\leqslant m$ and $j_1, \dots, j_n$ are different positive integers $\leqslant m$, let $b^{i_1, \dots, i_n}_{j_1, \dots, j_n}$ denote the corresponding algebraical minor of the matrix $\{c^i_j\}$ (i. e. the determinant obtained from $\{c^i_j\}$ by omitting the rows with indexes $j_1, \dots, j_n$ and the columns with indexes $i_1, \dots, i_n$, and multiplied respectively by $\pm 1$); otherwise let $b^{i_1, \dots, i_n}_{j_1, \dots, j_n} = 0$. For $n = 1, 2, \dots$ let

$$D_n\begin{pmatrix}\xi_1, \dots, \xi_n\\ x_1, \dots, x_n\end{pmatrix} = \overset{1}{a}_{j_1} \dots \overset{n}{a}_{j_n} b^{i_1 \dots i_n}_{j_1 \dots j_n} \underset{1}{a}^{j_1} \dots \underset{n}{a}^{j_n}$$

where $\xi_k = (\overset{k}{a}_1, \dots, \overset{k}{a}_m)$ and $x_k = (\underset{k}{a}^1, \dots, \underset{k}{a}^m)$. The sequence (22) so defined is a determinant system for $A$.

An example of a determinant system (under some hypotheses on $A$) in the case where $X$ and $\Xi$ are Banach spaces was given by Leżański [3] (see also Sikorski [9]) and (under more restrictive hypotheses on $A$) by Ruston [6] and Grothendieck [1].

Observe that if (22) is a determinant system for $A$, and $c \neq 0$, then

$$cD_0, cD_1, cD_2, \dots$$

is also a determinant system for $A$.

If (22) is a determinant system for $A$, and $c \neq 0$, then

$$D_0, \frac{1}{c} D_1, \frac{1}{c^2} D_2, \dots$$

is a determinant system for $cA$.

If (22) is a determinant system for $A$, and $B \in \mathfrak{B}$ has the inverse $B^{-1}$, then

$$D_n\begin{pmatrix}\xi_1 B^{-1}, \dots, \xi_n B^{-1}\\ x_1, \dots, x_n\end{pmatrix} \qquad (n = 0, 1, 2, \dots)$$

is a determinant system for $AB$, and

$$D_n\begin{pmatrix}\xi_1, \dots, \xi_n\\ B^{-1}x_1, \dots, B^{-1}x_n\end{pmatrix} \qquad (n = 0, 1, 2, \dots)$$

is a determinant system for $BA$.

The unit $I$ always has a determinant system. In fact, let

$$\theta_0 = 1 \quad \text{and} \quad \theta_n\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = \begin{vmatrix} \xi_1 x_1, & \ldots, & \xi_1 x_n \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \xi_n x_1, & \ldots, & \xi_n x_n \end{vmatrix} \quad \text{for } n = 1, 2, \ldots$$

Then $\theta_0, \theta_1, \theta_2, \ldots$ is a determinant system for $I$.

It follows immediately from the last two remarks that if $A$ has the inverse $A^{-1}$, then the formulas

$$(23) \qquad \mathcal{D}_0 = 1, \quad \mathcal{D}_n\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = \theta_n\begin{pmatrix}\xi_1 A^{-1}, \ldots, \xi_n A^{-1} \\ x_1, \ldots, x_n\end{pmatrix}$$

$$= \theta_n\begin{pmatrix}\xi_1, \ldots, \xi_n \\ A^{-1}x_1, \ldots, A^{-1}x_n\end{pmatrix} = \begin{vmatrix} \xi_1 A^{-1}x_1, & \ldots, & \xi_1 A^{-1}x_n \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \xi_n A^{-1}x_1, & \ldots, & \xi_n A^{-1}x_n \end{vmatrix} \quad \text{for} \quad n = 1, 2, \ldots$$

define a determinant system $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \ldots$ for $A$.

We quote also the following example of a determinant system (a slight generalization of a definition in Ruston [7]). Suppose $X$ and $\Xi$ are complex Banach spaces, $\Lambda$ is an open set of complex numbers, $\mathfrak{B}$ is an algebra of continuous bilinear functionals on $\Xi \times X$. Let $A(\lambda)$ be a holomorphic function of a complex variable $\lambda \in \Lambda$ with values in $\mathfrak{B}$. Suppose that, for every $\lambda \in \Lambda$ except an isolated set of points $\lambda_1, \lambda_2, \ldots$, the inverse $A(\lambda)^{-1}$ exists. Let $\mathcal{D}_n(\lambda)$ be the $2n$-linear functional defined by (23) where $A$ is replaced by $A(\lambda)$. $\mathcal{D}_n(\lambda)$ is a holomorphic function of $\lambda \in \Lambda - (\lambda_1, \lambda_2, \ldots)$ with values in the Banach space of all $2n$-linear continuous functionals on $\Xi \times \ldots \times \Xi \times X \times \ldots \times X$. Suppose that, for every $k$, there exists an integer $r_k > 0$ such that all $\mathcal{D}_n(\lambda)$ have at most a pole of order $\leqslant r_k$ at the point $\lambda_k$, and one of them has a pole exactly of order $r_k$. Let $D_0(\lambda)$ be a complex-valued holomorphic function in $\Lambda$ such that $D_0(\lambda)$ has a zero of order $r_k$ for $\lambda = \lambda_k$ $(k = 1, 2, \ldots)$ and $D_0(\lambda) \neq 0$ otherwise. Then all $D_n(\lambda) = D_0(\lambda)\mathcal{D}_n(\lambda)$ $(n = 1, 2, \ldots)$ are holomorphic in the whole $\Lambda$ and, for every $\lambda \in \Lambda$, $D_0(\lambda), D_1(\lambda), D_2(\lambda), \ldots$ is a determinant system for $A(\lambda)$.

Some criterions for $A(\lambda) = I + \lambda T$ (where $T \in \mathfrak{B}$) to satisfy the hypotheses mentioned above (in the case where $\Lambda$ is the whole plane) were given by Ruston [7].

**6. Fundamental theorems.** Let $X$ and $\Xi$ be conjugate and let $\mathfrak{B}$ satisfy the hypotheses mentioned in § 4.



THEOREM II. *If a bilinear functional $A \in \mathfrak{B}$ has in $\mathfrak{B}$ a determinant system (8) of order $r$, then $A$ is Fredholm of order $r$.*

More exactly:

*If $\eta_1, \ldots, \eta_r \in \Xi$ and $y_1, \ldots, y_r \in X$ are such that*

$$D_r\begin{pmatrix}\eta_1, \ldots, \eta_r \\ x_1, \ldots, x_r\end{pmatrix} \neq 0,$$

*then there exist elements $\zeta_1, \ldots, \zeta_r \in \Xi$ and $z_1, \ldots, z_r \in X$ such that*

$$(24) \qquad \zeta_i x = \frac{D_r\begin{pmatrix}\eta_1, \ldots\ldots\ldots\ldots, \eta_r \\ y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_r\end{pmatrix}}{D_r\begin{pmatrix}\eta_1, \ldots, \eta_r \\ y_1, \ldots, y_r\end{pmatrix}} \quad \text{for every} \quad x \in X$$

*and*

$$(24') \qquad \xi z_i = \frac{D_r\begin{pmatrix}\eta_1, \ldots, \eta_{i-1}, \xi, \eta_{i+1}, \ldots, \eta_r \\ y_1, \ldots\ldots\ldots\ldots, y_r\end{pmatrix}}{D_r\begin{pmatrix}\eta_1, \ldots, \eta_r \\ y_1, \ldots, y_r\end{pmatrix}} \quad \text{for every} \quad \xi \in \Xi.$$

*The elements $\zeta_1, \ldots, \zeta_r$ are linearly independent and are solutions of the equation*

$$(25) \qquad \xi A = 0.$$

*The elements $z_1, \ldots, z_r$ are linearly independent and are solutions of the equation*

$$(25') \qquad Ax = 0.$$

*Conversely, every solution $\xi$ of (25) is a linear combination of $\zeta_1, \ldots, \zeta_r$, and every solution $x$ of (25') is a linear combination of $z_1, \ldots, z_r$.*

*The equation*

$$(26) \qquad \xi A = \xi_0$$

*has a solution $\xi$ if and only if $\xi_0 z_i = 0$ for $i = 1, \ldots, r$. The equation*

$$(26') \qquad Ax = x_0$$

*has a solution $x$ if and only if $\zeta_i x_0 = 0$ for $i = 1, \ldots, r$.*

*The bilinear functional $B$ defined by the formula*

$$\xi B x = \frac{D_{r+1}\begin{pmatrix}\xi, \eta_1, \ldots, \eta_r \\ x, y_1, \ldots, y_r\end{pmatrix}}{D_r\begin{pmatrix}\eta_1, \ldots, \eta_r \\ y_1, \ldots, y_r\end{pmatrix}}$$

*is a quasi-inverse of $A$. If $\xi_0$ is orthogonal to all $z_1, \ldots, z_r$, then $\xi_0 B$ is the only solution of (26), orthogonal to $y_1, \ldots, y_r$. Analogously, if $x_0$ is orthogonal to $\zeta_1, \ldots, \zeta_r$, then $Bx_0$ is the only solution of (26′), orthogonal to $\eta_1, \ldots, \eta_r$.*

In fact, the conditions $\zeta_i \epsilon \Xi$ and $z_i \epsilon X$ $(i = 1, \ldots, r)$ follow directly from $(d_3)$. It follows immediately from the definition (24) that $\zeta_i y_j = \delta_{i,j}$. Thus $\zeta_1, \ldots, \zeta_r$ are linearly independent, and so are $y_1, \ldots, y_r$. Similarly, (24′) implies that $\eta_i z_j = \delta_{i,j}$. Thus $z_1, \ldots, z_r$ are linearly independent, and so are $\eta_1, \ldots, \eta_r$.

Let $L$ and $\mathit{Ł}$ be defined as in § 3:

$$L = \sum_{i=1}^{r} y_i \cdot \eta_i, \qquad \mathit{Ł} = \sum_{i=1}^{r} z_i \cdot \zeta_i.$$

The identity $(D_{r-1})$ (see p. 190) and the skew symmetry $(d_2)$ imply $\zeta_i Ax = 0$ for every $x \epsilon X$, i. e. (see $(c_2)$) $\zeta_i A = 0$ for $i = 1, \ldots, r$. Similarly $(D'_{r-1})$, $(d_2)$ and $(c_2)$ imply $Az_i = 0$ for $i = 1, \ldots, r$. Hence $A\mathit{Ł} = 0 = \mathit{Ł}A$.

The skew symmetry $(d_2)$ implies that $By_i = 0$ and $\eta_i B = 0$ for $i = 1, \ldots, r$. Hence $BL = 0 = LB$.

Setting in $(D_n)$ $n = r$ and replacing $\xi_0, \xi_1, \ldots, \xi_r, x_0, x_1, \ldots, x_r$ by $\xi, \eta_1, \ldots, \eta_r, x, y_1, \ldots, y_r$ respectively, we obtain, on account of the skew symmetry $(d_2)$, $AB = I - L\mathit{Ł}$. Analogously we obtain, from $(D'_r)$ on account of $(d_2)$, $BA = I - \mathit{Ł}L$.

Since (see (2)) $L\mathit{Ł}L = L$ and $\mathit{Ł}L\mathit{Ł} = \mathit{Ł}$, all the hypotheses of Theorem I are satisfied. Theorem II immediately follows from Theorem I.

THEOREM III. *If $A \epsilon \mathfrak{B}$ is Fredholm, then $A$ has a determinant system in $\mathfrak{B}$. This system is determined by $A$ uniquely up to a scalar factor $\neq 0$.*

Let $r$ be the order of $A$.

Consider first the case where $r = 0$, i. e. $A$ has the inverse $A^{-1}$. It follows from § 5 that $A$ then has a determinant system, viz. the system $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \ldots$ defined by (23). Conversely, if $D_0, D_1, D_2, \ldots$ is any determinant system of $A$, then it follows from the identities $(D_n)$, $(D'_n)$ for $\{D_n\}$ and $\{\mathcal{D}_n\}$, by induction on $n$, that (Ruston [7])

$$D_n = D_0 \cdot \mathcal{D}_n \quad \text{for} \quad n = 0, 1, 2, \ldots$$

This completes the proof of Theorem II in the case $r = 0$.

Suppose now that $r > 0$. Let $X_1, X_2, \Xi_1, \Xi_2, y_1, \ldots, y_r, z_1, \ldots, z_r, \eta_1, \ldots, \eta_r, \zeta_1, \ldots, \zeta_r$ and $B$ have the same meaning as in § 3, i. e. $X_1$, $\Xi_1$ are defined in (f) and (f′); $z_1, \ldots, z_r$ and $\zeta_1, \ldots, \zeta_r$ satisfy $(f_1)$, $(f_2)$, $(f'_1)$, $(f'_2)$; $y_1, \ldots, y_r$ and $\eta_1, \ldots, \eta_r$ satisfy (18) and (18′); $X_2$ and $\Xi_2$ are subspaces spanned by $y_1, \ldots, y_r$ and $\eta_1, \ldots, \eta_r$ respectively; and $B$ is a quasi-inverse of $A$.



Let

$$(27) \qquad \mathcal{D}_n = 0 \quad \text{for} \quad n = 0, \ldots, r-1,$$

$$(28) \qquad \mathcal{D}_r\begin{pmatrix}\xi_1, \ldots, \xi_r\\ x_1, \ldots, x_r\end{pmatrix} = \begin{vmatrix}\xi_1 z_1, & \ldots, & \xi_1 z_r\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \xi_r z_1, & \ldots, & \xi_r z_r\end{vmatrix} \cdot \begin{vmatrix}\zeta_1 x_1, & \ldots, & \zeta_1 x_r\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \zeta_r x_1, & \ldots, & \zeta_r x_r\end{vmatrix},$$

and for $k = 1, 2, \ldots$ let

$$(29) \qquad \mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots, \xi_{r+k}\\ x_1, \ldots, x_{r+k}\end{pmatrix} = \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p} \cdot \operatorname{sgn}\mathfrak{q} \cdot \begin{vmatrix}\xi_{p_1} Bx_{q_1}, & \ldots, & \xi_{p_1} Bx_{q_k}\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \xi_{p_k} Bx_{q_1}, & \ldots, & \xi_{p_k} Bx_{q_k}\end{vmatrix} \cdot \mathcal{D}_r\begin{pmatrix}\xi_{p_{k+1}}, \ldots, \xi_{p_{k+r}}\\ x_{q_{k+1}}, \ldots, x_{q_{k+r}}\end{pmatrix}$$

where $\sum_{\mathfrak{p},\mathfrak{q}}$ is extended over all permutations $\mathfrak{p} = (p_1, \ldots, p_{r+k})$ and $\mathfrak{q} = (q_1, \ldots, q_{r+k})$ of the integers $1, \ldots, r+k$, such that

$$p_1 < p_2 < \ldots < p_k, \quad p_{k+1} < p_{k+2} < \ldots < p_{k+r},$$
$$q_1 < q_2 < \ldots < q_k, \quad q_{k+1} < q_{k+2} < \ldots < q_{k+r}.$$

Of course, if the field $F$ of scalars has the characteristic zero, then (29) can also be written in the form

$$\mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots, \xi_{r+k}\\ x_1, \ldots, x_{r+k}\end{pmatrix} = \sum_{\mathfrak{p},\mathfrak{q}} \frac{\operatorname{sgn}\mathfrak{p} \cdot \operatorname{sgn}\mathfrak{q}}{(k!)^2 (r!)^2} \begin{vmatrix}\xi_{p_1} Bx_{q_1}, & \ldots, & \xi_{p_1} Bx_{q_k}\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \xi_{p_k} Bx_{q_1}, & \ldots, & \xi_{p_k} Bx_{q_k}\end{vmatrix} \cdot \mathcal{D}_r\begin{pmatrix}\xi_{p_{k+1}}, \ldots, \xi_{p_{k+r}}\\ x_{q_{k+1}}, \ldots, x_{q_{k+r}}\end{pmatrix}$$

where $\sum_{\mathfrak{p},\mathfrak{q}}$ is extended over all permutations $\mathfrak{p} = (p_1, \ldots, p_{r+k})$ and $\mathfrak{q} = (q_1, \ldots, q_{r+k})$ of the integers $1, \ldots, r+k$.

We shall prove that the sequence

$$(30) \qquad \mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \ldots$$

defined by (27), (28) and (29) is a determinant system for $A$. It is evident that (30) satisfies conditions ($d_1$), ($d_2$), ($d_3$). It follows from (28) and (18), (18′) that

$$\mathcal{D}_r\begin{pmatrix}\eta_1, \ldots, \eta_r \\ y_1, \ldots, y_r\end{pmatrix} = 1,$$

i. e. ($d_4$) is also satisfied.

It follows immediately from (27) that identities ($D_n$) and ($D'_n$) hold for $n = 0, \ldots, n-2$. They also hold for $n = r-1$. In fact, the right side is then equal to zero by (27) and so is the left side on account of (28). More exactly, if either one of the points $\xi_1, \ldots, \xi_2$ belongs to $\Xi_1$ (i. e. is of the form $\xi A$) or one of the points $x_1, \ldots, x_r$ belongs to $X_1$ (i. e. is of the form $Ax$), then

$$\mathcal{D}_r\begin{pmatrix}\xi_1, \ldots, \xi_r \\ x_1, \ldots, x_r\end{pmatrix} = 0 \tag{31}$$

since points in $\Xi_1$ are orthogonal to all $z_1, \ldots, z_r$ and points in $X_1$ are orthogonal to all $\zeta_1, \ldots, \zeta_r$.

Since the $\xi$'s and the $x$'s play a symmetric role everywhere, in order to prove ($d_5$) it suffices to prove only ($D_n$) for $n = r+k$, $k \geqslant 0$.

We have

$$\mathcal{D}_{r+k+1}\begin{pmatrix}\xi_0, \xi_1, \ldots, \xi_{r+k} \\ x_0, x_1, \ldots, x_{r+k}\end{pmatrix}$$

$$= \sum_{\mathfrak{p}',\mathfrak{q}} \operatorname{sgn}\mathfrak{p}' \cdot \operatorname{sgn}\mathfrak{q} \cdot \begin{vmatrix} \xi_0 Bx_{q_0}, \ldots, \xi_0 Bx_{q_k} \\ \xi_{p_1} Bx_{q_0}, \ldots, \xi_{p_1} Bx_{q_k} \\ \cdots\cdots\cdots \\ \xi_{p_k} Bx_{q_0}, \ldots, \xi_{p_k} Bx_{q_k} \end{vmatrix} \cdot \mathcal{D}_r\begin{pmatrix}\xi_{p_{k+1}}, \ldots, \xi_{p_{k+r}} \\ x_{q_{k+1}}, \ldots, x_{q_{k+r}}\end{pmatrix} +$$

$$+ \sum_{\mathfrak{p}'',\mathfrak{q}} \operatorname{sgn}\mathfrak{p}'' \cdot \operatorname{sgn}\mathfrak{q} \cdot \begin{vmatrix} \xi_{p_0} Bx_{q_0}, \ldots, \xi_{p_0} Bx_{q_k} \\ \xi_{p_1} Bx_{q_0}, \ldots, \xi_{p_1} Bx_{q_k} \\ \cdots\cdots\cdots \\ \xi_{p_k} Bx_{q_0}, \ldots, \xi_{p_k} Bx_{q_k} \end{vmatrix} \cdot \mathcal{D}_r\begin{pmatrix}\xi_0, \xi_{p_{k+2}}, \ldots, \xi_{p_{k+r}} \\ x_{q_{k+1}}, x_{q_{k+2}}, \ldots, x_{q_{k+r}}\end{pmatrix},$$

where $\mathfrak{p}'$, $\mathfrak{p}''$, $\mathfrak{q}$ denote arbitrary permutations (of the integers $0, 1, \ldots, r+k$) of the form:

$$\begin{aligned} \mathfrak{p}' &= (0, p_1, \ldots, p_k, p_{k+1}, \ldots, p_{k+r}), && p_1 < \ldots < p_k,\ p_{k+1} < \ldots < p_{k+r}; \\ \mathfrak{p}'' &= (p_0, p_1, \ldots, p_k, 0, p_{k+2}, \ldots, p_{k+r}), && p_0 < \ldots < p_k,\ p_{k+2} < \ldots < p_{k+r}; \\ \mathfrak{q} &= (q_0, q_1, \ldots, q_k, q_{k+1}, \ldots, q_{k+r}), && q_0 < \ldots < q_k,\ q_{k+1} < \ldots < q_{k+r}. \end{aligned}$$



It follows from (31) that

$$\mathcal{D}_{r+k+1}\begin{pmatrix}\xi_0 A, \xi_1, \ldots, \xi_{r+k} \\ x_0, x_1, \ldots, x_{r+k}\end{pmatrix}$$

$$= \sum_{\mathfrak{p}',\mathfrak{q}} \operatorname{sgn}\mathfrak{p}' \cdot \operatorname{sgn}\mathfrak{q} \cdot \begin{vmatrix} \xi_0 ABx_{q_0}, \ldots, \xi_0 ABx_{q_k} \\ \xi_{p_1} Bx_{q_0}, \ldots, \xi_{p} Bx_{q_k} \\ \cdots\cdots\cdots \\ \xi_{p_k} Bx_{q_0}, \ldots, \xi_{p_k} Bx_{q_k} \end{vmatrix} \cdot \mathcal{D}_r\begin{pmatrix}\xi_{p_{k+1}}, \ldots, \xi_{p_{k+r}} \\ x_{q_{k+1}}, \ldots, x_{q_{k+r}}\end{pmatrix}$$

$$= \sum_{\mathfrak{p}',\mathfrak{q}} \sum_{j=0}^{k} (-1)^j \cdot \operatorname{sgn}\mathfrak{p} \cdot \operatorname{sgn}\mathfrak{q} \cdot \xi_0 ABx_{q_j} \cdot$$

$$\cdot \begin{vmatrix} \xi_{p_1} Bx_{q_0}, \ldots, \xi_{p_1} Bx_{q_{j-1}}, \xi_{p_1} Bx_{q_{j+1}}, \ldots, \xi_{p_1} Bx_{q_k} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xi_{p_k} Bx_{q_0}, \ldots, \xi_{p_k} Bx_{q_{j-1}}, \xi_{p_k} Bx_{q_{j+1}}, \ldots, \xi_{p_k} Bx_{q_k} \end{vmatrix} \cdot \mathcal{D}_r\begin{pmatrix}\xi_{p+1}, \ldots, \xi_{p_{k+r}} \\ x_{q_{k+1}}, \ldots, x_{q_{k+r}}\end{pmatrix}$$

$$= \sum_{i=0}^{r+k} (-1)^i \cdot \xi_0 ABx_i \cdot \sum_{\mathfrak{p},\mathfrak{q}_i} \operatorname{sgn}\mathfrak{p} \cdot \operatorname{sgn}\mathfrak{q}_i \cdot$$

$$\cdot \begin{vmatrix} \xi_{p_1} Bx_{q_{i,1}}, \ldots, \xi_{p_1} Bx_{q_{i,k}} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \xi_{p_k} Bx_{q_{i,1}}, \ldots, \xi_{p_k} Bx_{q_{i,k}} \end{vmatrix} \cdot \mathcal{D}_r\begin{pmatrix}\xi_{p_{k+1}}, \ldots, \xi_{p_{k+r}} \\ x_{q_{i,k+1}}, \ldots, x_{q_{i,k+r}}\end{pmatrix}$$

where $\mathfrak{p} = (p_1, \ldots, p_{k+r})$ is any permutation of the integers $1, \ldots, k+r$ such that $p_1 < \ldots < p_k$, $p_{k+1} < \ldots < p_{k+r}$, and $\mathfrak{q}_i = (q_{i,1}, \ldots, q_{i,k+r})$ is any permutation of the integers $0, \ldots, i-1, i+1, \ldots, k+r$ such that $q_{i,1} < \ldots < q_{i,k}$, $q_{i,k+1} < \ldots < q_{i,k+r}$. The factor $(-1)^j$ is now replaced by $(-1)^i$ since if $q_j = i$, $\mathfrak{p}$ is the permutation obtained from $\mathfrak{p}'$ by omitting the integer 0, and $\mathfrak{q}_i$ is the permutation obtained from $\mathfrak{q}$ by omitting the integer $i$, then the corresponding $\pm 1$ - coefficients satisfy the equality

$$(-1)^j \cdot \operatorname{sgn}\mathfrak{p}' \cdot \operatorname{sgn}\mathfrak{q} = (-1)^i \cdot \operatorname{sgn}\mathfrak{p} \cdot \operatorname{sgn}\mathfrak{q}_i.$$

The sum $\sum_{\mathfrak{p},\mathfrak{q}_i}$ being equal to

$$\mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots, \xi_{r+k} \\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+k}\end{pmatrix},$$

we obtain

$$\mathcal{D}_{r+k+1}\begin{pmatrix}\xi_0 A, \xi_1, \ldots, \xi_{r+k}\\ x_0, x_1, \ldots, x_{r+k}\end{pmatrix}$$

$$= \sum_{i=0}^{r+k}(-1)^i\cdot\xi_0 ABx_i\cdot D_{r+k}\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots\ldots, \xi_{r+k}\\ x_0, \ldots, x_{i+1}, x_{i+1}, \ldots, x_{r+k}\end{pmatrix}.$$

Thus, in order to complete the proof of $(\mathrm{D}_{r+k})$ we have to prove that

$$(32)\qquad \sum_{i=k}^{r+k}(-1)^i\cdot\xi_0 ABx_i\cdot\mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots\ldots, \xi_{r+k}\\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+k}\end{pmatrix}$$

$$= \sum_{i=0}^{r+k}(-1)^i\cdot\xi_0 x_i\cdot\mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots\ldots, \xi_{r+k}\\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+k}\end{pmatrix}.$$

Every point $x\in X$ can be represented in the form

$$x = x'+a_1y_1+\ldots+a_ry_r$$

where $x'\in X_1$ (see § 2). The left and the right sides of (32) are linear in each of the variables $x_0, x_1, \ldots, x_{r+k}$. Therefore, in order to prove (32) it suffices to show that this equality holds if each of points $x_0, x_1, \ldots, x_{r+k}$ is either equal to one of the points $y_1, \ldots, y_r$ or belongs to $X_1$. The proof is based on the following remark, which follows directly from (29) and (31):

$$(33)\qquad \mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots, \xi_{r+k}\\ x_1, \ldots, x_{r+k}\end{pmatrix} = 0$$

whenever more than $k$ points among $x_1, \ldots, x_{r+k}$ belong to $X_1$.

Consider first the case where each of points $y_1, \ldots, y_r$ appears at most once in the sequence $x_0, x_1, \ldots, x_{r+k}$. Then at least $k+1$ points among $x_0, x_1, \ldots, x_{r+k}$ belong to $X_1$.

If $x_i\in X_1$, then $ABx_i = x_i$ (see (q)) and consequently

$$(34)\qquad (-1)^i\cdot\xi_0 ABx_i\cdot\mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots\ldots, \xi_{r+k}\\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+k}\end{pmatrix}$$

$$= (-1)^i\cdot\xi_0 x_i\cdot\mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots\ldots, \xi_{r+k}\\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+k}\end{pmatrix}.$$



If $x_i = y_j$, then equality (34) also holds since then

$$(35)\qquad \mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots\ldots, \xi_{r+k}\\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+k}\end{pmatrix} = 0$$

on account of (33).

Suppose now that an element $y_j$ appears in the sequence $x_0, x_1, \ldots, x_{r+k}$ at least twice, i. e. $x_{i_1} = y_j = x_{i_2}$ $(i_1\neq i_2)$. Then, by the skew symmetry of $\mathcal{D}_{r+k}$,

$$(-1)^{i_1}\cdot\xi_0 ABx_{i_1}\cdot\mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots\ldots, \xi_{r+k}\\ x_0, \ldots, x_{i_1-1}, x_{i_1+1}, \ldots, x_{r+k}\end{pmatrix} +$$

$$+(-1)^{i_2}\cdot\xi_0 ABx_{i_2}\cdot\mathcal{D}_{r+k}\begin{pmatrix}\xi_0, \ldots\ldots\ldots\ldots\ldots\ldots, \xi_{r+k}\\ x_0, \ldots, x_{i_2-1}, x_{i_2+1}, \ldots, x_{r+k}\end{pmatrix} = 0,$$

and analogously

$$(-1)^{i_1}\cdot\xi_0 x_{i_1}\cdot\mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots\ldots, \xi_{r+k}\\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+k}\end{pmatrix} +$$

$$+(-1)^{i_2}\cdot\xi_0 x_{i_2}\cdot\mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots\ldots, \xi_{r+k}\\ x_0, \ldots, x_{i_2-1}, x_{i_2+1}, \ldots, x_{r+k}\end{pmatrix} = 0.$$

If $i_1\neq i\neq i_2$, then (34) holds. In fact, (35) is then satisfied by the skew symmetry, since $y_j$ appears twice in the sequence $x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r$. This proves (32).

Suppose now that $D_0, D_1, D_2, \ldots$ is any determinant system for $A$. We shall prove that there exists a scalar $c\neq 0$ such that

$$(36)\qquad D_n = c\mathcal{D}_n \quad\text{for}\quad n = 0, 1, 2, \ldots$$

where $\{\mathcal{D}_n\}$ is defined by (27), (28), (29).

It follows from Theorem II that $r$ is the order of $\{D_n\}$, i. e.

$$(37)\qquad D_n = 0 \quad\text{for}\quad n = 0, \ldots, r-1$$

and $D_r\neq 0$. It follows from $(\mathrm{d}_2)$, $(\mathrm{D}'_{r-1})$, $(\mathrm{D}_{r-1})$ and from (37) that

$$(38)\qquad D_r\begin{pmatrix}\xi_1, \ldots, \xi_r\\ x_1, \ldots, x_r\end{pmatrix} = 0$$

if either one of the points $x_1, \ldots, x_r$ belongs to $X_1$ (i. e. is of the form $Ax$) or one of the points $\xi_1, \ldots, \xi_r$ belongs to $\Xi_1$ (i. e. is of the form $\xi A$).

Every point $x_i \in X$ can be uniquely represented in the form $x_i = x_i' + x_i''$ where $x_i' \in X_1$ and $x_i'' \in X_2$ and, analogously, every point $\xi_i \in \Xi$ can be uniquely represented in the form $\xi_i = \xi_i' + \xi_i''$ where $\xi_i' \in \Xi_1$ and $\xi_i'' \in \Xi_2$. Property (38) implies that for arbitrary $x_i \in X$ and $\xi_i \in \Xi$ $(i = 1, \dots, r)$,

$$(39) \qquad D_r\begin{pmatrix}\xi_1, \dots, \xi_r\\ x_1, \dots, x_r\end{pmatrix} = D_r\begin{pmatrix}\xi_1'', \dots, \xi_r''\\ x_1'', \dots, x_r''\end{pmatrix},$$

i. e. $D_r$ is completely determined by its values on subspaces $\Xi_2$ and $X_2$. The same is true for $\mathcal{D}_r$, i. e.

$$(40) \qquad \mathcal{D}_r\begin{pmatrix}\xi_1, \dots, \xi_r\\ x_1, \dots, x_r\end{pmatrix} = \mathcal{D}_r\begin{pmatrix}\xi_2'', \dots, \xi_r''\\ x_1'', \dots, x_r''\end{pmatrix}.$$

Since $\Xi_2$ and $X_2$ are $r$-dimensional, all $2r$-linear functionals defined on $\underbrace{\Xi_2 \times \dots \times \Xi_2}_{r\text{ times}} \times \underbrace{X_2 \times \dots \times X_2}_{r\text{ times}}$ and skew symmetric in $\xi_1'', \dots, \xi_r'' \in \Xi_2$ and $x_1'', \dots, x_2'' \in X_2$ differ only by a scalar factor. In particular, there exists a scalar $c \neq 0$ (since $D_r \neq 0 \neq \mathcal{D}_r$) such that

$$D_r\begin{pmatrix}\xi_1'', \dots, \xi_r''\\ x_1'', \dots, x_r''\end{pmatrix} = c \cdot \mathcal{D}_r\begin{pmatrix}\xi_1'', \dots, \xi_r''\\ x_1'', \dots, x_r''\end{pmatrix}$$

for $\xi_1'', \dots, \xi_r'' \in \Xi_2$ and $x_1'', \dots, x_2'' \in X_2$. This proves, by (39) and (40), that

$$D_r\begin{pmatrix}\xi_1, \dots, \xi_r\\ x_1, \dots, x_r\end{pmatrix} = c \cdot \mathcal{D}_r\begin{pmatrix}\xi_1, \dots, \xi_r\\ x_1, \dots, x_r\end{pmatrix}$$

for arbitrary $\xi_1, \dots, \xi_r \in \Xi$ and $x_1, \dots, x_r \in X$. This and (27), (37) imply that (36) is true for $n = 0, \dots, r$.

We shall prove, by induction with respect to $k$, that (36) is true for $n = r+k$, $k = 0, 1, 2, \dots$ The case $k = 0$ was just proved.

Suppose that (36) is true for $n = r+k$. We shall prove that it is true for $n+1 = r+k+1 > r$, i. e.

$$(41) \qquad D_{n+1}\begin{pmatrix}\xi_0, \xi_1, \dots, \xi_n\\ x_0, x_1, \dots, x_n\end{pmatrix} = c \cdot \mathcal{D}_{n+1}\begin{pmatrix}\xi_0, \xi_1, \dots, \xi_n\\ x_0, x_1, \dots, x_n\end{pmatrix}.$$

Since $D_{n+1}$ and $\mathcal{D}_{n+1}$ are linear in all variables, it suffices to prove (41) only in the case where each of the points $x_0, x_1, \dots, x_n$ either belongs to $X_1$ or is equal to one of the points $y_1, \dots, y_r$.

If the sequence $x_0, x_1, \dots, x_n$ contains one point of the form $Ax$ (i. e. belonging to $X_1$), then (41) holds. This follows from the induction hypothesis and from the identity $(\mathrm{D}_n')$ for $D_{n+1}$ and for $\mathcal{D}_{n+1}$ (see also $(\mathrm{d}_2)$).



If the sequence $x_0, x_1, \dots, x_n$ contains exclusively points $y_1, \dots, y_r$, one of them appears at least twice. Hence, by the skew symmetry $(\mathrm{d}_2)$, both sides of (41) are equal to 0.

Corollary. *$A \in \mathfrak{B}$ has a determinant system in $\mathfrak{B}$ if and only if $A$ is Fredholm. The determinant system (22) of $A$ is determined by $A$ uniquely up to a constant factor $c \neq 0$, viz. if the order of $A$ is zero, then*

$$(42_0) \qquad D_n = D_0 \cdot \mathcal{D}_n \quad \text{for} \quad n = 0, 1, 2, \dots$$

*where $\mathcal{D}_n$ are given by (23), and if the order of $A$ is positive, then*

$$(42) \qquad D_n = c \cdot \mathcal{D}_n \quad \text{for} \quad n = 0, 1, 2, \dots$$

*where $c \neq 0$ and $\mathcal{D}_n$ are defined by (27), (28), (29).*

*If $A \in \mathfrak{A}$ has a determinant system in $\mathfrak{A}$ and $A \in \mathfrak{B}$, then $A$ has also a determinant system in $\mathfrak{B}$ (viz. the same determinant system has this property).*

**§ 7. Determinant systems in product spaces.** Suppose that the spaces $\Xi'$, $X'$ are conjugate, and $\Xi''$, $X''$ are conjugate. Let $\Xi = \Xi' \times \Xi''$ and $X = X' \times X''$. For every point $x \in X$, $x'$ and $x''$ always denote such points in $X'$ and $X''$ respectively that $x = (x', x'')$. Analogously, if $\xi \in \Xi$, then $\xi'$ and $\xi''$ always denote such points in $\Xi'$ and $\Xi''$ respectively that $\xi = (\xi', \xi'')$.

Theorem IV. *If*

$$(43) \qquad D_0', D_1', D_2', \dots$$

*is a determinant system for a bilinear functional $A'$ on $\Xi' \times X'$, and*

$$(44) \qquad D_0'', D_1'', D_2'', \dots$$

*is a determinant system for a bilinear functional $A''$ on $\Xi'' \times X''$, then the formulas*

$$(45) \qquad D_n\begin{pmatrix}\xi_1, \dots, \xi_n\\ x_1, \dots, x_n\end{pmatrix} = \sum_{s=0}^{n} \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p} \cdot \operatorname{sgn}\mathfrak{q} \cdot D_s'\begin{pmatrix}\xi_{p_1}', \dots, \xi_{p_s}'\\ x_{q_1}', \dots, x_{q_s}'\end{pmatrix} \cdot D_{n-s}''\begin{pmatrix}\xi_{p_{s+1}}'', \dots, \xi_{p_n}''\\ x_{q_{s+1}}'', \dots, x_{q_n}''\end{pmatrix}$$

*(where $\sum_{\mathfrak{p},\mathfrak{q}}$ is extended over all the permutations $\mathfrak{p} = (p_1, \dots, p_n)$ and $\mathfrak{q} = (q_1, \dots, q_n)$ of the integers $1, \dots, n$, such that*

$$p_1 < p_2 < \dots < p_s, \quad p_{s+1} < p_{s+2} < \dots < p_n,$$
$$q_1 < q_2 < \dots < q_s, \quad q_{s+1} < q_{s+2} < \dots < q_n)$$

*define a determinant system (in $\mathfrak{A}$)*

$$D_0, D_1, D_2, \dots \tag{46}$$

*for the bilinear functional* $A = (A', A'')$ *on* $\Xi \times X$ (see definition (19)).

It is evident that (46) has the properties ($\mathrm{d}_1$), ($\mathrm{d}_2$) and ($\mathrm{d}_3$). It also has the property ($\mathrm{d}_4$) since if $r'$ and $r''$ are orders of (43) and (44) respectively, and points $x'_1, \dots, x'_{r'} \in X'$, $x''_{r'+1}, \dots, x''_{r'+r''} \in X''$, $\xi'_1, \dots, \xi'_{r'} \in \Xi$, $\xi''_{r'+1}, \dots, \xi_{r'+r''}$ are such that

$$D'_{r'}\begin{pmatrix}\xi'_1, \dots, \xi'_{r'}\\ x'_1, \dots, x'_{r'}\end{pmatrix} \neq 0, \qquad D''_{r''}\begin{pmatrix}\xi''_{r'+1}, \dots, \xi''_{r'+r''}\\ x''_{r'+1}, \dots, x''_{r'+r''}\end{pmatrix} \neq 0, \tag{47}$$

then $D_{r'+r''}\begin{pmatrix}\xi_1, \dots, \xi_{r'+r''}\\ x_1, \dots, x_{r'+r''}\end{pmatrix}$ is different from zero for

$$x_i = (x'_i, 0), \qquad \xi_i = (\xi'_i, 0) \qquad (i = 1, \dots, r'),$$

$$x_{r'+i} = (0, x''_{r'+i}), \qquad \xi_i = (0, \xi''_{r'+i}) \qquad (i = 1, \dots, r''),$$

since its value is then equal to the product of scalars (47).

We have to prove that (46) satisfies the identities ($\mathrm{D}_n$) and ($\mathrm{D}'_n$). Since $\xi$'s and $x$'s play a symmetric role everywhere, it suffices to prove only ($\mathrm{D}_n$).

By definition (45), we have

$$D_{n+1}\begin{pmatrix}\xi_0, \xi_1, \dots, \xi_n\\ x_0, x_1, \dots, x_n\end{pmatrix}$$

$$= \sum_{s=0}^{n}\sum_{\mathfrak{p}', \mathfrak{q}'} \operatorname{sgn}\mathfrak{p}' \cdot \operatorname{sgn}\mathfrak{q}' \cdot D'_{s+1}\begin{pmatrix}\xi'_0, \xi'_{p_1}, \dots, \xi'_{p_s}\\ x'_{q_0}, x'_{q_1}, \dots, x'_{q_s}\end{pmatrix} \cdot D''_{n-s}\begin{pmatrix}\xi''_{p_{s+1}}, \dots, \xi''_{p_n}\\ x''_{q_{s+1}}, \dots, x''_{q_n}\end{pmatrix} +$$

$$+ \sum_{s=0}^{n}\sum_{\mathfrak{p}'', \mathfrak{q}''} \operatorname{sgn}\mathfrak{p}'' \cdot \operatorname{sgn}\mathfrak{q}'' \cdot D'_{s}\begin{pmatrix}\xi'_{p_1}, \xi'_{p_2}, \dots, \xi'_{p_s}\\ x'_{q_0}, x'_{q_1}, \dots, x'_{q_{s-1}}\end{pmatrix} \cdot D''_{n-s+1}\begin{pmatrix}\xi''_0, \xi''_{p_{s+1}}, \dots, \xi''_{p_n}\\ x''_{q_s}, x''_{q_{s+1}}, \dots, x''_{q_n}\end{pmatrix}$$

where $\mathfrak{p}'$, $\mathfrak{p}''$, $\mathfrak{q}'$, $\mathfrak{q}''$ are any permutations of numbers $0, 1, \dots, n$ such that

$$\mathfrak{p}' = (0, p_1, \dots, p_n), \qquad p_1 < \dots < p_s, \qquad p_{s+1} < \dots < p_n, \tag{48}$$

$$\mathfrak{p}'' = (p_1, \dots, p_s, 0, p_{s+1}, \dots, p_n), \qquad p_1 < \dots < p_s, \qquad p_{s+1} < \dots < p_n, \tag{49}$$

$$\mathfrak{q}' = (q_0, \dots, q_n), \qquad q_0 < \dots < q_s, \qquad q_{s+1} < \dots < q_n, \tag{50}$$

$$\mathfrak{q}'' = (q_0, \dots, q_n), \qquad q_0 < \dots < q_{s-1}, \qquad q_s < \dots < q_n. \tag{51}$$



By ($\mathrm{D}_s$) and ($\mathrm{D}_{n-s}$),

$$D_{n+1}\begin{pmatrix}\xi_0 A, \xi_1, \dots, \xi_n\\ x_0, x_1, \dots, x_n\end{pmatrix}$$

$$= \sum_{s=0}^{n}\sum_{\mathfrak{p}', \mathfrak{q}'}\sum_{j=0}^{s} (-1)^j \cdot \operatorname{sgn}\mathfrak{p}' \cdot \operatorname{sgn}\mathfrak{q}' \cdot \xi'_0 A' x'_{q_j} \cdot D'_s\begin{pmatrix}\xi'_{p_1}, \dots\dots\dots\dots, \xi'_{p_s}\\ x'_{q_0}, \dots, x'_{q_{j-1}}, x'_{q_{j+1}}, \dots, x'_{q_s}\end{pmatrix} \cdot$$

$$\cdot D''_{n-s}\begin{pmatrix}\xi''_{p_{s+1}}, \dots, \xi''_{p_n}\\ x''_{q_{s+1}}, \dots, x''_{q_n}\end{pmatrix} +$$

$$+ \sum_{s=0}^{n}\sum_{\mathfrak{p}'', \mathfrak{q}''}\sum_{j=0}^{n-s} (-1)^j \cdot \operatorname{sgn}\mathfrak{p}'' \cdot \operatorname{sgn}\mathfrak{q}'' \cdot \xi''_0 A'' x''_{q_{s+j}} \cdot D'_s\begin{pmatrix}\xi'_{p_1}, \dots, \xi'_{p_s}\\ x'_{q_0}, \dots, x'_{q_{s-1}}\end{pmatrix} \cdot$$

$$\cdot D''_{n-s}\begin{pmatrix}\xi''_{p_{s+1}}, \dots\dots\dots\dots, \xi''_{p_n}\\ x''_{q_s}, \dots, x''_{q_{s+j-1}}, x''_{q_{s+j+1}}, \dots, x''_{q_n}\end{pmatrix}$$

$$= \sum_{i=0}^{n} (-1)^i \cdot \xi'_0 A' x'_i \cdot \sum_{s=0}^{n}\sum_{\mathfrak{p}'_0, \mathfrak{q}'_i} \operatorname{sgn}\mathfrak{p}'_0 \cdot \operatorname{sgn}\mathfrak{q}'_i \cdot D'_s\begin{pmatrix}\xi'_{p_1}, \dots, \xi'_{p_s}\\ x'_{q_{i,1}}, \dots, x'_{q_{i,s}}\end{pmatrix} \cdot$$

$$\cdot D''_{n-s}\begin{pmatrix}\xi''_{p_{s+1}}, \dots, \xi''_{p_n}\\ x''_{q_{i,s-1}}, \dots, \xi''_{q_{i,n}}\end{pmatrix} +$$

$$+ \sum_{i=1}^{n} (-1)^i \cdot \xi''_0 A'' x''_i \cdot \sum_{s=0}^{n}\sum_{\mathfrak{p}''_0, \mathfrak{q}''_i} \operatorname{sgn}\mathfrak{p}''_0 \cdot \operatorname{sgn}\mathfrak{q}''_i \cdot D'_s\begin{pmatrix}\xi'_{p_1}, \dots, \xi'_{p_s}\\ x'_{q_{i,1}}, \dots, x'_{q_{i,s}}\end{pmatrix} \cdot$$

$$\cdot D''_{n-s}\begin{pmatrix}\xi''_{p_{s+1}}, \dots, \xi''_{p_n}\\ x''_{q_{i,s+1}}, \dots, x''_{q_{i,n}}\end{pmatrix}.$$

In the last formula, $\mathfrak{p}'_0 = (p_1, \dots, p_n)$ and $\mathfrak{p}''_0 = (p_1, \dots, p_n)$ are permutations (of the integers $1, \dots, n$) obtained from $\mathfrak{p}'$ and $\mathfrak{p}''$ respectively (see (48) and (49)) by omitting the number zero; consequently

$$p_1 < \dots < p_s \quad \text{and} \quad p_{s+1} < \dots < p_n.$$

Similarly $\mathfrak{q}'_i = (q_{i,1}, \dots, q_{i,n})$ and $\mathfrak{q}''_i = (q_{i,1}, \dots, q_{i,n})$ are permutations (of numbers $0, \dots, i-1$, $i+1, \dots, n$) obtained respectively from $q'$ and $q''$ (see (50) and (51) by omitting the number $i$; consequently

$$q_{i,1} < \dots < q_{i,s} \quad \text{and} \quad q_{i,s+1} < \dots < q_{i,n}.$$

The factor $(-1)^j$ is now replaced by $(-1)^i$ since if $q_j = i$, then the corresponding $\pm 1$-coefficients at $\xi'_0 A' x'_{qj}$, $\xi'_0 A' x'_i$ satisfy the equality

$$(-1)^j \cdot \operatorname{sgn}\mathfrak{p}' \cdot \operatorname{sgn}\mathfrak{q}' = (-1)^i \cdot \operatorname{sgn}\mathfrak{p}'_0 \cdot \operatorname{sgn}\mathfrak{q}'_i;$$

analogously, if $q_{s+j} = i$, then the corresponding $\pm 1$-coefficients at $\xi''_0 A'' x''_{q+j}$, $\xi''_0 A'' x''_i$ satisfy the equality

$$(-1)^j \cdot \operatorname{sgn}\mathfrak{p}'' \cdot \operatorname{sgn}\mathfrak{q}'' = (-1)^i \cdot \operatorname{sgn}\mathfrak{p}''_0 \cdot \operatorname{sgn}\mathfrak{q}''_i.$$

Consequently we have

$$D_{n+1}\begin{pmatrix}\xi_0 A, \xi_1, \ldots, \xi_n\\ x_0, x_1, \ldots, x_n\end{pmatrix}$$

$$= \sum_{i=0}^{n}(-1)^i\cdot \xi_0' A' x_i'\cdot D_n\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots, \xi_n\\ x_0, \ldots, x_{i-1}, x_{i+1}\ldots, x_n\end{pmatrix}+$$

$$+\sum_{i=0}^{n}(-1)^i\cdot \xi_0'' A'' x_i''\cdot D_n\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots, \xi_n\\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\end{pmatrix}$$

$$= \sum_{i=0}^{n}(-1)^i\cdot \xi_0 A x_i\cdot D_n\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots, \xi_n\\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\end{pmatrix}$$

since $\xi_0' A' x_i' + \xi_0'' A'' x_i'' = \xi_0 A x_i$ by the definition (19) of $A$.

## References


[1] A. Grothendieck, *La théorie de Fredholm*, Bull. Soc. Math. France 84 (1956), p. 319-384.

[2] — *Produits tensoriels topologiques et espaces nucléaires*, Memoires of Amer. Math. Soc. 1955.

[3] T. Leżański, *The Fredholm theory of linear equations in Banach spaces*, Studia Math. 13(1953), p. 244-276.

[4] — *Sur les fonctionelles multiplicatives*, ibidem 14 (1953), p. 13-23.

[5] A. F. Ruston, *On the Fredholm theory of integral equations for operators belonging to the trace class of a general Banach space*, Proc. London Math. Soc. (2) 53 (1951), p. 109-124.

[6] — *Direct products of Banach spaces and linear functional equations*, ibidem (3) 1 (1951), p. 327-384.

[7] — *Formulae of Fredholm type for compact linear operators on a general Banach space*, ibidem (3) 3 (1953), p. 368-377.

[8] — *Operators with a Fredholm theory*, Journal of London Math. Soc. 29 (1954), p. 318-326.

[9] R. Sikorski, *On Leżański's determinants of linear equations in Banach spaces*, Studia Math. 14 (1953), p. 24-48.

[10] — *On determinants of Leżański and Ruston*, ibidem 16 (1957), p. 99-112.


INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

*Reçu par la Rédaction le 30. 7. 1958*

STUDIA MATHEMATICA, T. XVIII. (1959).

# On Leżański endomorphisms

by

R. SIKORSKI (Warszawa)

This paper is a supplement to my paper [5](1).

An endomorphism $T$ of a complex Banach space $X$ is said to be a *Leżański endomorphism* provided the functional

(1) $$F_0(K) = \operatorname{trace} KT \qquad (K \in \mathfrak{K}_0)$$

is continuous on the space $\mathfrak{K}_0$ of all finitely dimensional endomorphisms $K$ in $X$ (with respect to the usual norm of $K$), i. e. if it satisfies the hypotheses of Leżański's [1], [2] determinant theory of the linear equation

(2) $$x+\lambda Tx = x_0.$$

In [5] I quoted an example of a Leżański endomorphism $T$ (in the space $L$) which was not compact (= completely continuous). However, the endomorphism $T^2$ was compact. The subject of this paper is to prove that this is true for every Leżański endomorphism. More precisely:

THEOREM. *If $T$ is a Leżański endomorphism in $X$, then $T^2$ (and, consequently, $T^n$ for $n = 2, 3, \ldots$) is the limit (in the norm) of a sequence of finitely dimensional endomorphisms.*

Let $F$ be any continuous linear extension of $F_0$ (see (1)) over the space $\mathfrak{K}$ of all linear continuous endomorphisms (with the usual norm) in $X$. Let $D_0(\lambda)$ be the Leżański determinant of (2), determined by $F$. $D_0(\lambda)$ is an entire function of $\lambda$ and, for small $\lambda$,

(3) $$D_0(\lambda) = \exp\left(\frac{\sigma_1\lambda}{1} - \frac{\sigma_2\lambda^2}{2} + \frac{\sigma_3\lambda^3}{3} - \frac{\sigma_4\lambda^4}{4} + \ldots\right)$$

where

(4) $$\sigma_n = F(T^{n-1}) \quad \text{for} \quad n = 1, 2, \ldots$$

(1) Errata to [5]. In footnote (4) on p. 106 instead of "$\mathfrak{K}^*$ is identical with $\mathfrak{K}_*$" we should have "$K_*$ is identical with the class of all $T$ satisfying (*)".

Errata to [4]. The lines 18-30 on p. 46 should be omitted.